Pólya Formula for Jordan Blocks

Changrim Ahn

Department of Physics
Ewha Womans University
Seoul, South Korea

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Based on Collaborations with

- Matthias Staudacher : "The integrable (hyper)eclectic spin chain", JHEP 02 (2021) 019
- Luke Corcoran and Matthias Staudacher : "Combinatorial solution of the eclectic spin chain", JHEP 03 (2022) 028
- Matthias Staudacher : "Spectrum of the Hypereclectic Spin Chain and Pólya Counting", arXiv:2207.02885

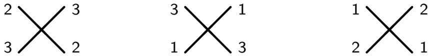
Strong twisted SYM theory

Start with planar, integrable, three parameter γ-deformed SYM and take Double Scaling limits to find simpler conformal field theories "(dynamical) Fishnet" models [Gürdoğan,Kazakov'15;Sieg,Wilhelm'16; Caetano,Gürdoğan,Kazakov'18]

$$g = \frac{\sqrt{\lambda}}{4\pi} \to 0 \quad ; \quad \xi_j = g q_j^{\pm 1} = \text{ finite}, \quad j = 1, 2, 3$$

Among 2^3 possibilities, focus on $(+, +, +)$ which leads to

$$\mathcal{L}_{\text{int}} = N_c \text{Tr} \left[\xi_1 \phi_2^\dagger \phi_3^\dagger \phi^2 \phi^3 + \xi_2 \phi_3^\dagger \phi_1^\dagger \phi^3 \phi^1 + \xi_3 \phi_1^\dagger \phi_2^\dagger \phi^1 \phi^2 \right] + \text{fermions}$$



Eclectic spin chain

[Ipsen,Staudacher,Zippelius '19]

One-loop dilatation operators of single trace composite operators made of $\{\phi_1, \phi_2, \phi_3\}$ are given by "Eclectic" spin chain Hamiltonian

$$\mathbf{H} = \sum_{n=1}^{L} \left[\xi_3 \mathbb{P}_{\mathbf{21}}^{n,n+1} + \xi_2 \mathbb{P}_{\mathbf{13}}^{n,n+1} + \xi_1 \mathbb{P}_{\mathbf{32}}^{n,n+1} \right], \qquad \mathbb{P}^{L,L+1} \equiv \mathbb{P}^{L,1}$$

acting on cyclic states e.g. $(\phi_1 \equiv \mathbf{1},\ \phi_2 \equiv \mathbf{2},\ \phi_3 \equiv \mathbf{3})$

$$|\cdots \mathbf{123212121312} \cdots\rangle_{\text{cyclic}}$$

by a rule that only non-vanishing actions are

$$\mathbb{P}_{\mathbf{21}}|\mathbf{21}\rangle = |\mathbf{12}\rangle, \quad \mathbb{P}_{\mathbf{13}}|\mathbf{13}\rangle = |\mathbf{31}\rangle, \quad \mathbb{P}_{\mathbf{32}}|\mathbf{32}\rangle = |\mathbf{23}\rangle$$

Integrability of Eclectic spin-chain

R-matrix: $\mathbf{R}(u) : V \otimes V \to V \otimes V, \quad V = (\mathbf{1}, \mathbf{2}, \mathbf{3})$

$$\mathbf{R}(u) = \left(\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & & & 1 & & & & & \\
 & & \xi_2 u & & & & 1 & & \\
\hline
 & 1 & & \xi_3 u & & & & & \\
 & & & & 1 & & & & \\
 & & & & & & & 1 & \\
\hline
 & & 1 & & & & & & \\
 & & & & & 1 & & \xi_1 u & \\
 & & & & & & & & 1
\end{array}\right)$$

- Yang-Baxter equation
- Monodromy and Transfer matrices

$$\mathbf{M}_a(u) = \mathbf{R}_{a,L}(u)\mathbf{R}_{a,L-1}(u)\cdots\mathbf{R}_{a,1}(u), \qquad \mathbf{T}(u) = \mathrm{Tr}_a\, \mathbf{M}_a(u)$$

- Integrability : $[\mathbf{T}(u), \mathbf{T}(u')] = 0$
- Hamiltonian : $\mathbf{H} = \mathbf{U} \cdot \mathbf{T}'(0)$, Shift op : $\mathbf{U} = \mathbf{T}(0)$

How to diagonalize $\mathbf{H}$?

Standard algebraic Bethe ansatz fails. FCRs becomes useless (ex)

$$\mathbf{R}_{21}^{12}(u-v)\mathbf{M}_{22}(v)\mathbf{M}_{12}(u)=\mathbf{M}_{22}(u)\mathbf{M}_{12}(v)\mathbf{R}_{22}^{22}(u-v)$$

Numerical analysis based on Matlab and Mathematica show that it is **non-diagonalizable** due to formation of rich Jordan Block spectrum.

