6/23

$$\boxed{U = (1-\phi^2)^2 (4-\phi^2)^2}$$

$1 \leq \phi \leq 2$

$x_0 \gg x$

$2 - e^{\underbrace{-b(x-x_0)+\ln a}_{-b(x - x_0 - \frac{\ln a}{b})}}$

$$\phi' = \sqrt{2U} = \sqrt{2}\,(\phi^2-1)(4-\phi^2)$$

$\phi^{(2)} \approx 1 + a e^{-b(x_0-x)}$

$\phi^{(2)\prime} = \cancel{a} b\, e^{\cancel{-b(x_0-x)}}$

$= \sqrt{2}\cdot 3 \cdot 2 \cancel{a}\, e^{\cancel{-b(x_0-x)}}$

$$\phi = \phi^{(1)} + \underbrace{\phi^{(2)}}_{a e^{-6\sqrt{2}(x_2-x)} \equiv \Delta} - 1$$

$\phi^{(1)} = 1 - a' e^{-6\sqrt{2}(x-x_1)}$

$x \gg x_1$

$\phi^{(1)} = 1 - a' e^{-b'(x-x_1)}$

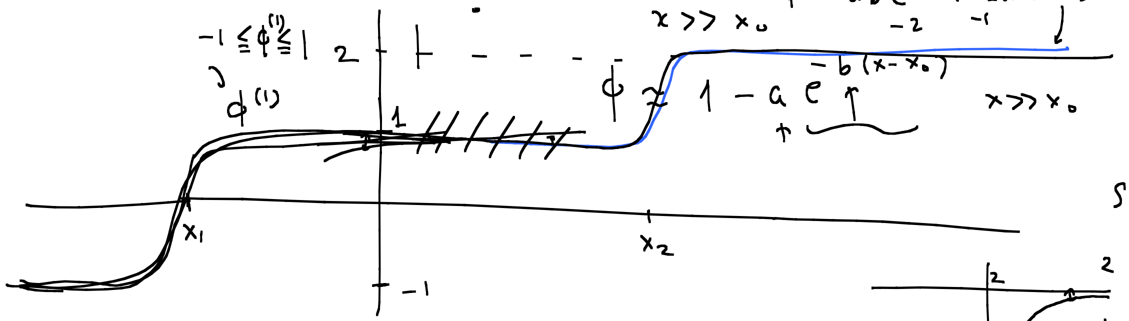
$\phi^{(1)\prime} = a' b' e^{\cdots} = \sqrt{2}\; 2 a' e^{-b'\cdots} \cdot 3$

$b = 6\sqrt{2}$.