Instead of being diagonalized, the Hamiltonian is reduced to **Jordan Normal Form (JNF)**

Jordan Normal Form

Assume, for simplicity, that a matrix $\mathbf{H}$ has only one eigenvalue E but several linearly independent *true* eigenvectors $|\psi_j^1\rangle$:

$$(\mathbf{H} - E)|\psi_j^1\rangle = 0, \qquad j = 1, \cdots, \gamma$$

where γ is known as *geometric multiplicity*

For each eigenvector, *generalized* eigenvectors are associated

$$(\mathbf{H} - E)^m|\psi_j^m\rangle = 0, \qquad m = 1, \cdots, N_j$$

Jordan chain is formed by successive action of $(\mathbf{H} - E)$

$$|\psi_j^{N_j}\rangle \to |\psi_j^{N_j-1}\rangle \to |\psi_j^{N_j-2}\rangle \to \cdots \to |\psi_j^2\rangle \to |\psi_j^1\rangle \to 0$$

which has one-to-one correspondence with one Jordan block in JNF

$$J_{N_j}(E) = \begin{pmatrix} E & 1 & & & \\ & E & 1 & & \\ & & \ddots & \ddots & \\ & & & E & 1 \\ & & & & E \end{pmatrix}, \qquad N_j \times N_j$$

Jordan Normal Form of Eclectic model

- There is only one eigenvalue E which is $E = 0$ (See later)
- $\mathbf{H}$ is reduced to direct sum of JNFs

$$\mathbf{S}^{-1} \cdot \mathbf{H} \cdot \mathbf{S} = [\overbrace{J_{N_1} \oplus \cdots \oplus J_{N_1}}^{n_1}] \oplus \cdots \oplus [\overbrace{J_{N_b} \oplus \cdots \oplus J_{N_b}}^{n_b}]$$

whose "Jordan spectrum" (sizes and multiplicities) is denoted by

$$N_1^{n_1} \; N_2^{n_2} \; \cdots N_b^{n_b}, \qquad N_1 < N_2 < \cdots < N_b$$

Notations for sectors

- $(L, M, K) = [L_1, M_1, K]$ sector:
$\underline{L_1 \equiv L - M}$ = # of $\mathbf{1}$'s, $\underline{M_1 \equiv M - K}$ = # of $\mathbf{2}$'s, K = # of $\mathbf{3}$'s

- Without loss of generality, we assume a filling condition $\underline{L_1 \geq M_1 \geq K}$

Hyper-eclectic model

- Set $\xi_1 = \xi_2 = 0, \xi_3 = 1$ with $L_1 \geq M_1 \geq K$,

$$\mathbf{H_3} = \sum_{n=1}^{L} \mathbb{P}^{n,n+1}_{\mathbf{21}}$$

- **Universality** : Satisfying the filling condition, this has the same Jordan spectrum as the eclectic model with generic ξ's (See later)

- It is easier to work with hypereclectic model to find Jordan spectrum numerically (ex) $M = 5, K = 1$

L	Sizes of Jordan Blocks
8	$1\ 5\ 7\ 9\ 13$
9	$1\ 5^2\ 9^2\ 11\ 13\ 17$
10	$1\ 5^2\ 7\ 9^2\ 11\ 13^2\ 15\ 17\ 21$
11	$1^2\ 5^2\ 7\ 9^3\ 11\ 13^3\ 15\ 17^2\ 19\ 21\ 25$
12	$1\ 5^3\ 7\ 9^3\ 11^2\ 13^3\ 15^2\ 17^3\ 19\ 21^2\ 23\ 25\ 29$
13	$1^2\ 5^3\ 7\ 9^4\ 11^2\ 13^4\ 15^2\ 17^4\ 19^2\ 21^3\ 23\ 25^2\ 27\ 29\ 33$
14	$1^2\ 5^3\ 7^2\ 9^4\ 11^2\ 13^5\ 15^3\ 17^4\ 19^3\ 21^4\ 23^2\ 25^3\ 27\ 29^2\ 31\ 33\ 37$
15	$1^2\ 5^4\ 7\ 9^5\ 11^3\ 13^5\ 15^3\ 17^6\ 19^3\ 21^5\ 23^3\ 25^4\ 27^2\ 29^3\ 31\ 33^2\ 35\ 37\ 41$
16	$1^2\ 5^4\ 7^2\ 9^5\ 11^3\ 13^6\ 15^4\ 17^6\ 19^4\ 21^6\ 23^4\ 25^5\ 27^3\ 29^4\ 31^2\ 33^3\ 35\ 37^2\ 39\ 41\ 45$
17	$1^3\ 5^4\ 7^2\ 9^6\ 11^3\ 13^7\ 15^4\ 17^7\ 19^5\ 21^7\ 23^4\ 25^7\ 27^4\ 29^5\ 31^3\ 33^4\ 35^2\ 37^3\ 39\ 41^2\ 43\ 45\ 49$
18	$1^2\ 5^5\ 7^2\ 9^6\ 11^4\ 13^7\ 15^5\ 17^8\ 19^5\ 21^8\ 23^6\ 25^7\ 27^5\ 29^7\ 31^4\ 33^5\ 35^3\ 37^4\ 39^2\ 41^3\ 43\ 45^2\ 47\ 49\ 53$

JB spectrum for $M = 4, K = 1$

L	Sizes of Jordan Blocks
6	3 7
10	$3\ 7^2\ 9\ 11\ 13\ 15\ 19$
14	$3\ 7^2\ 9\ 11^2\ 13^2\ 15^2\ 17\ 19^2\ 21\ 23\ 25\ 27\ 31$
18	$3\ 7^2\ 9\ 11^2\ 13^2\ 15^2\ 17^2\ 19^2\ 21^2\ 23^2\ 25^2\ 27^2\ 29\ 31^2\ 33\ 35\ 37\ 39\ 43$
8	1 5 7 9 13
12	$1\ 5\ 7\ 9^2\ 11\ 13^2\ 15\ 17\ 19\ 21\ 25$
16	$1\ 5\ 7\ 9^2\ 11\ 13^3\ 15^2\ 17^2\ 19^2\ 21^2\ 23\ 25^2\ 27\ 29\ 31\ 33\ 37$
20	$1\ 5\ 7\ 9^2\ 11\ 13^3\ 15^2\ 17^3\ 19^3\ 21^3\ 23^2\ 25^3\ 27^2\ 29^2\ 31^2\ 33^2\ 35\ 37^2\ 39\ 41\ 43\ 45\ 49$
7	4 6 10
11	$4\ 6\ 8\ 10^2\ 12\ 14\ 16\ 18\ 22$
15	$4\ 6\ 8\ 10^2\ 12^2\ 14^2\ 16^2\ 18^2\ 20\ 22^2\ 24\ 26\ 28\ 30\ 34$
19	$4\ 6\ 8\ 10^2\ 12^2\ 14^2\ 16^3\ 18^3\ 20^2\ 22^3\ 24^2\ 26^2\ 28^2\ 30^2\ 32\ 34^2\ 36\ 38\ 40\ 42\ 46$
9	4 6 8 10 12 16
13	$4\ 6\ 8\ 10^2\ 12^2\ 14\ 16^2\ 18\ 20\ 22\ 24\ 28$
17	$4\ 6\ 8\ 10^2\ 12^2\ 14^2\ 16^3\ 18^2\ 20^2\ 22^2\ 24^2\ 26\ 28^2\ 30\ 32\ 34\ 36\ 40$
21	$4\ 6\ 8\ 10^2\ 12^2\ 14^2\ 16^3\ 18^3\ 20^3\ 22^3\ 24^3\ 26^2\ 28^3\ 30^2\ 32^2\ 34^2\ 36^2\ 38\ 40^2\ 42\ 44\ 46\ 48\ 52$

- Some regularities can be noticed but why? Can we predict Jordan spectrum for any L, M, K?

The first approach : Algebraic Bethe ansatz

Algebraic Bethe ansatz with finite q_j's

$$\Lambda(u) = \frac{(-1)^M {q_2}^K}{{q_3}^{M-K}}(u+1)^L \prod_{n=1}^{M} \frac{u_n - u + 1}{u - u_n} + \frac{{q_1}^{M-K}}{{q_2}^{L-M}} u^L \prod_{j=1}^{K} \frac{u - v_j + 1}{u - v_j}$$

$$+ (-1)^K \frac{{q_3}^{L-M}}{{q_1}^K} u^L \prod_{n=1}^{M} \frac{u - u_n + 1}{u - u_n} \prod_{j=1}^{K} \frac{v_j - u + 1}{u - v_j}$$

$$\left(\frac{u_m + 1}{u_m}\right)^L = \frac{{q_3}^L}{(q_1 q_2 q_3)^K} \prod_{\substack{n=1 \\ n \neq m}}^{M} \frac{u_m - u_n + 1}{u_m - u_n - 1} \prod_{j=1}^{K} \frac{u_m - v_j - 1}{u_m - v_j}$$

$$1 = \frac{(q_2 q_3)^L}{(q_1 q_2 q_3)^M} \prod_{n=1}^{M} \frac{v_k - u_n + 1}{v_k - u_n} \prod_{\substack{j=1 \\ j \neq k}}^{K} \frac{v_k - v_j - 1}{v_k - v_j + 1}$$

Bethe vectors

$$|\psi\rangle = \mathbf{M}_{13}(v_1) \cdots \mathbf{M}_{13}(v_K) \mathbf{M}_{12}(u_1) \cdots \mathbf{M}_{12}(u_M) |0\rangle$$

Take a strong twist limit

$$q_k \equiv \frac{\xi_k}{\varepsilon}, \quad u \to \varepsilon \bar{u}, \quad \text{with} \quad \varepsilon \to 0$$