$b' = 6\sqrt{2}$

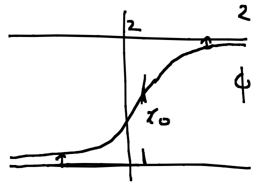
$-1 \leq \phi^{(1)} \leq 1$

$\phi^{(1)}$



$\phi \approx 1 - a e^{-b(x-x_0)}$

S-S



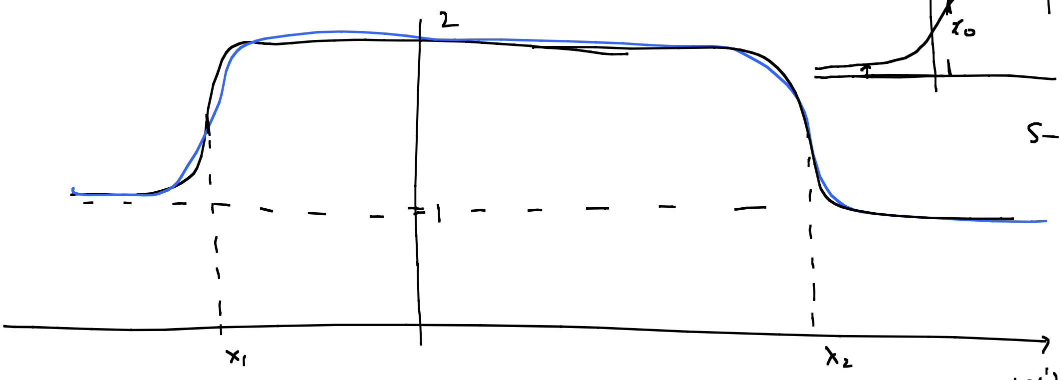
$x \gg x_0$

$\phi \approx 2 - a e^{-b(x-x_0)}$

$x \ll x_0$

$1 + c e^{+b'(x-x_0)}$

S-A



$-96 a' e^{\cdots}$

$\cancel{-24}$

$= -12 a e^{\cdot}$

$36(-2a')$

$+ \overbrace{2(\varphi^2-4)\,2\varphi}$

$V' = 2(\varphi^2-1)\,2\varphi \cdot 9$

$V = (\varphi^2-1)^2 \overbrace{(\varphi^2-4)}^{2}$

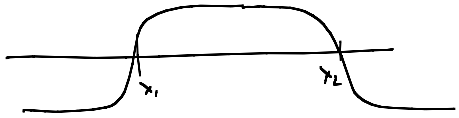
$\varphi_1 = 1 - a' e^{-6\sqrt{2}(x-x_1)}$

$\Delta = a\, e^{-6\sqrt{2}(x_2-x)}$

$$-\varphi_1' \Delta' + V'(\varphi_1)\Delta$$

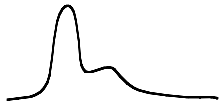
$$= -6\sqrt{2}\, a' \underbrace{e^{-6\sqrt{2}(x-x_1)} (6\sqrt{2}\, a)\, e^{-6\sqrt{2}(x_2-x)}}_{e^{-6\sqrt{2}R}} + -96\, a a' e^{-6\sqrt{2}(x_2-x_1)}$$

$$= -168\, a a'\, e^{-6\sqrt{2}(R + c)}$$



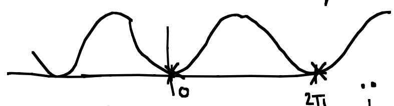
$x_1 \ll x_2$

$x_1 \sim x_2$



특별한 경우만 multi-soliton 해가 가능.

(KdV, sine-Gordon) ✓	(φ^4, φ^6, ...) no.
integrable	non-integrable



$$V(\phi) = \frac{m^2}{\lambda}(1-\cos(\lambda\phi)) \Rightarrow \ddot{\phi} - \phi'' = -m^2 \sin(\lambda\phi)$$

$m=1$, $\lambda=1$ $\quad \ddot{\phi} - \phi'' + \sin\phi = 0$

static sol. $\quad \phi(x,t) \xrightarrow{\text{static}} \dot{\phi} = 0 \quad \to \phi'' = \sin\phi \to \int \phi'\phi'' = \int \phi' \sin\phi$

$$\frac{1}{2}\phi'^2 = -\cos\phi + C$$

$\phi' = 0$ when $\phi \to 2\pi \cdot n \to C = 1$

$$\phi'^2 = 2(1-\cos\phi) = 4\sin^2\frac{\phi}{2}$$

$$\underline{\phi' = \pm 2\sin\frac{\phi}{2} = \frac{d\phi}{dx}}$$

$$\int \frac{d\phi/2}{\sin\frac{\phi}{2}} = \pm \int dx$$

$$\frac{\sin t\, dt}{\underbrace{\sin^2 t}_{1-\cos^2 t}} = \int \frac{-dy}{1-y^2} = -\frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)$$

$\cos\frac{\phi}{2} \equiv y$ $\quad = -\frac{1}{2}\ln\frac{1+y}{1-y} = \pm(x-x_0)$

$$\frac{1+y}{1-y} = e^{\pm 2(x-x_0)}$$

$$\frac{1+\cos\frac{\phi}{2}}{1-\cos\frac{\phi}{2}} = \frac{\cancel{2}\cos^2\frac{\phi}{4}}{\cancel{2}\sin^2\frac{\phi}{4}}$$

$$\boxed{\phi_0 = 4\tan^{-1}\left(e^{\pm(x-x_0)}\right)}$$

static $\longrightarrow$ moving $\quad \phi = \phi_0(\gamma(x-vt))$

$\ddot{\phi} = (\gamma v)^2 \phi_0''$

$\phi'' = \gamma^2 \phi_0''$

$$\to -\phi_0'' \underbrace{\gamma^2(1-v^2)}_{1} + \sin\phi_0 = 0$$

$$-\phi_0'' + \sin\phi_0 = 0$$

$$\therefore \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$\underline{\phi = \phi_0\left(\frac{1}{\sqrt{1-v^2}}(x-vt)\right)}$$