BAE

$$\left(\frac{u_m + 1}{u_m}\right)^L = \frac{\varepsilon^{3K-L} \cdot {\xi_3}^L}{(\xi_1\xi_2\xi_3)^K} \prod_{\substack{n=1 \\ n\neq m}}^{M} \frac{u_m - u_n + 1}{u_m - u_n - 1} \prod_{j=1}^{K} \frac{u_m - v_j - 1}{u_m - v_j}$$

$$1 = \frac{\varepsilon^{3M-2L} \cdot (\xi_2\xi_3)^L}{(\xi_1\xi_2\xi_3)^M} \prod_{n=1}^{M} \frac{v_k - u_n + 1}{v_k - u_n} \prod_{\substack{j=1 \\ j\neq k}}^{K} \frac{v_k - v_j - 1}{v_k - v_j + 1}$$

With **exact solutions**

$$\begin{aligned} u_n &= 0 + \varepsilon^\alpha \hat{u}_n, \quad n = 1, \cdots, M_1 \\ u_{M_1+k} &= -1 + \varepsilon^\beta \hat{w}_k, \quad k = 1, \cdots, K \\ v_k &= -2 + \varepsilon^\beta \hat{w}_k + \varepsilon^\gamma \hat{v}_k, \quad k = 1, \cdots, K \end{aligned}$$

$$\alpha = \frac{L-M-K}{L-M+K}, \quad \beta = \frac{L-3(M-K)}{L-M+K}, \quad \gamma = 2L-3M-\beta(K-1)$$

$$\hat{w}_k = -\frac{(\xi_1\xi_3)^{\frac{M_1}{L-M_1}}}{\xi_2}\, \omega_{L-M_1}^{n_k+\frac{K-1}{2}}, \quad n_k = \{1,\cdots,L-M_1\}, \quad \omega_n \equiv e^{\frac{2\pi i}{n}}$$

$$\hat{u}_n = \left(\frac{(\xi_1\xi_2\xi_3)^K}{{\xi_3}^L}(-1)^{M-1}\prod_{k=1}^{K}\hat{w}_k\right)^{\frac{1}{L}} \omega_L^{i_n}, \quad i_n = \{1,\cdots,L\}$$

- This result does not apply for $3M_1 \geq L > 2M-K$ since $\alpha, \beta > 0$.
- Eigenvalues of Transfer matrix: the L-th roots of unity

$$\boxed{\Lambda(u) = \exp\frac{2\pi i}{L}\left[\sum_{k=1}^{K} n_k - \sum_{m=1}^{M_1} i_m + \frac{1}{2}M_1(M_1-1) + \frac{1}{2}K(K-1)\right]}$$

for **Cyclic states** : $[\cdots] = 0 \bmod L$
Independent of ξ's : already shows a glimpse of universality

George Pólya (Budapest, 1887 - Palo Alto, 1985)

"I am not good enough for physics and too good for philosophy; mathematics is in between."

Pólya's Formula I

Number of inequivalent necklaces of L beads with colors $\mathcal{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$

Physicists' derivation following [Spradlin, Volovich '05]
Let $\hat{\ell}, \hat{m}, \hat{k}$ count # of beads with colors $\mathbf{1}, \mathbf{2}, \mathbf{3}$ in $|\mathcal{A}_1 \cdots \mathcal{A}_L\rangle$, resp.
The generating function defined by

$$Z(x,y,z) = \sum_{L=1}^{\infty} \mathrm{Tr}_{\mathcal{A}^{\otimes L}} \left[\mathbf{P}_{\mathrm{cyc}} x^{\hat{\ell}} y^{\hat{m}} z^{\hat{k}} \right], \qquad \mathbf{P}_{\mathrm{cyc}} = \frac{1}{L} \sum_{j=1}^{L} \mathbf{U}^j$$

will count # of cyclic states (necklaces)
- If $j = 1$, $\langle \mathcal{A}_1 \cdots \mathcal{A}_L | \mathcal{A}_2 \cdots \mathcal{A}_1 \rangle$ is non-zero only if $\mathcal{A}_1 = \cdots = \mathcal{A}_L$
- If the greatest common divisor $(j, L) = 2$, all $\mathcal{A}_{\mathrm{even}}$ (and independently $\mathcal{A}_{\mathrm{odd}}$) should have the same colors
- Similarly for $(j, L) = 3, 4, \cdots, L$, the counting function becomes

$$Z(x,y,z) = \sum_{L=1}^{\infty} \frac{1}{L} \sum_{j=1}^{L} \left(\mathrm{Tr}_{\mathcal{A}} \left[x^{L/(j,L)\hat{\ell}} y^{L/(j,L)\hat{m}} z^{L/(j,L)\hat{k}} \right] \right)^{(j,L)}$$

- # of j's with $(j, L) = p$ is $\phi(L/p)$ where Euler's totient function $\phi(n)$ counts # of coprimes to n which is less than n

G. Pólya's formula ($L = np$)

$$\boxed{Z(x,y,z) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{\phi(n)}{np}\zeta(x^n,y^n,z^n)^p = -\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\ln\left[1-\zeta(x^n,y^n,z^n)\right]}$$

where

$$\zeta(x,y,z) = \mathrm{Tr}_{\mathcal{A}}\left[x^{\hat{\ell}}y^{\hat{m}}z^{\hat{k}}\right] = x+y+z$$

Expanding in x, y, z,

$$Z(x,y,z) = \sum_{L=1}^{\infty}\sum_{M=1}^{L}\sum_{K=1}^{M} d(L,M,K)\cdot x^{L-M}y^{M-K}z^{K}$$

$d(L, M, K)$ counts # of cyclic states in (L, M, K) sector

L	M	K	naive counting	Pólya formula	Bethe ansatz
14	6	2	6435/2	3225	3225
16	6	2	15015/2	7518	7518
18	6	2	15470	15484	15484
20	6	2	29070	29088	29088
20	8	2	176358	176400	176400
20	10	4	1939938	1940064	1940064
21	9	3	1175720	1175730	1175730
22	6	2	101745/2	50895	50895
22	8	2	406980	407040	407040
22	10	4	6172530	6172740	6172740
24	6	2	168245/2	84150	84150
24	8	2	1716099/2	858132	858132
24	9	3	4576264	4576278	4576278
24	10	4	17160990	17161320	17161320

naive counting : $\frac{1}{L}\frac{L!}{(L-M)!(M-K)!K!}$

However, Bethe ansatz fails for Jordan spectrum

All Bethe states collapse into only one state called "Locked state"

$$
\begin{aligned}
|\psi\rangle &= \mathcal{M}_{13}(v_1)\cdots\mathcal{M}_{13}(v_K)\mathcal{M}_{12}(u_M)\mathcal{M}_{12}(u_1)\cdots\mathcal{M}_{12}(u_M)|0\rangle \\
&\to \sum_{n=1}^{L} |\cdots 1\ 1\ \overset{[n]}{\overset{\downarrow}{\mathbf{2}}}\ \mathbf{2}\cdots\mathbf{2}\ \mathbf{3}\cdots\mathbf{3}\ 1\ 1\cdots\rangle \qquad \text{as} \quad \varepsilon \to 0
\end{aligned}
$$

The same conclusion has been obtained using coordinate BA
In addition, generalized eigenvectors associated with the Locked state have been obtained

[Nieto García,Wyss '21; Nieto García '22]

But Integrability can not provide **other true eigenvectors** which exist for each Jordan block

Understanding the Jordan spectrum

The second approach : Combinatorics and Linear Algebra

Hyper-eclectic model with $K = 1$

Consider a generic cyclic state

$$|\underbrace{1\cdots1}_{n_0}\mathbf{2}\underbrace{1\cdots1}_{n_1}\mathbf{2}\underbrace{1\cdots1}_{n_2}\cdots\mathbf{2}\underbrace{1\cdots1}_{n_{M_1}}\mathbf{3}\rangle$$

Define a level S as sum of # of **1** on RHS of each **2**

$$\Rightarrow \quad \mathrm{S} = (n_1 + \cdots + n_{M_1}) + (n_2 + \cdots + n_{M_1}) + \cdots + n_{M_1}$$

Hamiltonian acting on this state

$$\mathbf{H_3} = \sum_{n=1}^{L} \mathbb{P}_{\mathbf{21}}^{n,n+1}$$

moves each **2** to Right by one step, hence

$$\mathbf{H_3} : \mathrm{S} \ \rightarrow \ \mathrm{S} - 1$$

$S_{\max} = L_1 M_1$ is given by "anti-Locked" state $|2 \cdots 21 \cdots 13\rangle$

$S_{\min} = 0$ is given by the Locked state $|1 \cdots 12 \cdots 23\rangle$

A Jordan chain with length $L_1 M_1 + 1$ is formed by acting $\mathbf{H_3}$

$$|2 \cdots 21 \cdots 13\rangle \to |2 \cdots 2121 \cdots 13\rangle \to \cdots \to |1 \cdots 12 \cdots 23\rangle \to 0$$

(Ex) $M = 5, K = 1$

L	Sizes of Jordan Blocks																								
8	1	5	7	9		**13**																			
9	1	5^2		9^2	11	13		**17**																	
10	1	5^2	7	9^2	11	13^2	15	17		**21**															
11	1^2	5^2	7	9^3	11	13^3	15	17^2	19	21		**25**													
12	1	5^3	7	9^3	11^2	13^3	15^2	17^3	19	21^2	23	25		**29**											
13	1^2	5^3	7	9^4	11^2	13^4	15^2	17^4	19^2	21^3	23	25^2	27	29		**33**									
14	1^2	5^3	7^2	9^4	11^2	13^5	15^3	17^4	19^3	21^4	23^2	25^3	27	29^2	31	33		**37**							
15	1^2	5^4	7	9^5	11^3	13^5	15^3	17^6	19^3	21^5	23^3	25^4	27^2	29^3	31	33^2	35	37		**41**					
16	1^2	5^4	7^2	9^5	11^3	13^6	15^4	17^6	19^4	21^6	23^4	25^5	27^3	29^4	31^2	33^3	35	37^2	39	41		**45**			
17	1^3	5^4	7^2	9^6	11^3	13^7	15^4	17^7	19^5	21^7	23^4	25^7	27^4	29^5	31^3	33^4	35^2	37^3	39	41^2	43	45		**49**	
18	1^2	5^5	7^2	9^6	11^4	13^7	15^5	17^8	19^5	21^8	23^6	25^7	27^5	29^7	31^4	33^5	35^3	37^4	39^2	41^3	43	45^2	47	49	**53**