Bäcklund transform

SG: $\dfrac{\partial^2\phi}{\partial t^2} - \dfrac{\partial^2\phi}{\partial x^2} + \sin\phi = 0$

light-cone 좌표계:

$$\left.\begin{aligned} x_+ &= t + x \\ x_- &= x - t \end{aligned}\right\} \quad t = \frac{x_+ - x_-}{2}, \quad x = \frac{x_+ + x_-}{2}$$

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial x_+}\overbrace{\frac{\partial x_+}{\partial t}}^{1} + \frac{\partial\phi}{\partial x_-}\overbrace{\frac{\partial x_-}{\partial t}}^{-1}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x_+}\overbrace{\frac{\partial x_+}{\partial x}}^{1} + \frac{\partial\phi}{\partial x_-}\overbrace{\frac{\partial x_-}{\partial x}}^{+1}$$

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial x_+} - \frac{\partial\phi}{\partial x_-}$$

$$\frac{\partial^2\phi}{\partial t^2} = \cancel{\frac{\partial^2\phi}{\partial x_+^2}}\overbrace{\frac{\partial x_+}{\partial t}}^{1} - \frac{\partial^2\phi}{\partial x_+\partial x_-}\overbrace{\frac{\partial x_-}{\partial t}}^{1} - \frac{\partial^2\phi}{\partial x_-\partial x_+} + \cancel{\frac{\partial^2\phi}{\partial x_-^2}}\overbrace{\frac{\partial x_-}{\partial t}}^{1}$$

$$-)\ \frac{\partial^2\phi}{\partial x^2} = \cancel{\frac{\partial^2\phi}{\partial x_+^2}} + \frac{\partial^2\phi}{\partial x_+\partial x_-} + \frac{\partial^2\phi}{\partial x_-\partial x_+} + \cancel{\frac{\partial^2\phi}{\partial x_-^2}}$$

$$\boxed{-4\frac{\partial^2\phi}{\partial x_+\partial x_-} + \sin\phi = 0}$$

$$\tilde{x}_\pm = \frac{x_\pm}{2} = \frac{x \pm t}{2}$$

$$\to -\frac{\partial^2\phi}{\partial\tilde{x}_+\partial\tilde{x}_-} + \sin\phi = 0$$

2nd order DE

2개의 1st-order DEs. $\quad \frac{\partial}{\partial x_+} \equiv \partial_+ \quad \frac{\partial}{\partial x_-} \equiv \partial_-$

$$-\partial_+ \partial_- \phi + \sin\phi = 0$$

$$\partial_+ \phi_- = a \sin\phi_+ \quad , \quad \partial_- \phi_+ = \frac{1}{a} \sin\phi_-$$

↑ 상수

$$\partial_-\partial_+\phi_- = \partial_-(a\sin\phi_+) = a\cos\phi_+ \underbrace{\partial_-\phi_+}_{\frac{1}{a}\sin\phi_-} = \cos\phi_+ \sin\phi_-$$

$$\pm)\ \partial_+\partial_-\phi_+ = \partial_+\left(\frac{1}{a}\sin\phi_-\right) = \frac{1}{a}\cos\phi_- \underbrace{(\partial_+\phi_-)}_{a\sin\phi_+} = \cos\phi_- \sin\phi_+$$

$$\partial_-\partial_+(\phi_- + \phi_+) = \sin(\phi_+ + \phi_-) \longrightarrow \partial_-\partial_+\phi_0 = \sin\phi_0$$

$$\partial_-\partial_+(\phi_- - \phi_+) = \sin(\phi_- - \phi_+) \qquad \partial_-\partial_+\phi_1 = \sin\phi_1$$

$\phi_+ , \phi_- \longrightarrow$ solution of SG.

$$\phi_\pm = \frac{\phi_0 \pm \phi_1}{2} \qquad \phi_+ + \phi_- = \phi_0 \quad \phi_+ - \phi_- = \phi_1$$

$$\partial_+\left(\frac{\phi_0 - \phi_1}{2}\right) = a\sin\left(\frac{\phi_0+\phi_1}{2}\right) \quad \& \quad \partial_-\left(\frac{\phi_0+\phi_1}{2}\right) = \frac{1}{a}\sin\left(\frac{\phi_0-\phi_1}{2}\right)$$

ϕ_0, ϕ_1 are 모두 SG 방정식의 해.