(Ex) $(7, 3, 1)$ sector

The first Jordan chain by $\mathbf{H_3}$ is from anti-Locked to Locked state

$$|2211113\rangle^{S=8} \to |2121113\rangle^{7} \to |1221113\rangle^{6} + |2112113\rangle^{6} \to$$

$$2|1212113\rangle^{5} + |2111213\rangle^{5} \to 3|1211213\rangle^{4} + 2|1122113\rangle^{4} + |2111123\rangle^{4}$$

$$\to 5|1121213\rangle^{3} + 4|1211123\rangle^{3} \to 9|1121123\rangle^{2} + 5|1112213\rangle^{2} \to$$

$$14|1112123\rangle^{1} \to 14|1111223\rangle^{0} \to 0$$

The second Jordan chain can exist if it starts at $S = 6$

$$a|1221113\rangle^{6} + b|2112113\rangle^{6} \to (a+b)|1212113\rangle^{5} + b|2111213\rangle^{5} \to$$

$$\cdots \to (3a+6b)|1121123\rangle^{2} + (2a+3b)|1112213\rangle^{2} \to (5a+9b)|1112123\rangle^{1}$$

With $5a + 9b = 0$, we can find the second true eigenvector and JB of size 5

$$-9|1221113\rangle^{6} + 5|2112113\rangle^{6} \to \cdots \to |1121123\rangle^{2} - |1112213\rangle^{2}$$

The third Jordan chain can start at $\mathsf{S} = 4$ since there are three vectors

$$a'|1211213\rangle^4 + b'|1122113\rangle^4 + c'|2111123\rangle^4 \rightarrow$$

$$(a' + b')|1121213\rangle^3 + (a' + c')|1211123\rangle^3$$

With $b' = c' = -a'$, we can find the third true eigenvector and JB of size 1

$$|1211213\rangle^4 - |1122113\rangle^4 - |2111123\rangle^4$$

- If top vector in a Jordan chain starts at level S, the bottom vector (true eigenvector) occurs at level $\mathsf{S}_{\max} - \mathsf{S} \equiv \overline{\mathsf{S}}$

- Length of the Jordan chain : $2\mathsf{S} - \mathsf{S}_{\max} + 1$

- The Jordan spectrum : $\mathsf{JNF}(7, 3, 1) = 1\ 5\ 9$

Jordan spectrum of $K = 1$

- (Def) $\mathbf{dim}_\mathsf{S}$ be the dimension of a vector space with level S i.e. # of linearly independent states at the level S

- New Jordan chains are formed at level S if $\mathbf{dim}_\mathsf{S} > \mathbf{dim}_{\mathsf{S}+1}$ for $\mathsf{S} \geq \frac{\mathsf{S}_{\max}}{2}$

- In principle, **unexpected shortening** may occur somewhere in the chain and a new chain may start again. Although we have no mathematical proof, we checked that this never occurs for many explicit cases

- Multiplicity of Jordan blocks : $\mathbf{dim}_\mathsf{S} - \mathbf{dim}_{\mathsf{S}+1}$ for $\mathsf{S} \geq \frac{\mathsf{S}_{\max}}{2}$

(Ex) (7,3,1) sector

S	8	7	6	5	4	3	2	1	0
$\mathbf{dim}_\mathsf{S}$	1	1	2	2	3	2	2	1	1
$\mathbf{dim}_\mathsf{S} - \mathbf{dim}_{\mathsf{S}+1}$	1	0	1	0	1	-	-	-	-

- Jordan spectrum is encoded in $\mathbf{dim}_\mathsf{S}$!