$\phi_0 \xmapsto{Ba} \phi_1$ Bäcklund 변환

(ex) $\underline{\phi_0 = 0} \xrightarrow{Ba} \underline{\underline{\phi_1}}$

$$\frac{\partial}{\partial x_+} = \partial_+\phi_1 = -2a\sin\left(\frac{\phi_1}{2}\right)$$

$$\frac{\partial}{\partial x_-} = \partial_-\phi_1 = -\frac{2}{a}\sin\left(\frac{\phi_1}{2}\right)$$

새로운 좌표 : $\xi = a x_+ + \frac{1}{a} x_- \qquad \eta = a x_+ - \frac{1}{a} x_-$

$$\frac{\partial\phi_1}{\partial x_+} = \underbrace{\frac{\partial\phi_1}{\partial\xi}}_{\equiv\, \partial_\xi\phi} \underbrace{\frac{\partial\xi}{\partial x_+}}_{a} + \frac{\partial\phi_1}{\partial\eta}\underbrace{\frac{\partial\eta}{\partial x_+}}_{a} = a(\partial_\xi\phi_1 + \partial_\eta\phi_1) = -2a\sin\frac{\phi_1}{2}$$

$$\frac{\partial\phi_1}{\partial x_-} = \frac{\partial\phi_1}{\partial\xi}\underbrace{\frac{\partial\xi}{\partial x_-}}_{\frac{1}{a}} + \frac{\partial\phi_1}{\partial\eta}\underbrace{\frac{\partial\eta}{\partial x_-}}_{-\frac{1}{a}} = \frac{1}{a}(\partial_\xi\phi_1 - \partial_\eta\phi_1) = -\frac{2}{a}\sin\frac{\phi_1}{2}$$

$$(\partial_\xi\phi_1 + \partial_\eta\phi_1) = -2\sin\frac{\phi_1}{2} \qquad (\partial_\xi\phi_1 - \partial_\eta\phi_1) = -2\sin\frac{\phi_1}{2}$$

$$\therefore\ \partial_\eta\phi_1 = 0 \qquad \partial_\xi\phi_1 = -2\sin\frac{\phi_1}{2}$$

$$\phi_1 = \phi_1(\underbrace{a x_+ + \tfrac{1}{a} x_-}) \qquad \left(\frac{\phi_1}{2}\right)' = -\sin\left(\frac{\phi_1}{2}\right)$$

$$\xi = a\frac{(x+t)}{2} + \frac{1}{a}\frac{(x-t)}{2}$$

$$= \frac{1}{2}\left(a+\frac{1}{a}\right)x - \frac{1}{2}\left(\frac{1}{a}-a\right)t$$

$$\phi_1 = 4\tan^{-1}\left(e^{\pm(\xi - \xi_0)}\right)$$

$$= \left(a+\frac{1}{a}\right)/2 \Big(x - \underbrace{\frac{\frac{1}{a}-a}{\frac{1}{a}+a}}_{\equiv\, v} t\Big)$$

$$1 - v^2 = 1 - \frac{\left(\frac{1}{a}-a\right)^2}{\left(\frac{1}{a}+a\right)^2} = \frac{4}{\left(\frac{1}{a}+a\right)^2}$$

$$\therefore\ \phi_1 = 4\tan^{-1}\left(e^{\pm\gamma(x - vt) + c}\right) \qquad \frac{2/2}{\sqrt{1-v^2}} = \gamma$$

$$v = \frac{\frac{1}{a} - a}{\frac{1}{a} + a}$$

② ϕ_1

$$\begin{tikzcd} & \phi_1 \arrow[dr, "B_{a_2}"] & \\ \phi_0 \arrow[ur, "B_{a_1}"] \arrow[dr, "B_{a_2}"'] & & \phi_{12} \\ & \phi_2 \arrow[ur, "B_{a_1}"'] & \end{tikzcd}$$