Gaussian binomial coefficients

$$\mathrm{S}(|1\cdots1\mathbf{2}\overbrace{1\cdots1}^{n_1}\mathbf{2}\overbrace{1\cdots1}^{n_2}\cdots\mathbf{2}\overbrace{1\cdots1}^{n_{M_1}}\mathbf{3}\rangle)$$
$$= (n_1+\cdots+n_{M_1}) + (n_2+\cdots+n_{M_1}) + \cdots + n_{M_1}$$

- $\mathbf{dim}_\mathrm{S} = \#$ of partitions of S into at most M_1 parts, each $\leq L_1$

- This restricted partition is generated by Gaussian binomial coefficients

$$\boxed{\begin{bmatrix} L_1+M_1 \\ M_1 \end{bmatrix}_q = \prod_{k=1}^{M-1} \frac{[L-k]_q}{[k]_q} = \sum_{\mathrm{S}=0}^{L_1 M_1} \mathbf{dim}_\mathrm{S}\, q^{\mathrm{S}-L_1 M_1/2}}$$

in terms of q-number defined by

$$[n]_q = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}} = q^{(n-1)/2} + \cdots q^{-(n-1)/2}$$

- One can show : $\mathbf{dim}_\mathrm{S} = \mathbf{dim}_{\bar{\mathrm{S}}}, \quad (\bar{\mathrm{S}} \equiv \mathrm{S}_{\max} - \mathrm{S})$

- (Def) Generating function as trace over all states in $(L, M, 1)$ sector

$$Z_{L,M}(q) = \mathrm{Tr}\left[q^{\mathsf{S}-\mathsf{S}_{\max}/2}\right]$$

- Contribution of a Jordan chain from S to $\overline{\mathsf{S}}$ to $Z_{L,M}$

$$q^{\mathsf{S}-\mathsf{S}_{\max}/2} + \cdots + q^{-\mathsf{S}+\mathsf{S}_{\max}/2} = [2\mathsf{S}+1-\mathsf{S}_{\max}]_q = [\text{Length of JB}]_q$$

- Generating function is the sum of all possible Jordan chains

$$\boxed{Z_{L,M}(q) = \prod_{k=1}^{M-1} \frac{[L-k]_q}{[k]_q} = \sum_{j=1}^{b} n_j [N_j]_q \quad \Rightarrow \quad \text{JNF} = N_1^{n_1} \cdots N_b^{n_b}}$$

Jordan spectrum of $K > 1$

- (Def) Partition function

$$\mathbf{bin}(x, y, z, q) = \mathrm{Tr}_{\mathcal{A}}\left[x^{\hat{\ell}} y^{\hat{m}} z^{1} q^{\mathsf{S}-\hat{\ell}\hat{m}/2}\right]$$

where $\mathcal{A}$ is **infinite "colors"**, a set of all cyclic states with single $\mathbf{3}$

$$\mathcal{A} = \{\mathbf{3}, 1\mathbf{3}, \mathbf{23}, 11\mathbf{3}, 1\mathbf{23}, \mathbf{21}\mathbf{3}, \mathbf{223}, 111\mathbf{3}, 11\mathbf{23}, 1\mathbf{21}\mathbf{3}, \mathbf{2}11\mathbf{3}, \cdots\}$$

- which can be easily computed to be

$$\boxed{\mathbf{bin}(x, y, z, q) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} Z_{\ell+m,m}(q)\, x^{\ell} y^{m} z}$$

(cf) Pólya I formula with colors $\mathcal{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$

$$\zeta(x, y, z) = \mathrm{Tr}_{\mathcal{A}}\left[x^{\hat{\ell}} y^{\hat{m}} z^{\hat{k}}\right] = x + y + z$$

Pólya's Formula II

Count necklaces with K beads of infinite colors $\mathcal{A} = \{\mathbf{3}, 1\mathbf{3}, \mathbf{23}, \cdots\}$
The generating function defined by

$$Z(x,y,z,q) = \sum_{K=1}^{\infty} \mathrm{Tr}_{\mathcal{A}^{\otimes K}} \left[\mathbf{P}_{\rm cyc} x^{\hat{\ell}} y^{\hat{m}} z^{\hat{k}} q^{S - \hat{\ell}\cdot\hat{m}/2} \right], \qquad \mathbf{P}_{\rm cyc} = \frac{1}{K} \sum_{j=1}^{K} \mathbf{U}^j$$

will count # of cyclic states (necklaces)
The rest steps are identical as before and we get

$$\boxed{Z(x,y,z,q) = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \ln\left[1 - \mathbf{bin}(x^n, y^n, z^n, q^n)\right]}$$

Expanding, we obtain generating function $Z_{L,M,K}(q)$

$$Z(x,y,z,q) = \sum_{L=1}^{\infty} \sum_{M=1}^{L} \sum_{K=1}^{M} Z_{L,M,K}(q) \cdot x^{L-M} y^{M-K} z^K$$

which can give the JNF spectrum

$$Z_{L,M,K}(q) = \sum_{j=1}^{b} n_j [N_j]_q \quad \Rightarrow \quad \mathrm{JNF} = N_1^{n_1} \cdots N_b^{n_b}$$

Examples: Using Mathematica, it is easy to read off

- $(8, 4, 2)$

$$Z_{8,4,2}(q) = 4[1]_q + 3[3]_q + 4[5]_q + [7]_q + [9]_q \quad \Rightarrow \quad \mathrm{JNF} = 1^4\ 3^3\ 5^4\ 7\ 9$$

- $(9, 6, 3)$

$$Z_{9,6,3}(q) = 10[1]_q + 8[2]_q + 10[3]_q + 14[4]_q + 4[5]_q + 3[6]_q + 4[7]_q + [10]_q$$

$$\Rightarrow \quad \mathrm{JNF} = 1^{10}\ 2^8\ 3^{10}\ 4^{14}\ 5^4\ 6^3\ 7^4\ 10$$

Universality

We claim that the eclectic and hyper-eclectic models have the same Jordan spectrum.