0 1-soliton 2-soliton

$$\tan\frac{\phi_1}{4} = e^{y_1}$$

$$\phi^{(a_1)} = 4\tan^{-1}\left(e^{\gamma_{a_1}(x - v_{a_1}t) + c_1}\right) \qquad v_{a_1} = \frac{\frac{1}{a_1} - a_1}{\frac{1}{a_1} + a_1}$$

$$\partial_+\left(\frac{\phi_0 - \phi_1}{2}\right) = a_1 \sin\left(\frac{\phi_0 + \phi_1}{2}\right) \quad ① \quad \& \quad \partial_-\left(\frac{\phi_0 + \phi_1}{2}\right) = \frac{1}{a_1}\sin\left(\frac{\phi_0 - \phi_1}{2}\right)$$

$$\partial_+\left(\frac{\phi_0 - \phi_2}{2}\right) = a_2 \sin\left(\frac{\phi_0 + \phi_2}{2}\right) \quad ② \quad \& \quad \partial_-\left(\frac{\phi_0 + \phi_2}{2}\right) = \frac{1}{a_2}\sin\left(\frac{\phi_0 - \phi_2}{2}\right)$$

$$\partial_+\left(\frac{\phi_1 - \phi_{12}}{2}\right) = a_2 \sin\left(\frac{\phi_1 + \phi_{12}}{2}\right) \quad ③ \quad \& \quad \partial_-\left(\frac{\phi_1 + \phi_{12}}{2}\right) = \frac{1}{a_2}\sin\left(\frac{\phi_1 - \phi_{12}}{2}\right)$$

$$\partial_+\left(\frac{\phi_2 - \phi_{12}}{2}\right) = a_1 \sin\left(\frac{\phi_2 + \phi_{12}}{2}\right) \quad ④ \quad \& \quad \partial_-\left(\frac{\phi_2 + \phi_{2}}{2}\right) = \frac{1}{a_1}\sin\left(\frac{\phi_2 - \phi_{12}}{2}\right)$$

$$\sin(\alpha+\beta) \qquad \sin(\alpha-\beta)$$

$$① - ④ : \partial_+\left(\frac{\phi_0 - \phi_1 - \phi_2 + \phi_{12}}{2}\right) = a_1\left(\sin\frac{\phi_0 + \phi_1}{2} - \sin\frac{\phi_2 + \phi_{12}}{2}\right) = 2a_1\cos\left(\frac{\phi_0 + \phi_1 + \phi_2 + \phi_{12}}{4}\right)\sin\left(\frac{\phi_0 + \phi_1 - \phi_2 - \phi_{12}}{4}\right)$$

$$② - ③ : \partial_+\left(\frac{\phi_0 - \phi_2 - \phi_1 + \phi_{12}}{2}\right) = a_2\left(\sin\frac{\phi_0 + \phi_2}{2} - \sin\frac{\phi_1 + \phi_{12}}{2}\right) = 2a_2\cos\left(\frac{\phi_0 + \phi_1 + \phi_2 + \phi_{12}}{4}\right)\sin\left(\frac{\phi_0 + \phi_2 - \phi_1 - \phi_{12}}{4}\right)$$

$$a_1 \sin\underbrace{\frac{\phi_0 + \phi_1 - \phi_2 - \phi_{12}}{4}}_{\sin\frac{\phi_0 - \phi_{12}}{4}\cos\frac{\phi_1 - \phi_2}{4} + \cos\frac{\phi_0 - \phi_{12}}{4}\sin\frac{\phi_1 - \phi_2}{4}} = a_2 \sin\underbrace{\frac{\phi_0 + \phi_2 - \phi_1 - \phi_{12}}{4}}_{\sin\frac{\phi_0 - \phi_{12}}{4}\cos\frac{\phi_1 - \phi_2}{4} - \cos\frac{\phi_0 - \phi_{12}}{4}\sin\frac{\phi_1 - \phi_2}{4}}$$

$$\alpha = \frac{\phi_0 - \phi_{12}}{4}, \quad \beta = \frac{\phi_1 - \phi_2}{4}$$

$$\therefore\ a_1(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = a_2(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$$\therefore\ (a_1 - a_2)\sin\alpha\cos\beta = -(a_1 + a_2)\cos\alpha\sin\beta$$

$$\tan\frac{\phi_0 - \phi_{12}}{4} = \tan\alpha = -\frac{a_1 + a_2}{a_1 - a_2}\tan\beta = -\frac{a_1 + a_2}{a_1 - a_2}\tan\frac{\phi_1 - \phi_2}{4}$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

$$\tan\frac{\phi_{12} - \overset{=0}{\phi_0}}{4} = +\frac{a_1 + a_2}{a_1 - a_2}\,\frac{\tan\frac{\phi_1}{4} - \tan\frac{\phi_2}{4}}{1 + \tan\frac{\phi_1}{4}\tan\frac{\phi_2}{4}}$$

$$\therefore\ \phi_{12} = 4\tan^{-1}\left[\frac{a_1 + a_2}{a_1 - a_2}\,\underbrace{\frac{\tan\frac{\phi_1}{4} - \tan\frac{\phi_2}{4}}{1 + \tan\frac{\phi_1}{4}\tan\frac{\phi_2}{4}}}\right]$$

$$\phi_1 = 4\tan^{-1}\left(e^{\overbrace{\gamma_{a_1}(x - v_{a_1}t) + c_1}^{y_1}}\right) \quad v_{a_1} = \frac{\frac{1}{a_1} - a_1}{\frac{1}{a_1} + a_1} \quad \rightarrow \quad \tan\frac{\phi_1}{4} = e^{y_1}, \quad \tan\frac{\phi_2}{4} = e^{y_2}$$

$$\boxed{\begin{aligned} y_1 &= \gamma_1(x - v_1 t) + c_1 \\ y_2 &= \gamma_2(x - v_2 t) + c_2 \end{aligned}}$$

$$\therefore\ \boxed{\phi_{12} = 4\tan^{-1}\left[\frac{a_1 + a_2}{a_1 - a_2}\,\frac{\sinh\frac{y_1 - y_2}{2}}{\cosh\frac{y_1 + y_2}{2}}\right]} \quad \frac{2\sinh\frac{y_1 - y_2}{2}}{2\cosh\frac{y_1 + y_2}{2}} = \frac{e^{y_1} - e^{y_2}}{1 + e^{y_1 + y_2}} = \frac{\cancel{e^{\frac{y_1 + y_2}{2}}}\left(e^{\frac{y_1 - y_2}{2}} - e^{-\frac{y_1 - y_2}{2}}\right)}{\cancel{e^{\frac{y_1 + y_2}{2}}}\left(e^{\frac{y_1 + y_2}{2}} + e^{-\frac{y_1 + y_2}{2}}\right)}$$

$$\gamma_{1,2} = \frac{1}{2}\left(a_{1,2} + \frac{1}{a_{1,2}}\right) \qquad v_{1,2} = \frac{\frac{1}{a_{1,2}} - a_{1,2}}{\frac{1}{a_{1,2}} + a_{1,2}}$$

$$\begin{tikzcd}
 & \phi_1 \arrow[r, "a_3"] \arrow[dr, "a_2"] & \phi_{13} \arrow[r, "a_2"] & \phi_{123} \\
\phi_0 \arrow[ur, "a_1"] \arrow[dr, "a_2"] & & \phi_{12} \arrow[ur, "a_3"] & \\
 & \phi_2 \arrow[ur, "a_1"] \arrow[r, "a_3"] & \phi_{23} \arrow[uur, "a_1"] &
\end{tikzcd}$$

Multi-soliton $^{(n)}$ ($\star$)

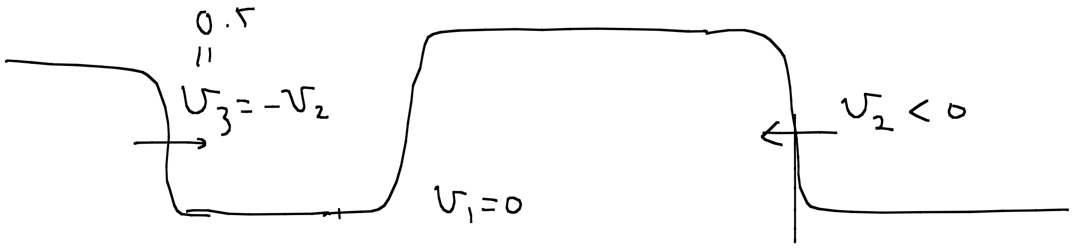
$$\boxed{\begin{aligned}
\cos\phi &= 1 - 2\underbrace{\partial_\mu \partial^\mu}_{\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)} \ln\{\mathrm{Det}(M)\} \\
&\text{n×n matrix} \\
M_{ij} &= \frac{2}{a_i + a_j}\cosh\frac{y_i + y_j}{2} \\
y_i &= \gamma_i (x - v_i t) + c_i
\end{aligned}}$$

H.W.

① ϕ_{123} 을 a_1, a_2, a_3, y_1, y_2, y_3 로 구하기.

② 2 결과를 $\star$ 와 비교.

<u>Math.</u>



-10 $x = 0$ 10 $t = 0$

next.