Hamiltonian of the eclectic model

$$\mathbf{H} = \mathbf{H_3} + \mathbf{H_1} + \mathbf{H_2}, \qquad \mathbf{H_1} = \xi_1 \sum_{n=1}^{L} \mathbb{P}_{\mathbf{32}}^{n,n+1}, \quad \mathbf{H_2} = \xi_2 \sum_{n=1}^{L} \mathbb{P}_{\mathbf{13}}^{n,n+1}$$

One can check that

$$\mathbf{H_3} : \ \mathsf{S} \to \mathsf{S} - 1, \quad \mathbf{H_2} : \ \mathsf{S} \to \mathsf{S} - M_1, \quad \mathbf{H_1} : \ \mathsf{S} \to \mathsf{S} - L_1$$

It is obvious that the eigenvector of the hyper-eclectic model at the level $\overline{\mathsf{S}} = \mathsf{S}_{\max} - \mathsf{S}$ becomes that of the eclectic model if $M_1 > \overline{\mathsf{S}}$

This means for cases with relatively large number of **2**'s, two models share the same Jordan blocks with relatively large sizes

If not, one can modify the top vector of the hyper-eclectic model which becomes an eigenvector by acting **H**

(Ex) $(7,3,1)$ again: We showed above that

$$\mathbf{H_3}^5(-9|1\mathbf{22}111\mathbf{3}\rangle^6 + 5|\mathbf{2}11\mathbf{2}11\mathbf{3}\rangle^6) = 0$$

This can be extended to

$$\mathbf{H}^5(-9|1\mathbf{22}111\mathbf{3}\rangle^6 + 5|\mathbf{2}11\mathbf{2}11\mathbf{3}\rangle^6 + \gamma|1\mathbf{2}1\mathbf{2}11\mathbf{3}\rangle^5) = 0 \quad \text{if } \gamma = 3\xi_2$$

Conjecture: One can construct top vectors of the eclectic model from those of the hyper-eclectic model by adding vectors with lower level S

But we have no rigorous mathematical proof yet in general context.

Summary and Future directions

- We have investigated the (hyper)-eclectic models, simple integrable models appeared in the context of strongly twisted $\mathcal{N} = 4$ SYM

- We showed the BAE can be consistent with Pólya formula which is a non-trivial check for the BAE and its solutions

- The integrability fails to explain Jordan spectrum

- We used mathematics (combinatorics and linear algebra) to obtain complete Jordan spectrum

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- Applications : Logarithmic CFTs, Correlation functions of Fishnet theory, etc.

- Complete mathematical proof of no unexpected shortening and universality assumptions

- Applying our Pólya formula to other non-Hermitian spin chain model with integrability

-Systematic construction of **eigenvectors**

(Ex) $(10, 5, 1)$
Jordan spectrum : $1\ 5^2\ 7\ 9^2\ 11\ 13^2\ 15\ 17\ 21$

Complete eigenvectors

```
{7 |1122122113> - 10 |1122211213> - 7 |1211222113> + 3 |1212121213> + 10 |1212211123> + |1221112213> -
  9 |1221121123> + 4 |2111221213> - 4 |2112112213> - 4 |2112121123> + 8 |2121112123> - 8 |2211111223>,
 2 |1112222113> - 2 |1121221213> + 3 |1122121123> + 2 |1211212213> - |1211221123> -
  3 |1212112123> + 5 |1221111223> - 2 |2111122213> + 2 |2111212123> - 2 |2112111223>,
 3 |1112221213> - 3 |1121212213> - 3 |1121221123> + 5 |1122112123> + 3 |1211122213> + |1211212123> -
  5 |1212111223> - 4 |2111122123> + 4 |2111211223>, 2 |1122112213> - 3 |1122121123> - 2 |1211212213> +
  3 |1211221123> + |1212112123> - 3 |1221111223> + 2 |2111122213> - 2 |2111212123> + 2 |2112111223>,
 |1112212213> - |1121122213> - |1121212123> + 3 |1122111223> + 2 |1211122123> - 2 |1211211223>,
 |1111222213> - |1112122123> + |1121121223> - |1211112223>,
 |1112221123> - |1121212123> + 2 |1122111223> + |1211122123> - |1211211223>,
 |1112212123> - |1121122123> - |1121211223> + 2 |1211121223> - 2 |2111112223>,
 |1111222123> - |1112121223> + |1121112223>,
 |1112211223> - |1121121223> + |1211112223>, |1111221223> - |1112112223>, |1111122223>}
```

Thank you for attention!