Jordan Blocks in Fishnet

of Strongly Twisted SYM theory

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Motivation

- Many physics problems are to diagonalize matrices
- However, some matrices are non-diagonalizable due to Jordan blocks (ex)

$$\begin{pmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{pmatrix} \to \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

- Integrability is a powerful tool for diagonalization of some large size matrices

Can integrability be useful even for Jordan blocks?

AdS/CFT duality

1. AdS/CFT correspondence: strings $\leftrightarrow$ SYM
2. Spectral problem:
   - Conformal dimensions of **non-BPS** operators in SYM

$$
\begin{aligned}
&\mathrm{Tr}\left[\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_{L-1}\mathcal{A}_L\right](x)\\
&\mathcal{A}\in\{\partial^{k_i}\phi_i,\partial^{k_i}\phi_i^\dagger,\partial^{k_j}\psi_j,\partial^{k_j}\overline{\psi}_j,\partial^{k_i}\mathcal{F},\partial^{k_i}\overline{\mathcal{F}}\}
\end{aligned}
$$

   - Energy of string configurations moving in $\mathrm{AdS}_5\times S^5$
3. Extended to higher-point correlation functions

Integrability

1. Weak coupling: Integrable quantum spin chain

$$H = \lambda^k \sum_{n=1}^{L} \mathcal{H}_{n,n+k}$$

2. Strong coupling: classical string theory described by classical integrable systems
3. Nonperturbative integrability in exact S-matrix [Beisert]

$$\mathbf{S}(p_1, p_2) = S_0(p_1, p_2)\ \mathcal{S}_{su(2|2)} \otimes \mathcal{S}_{su(2|2)}$$

Extended to wider class of AdS/CFT

1. γ-deformed SYM

$$\mathcal{L} = N_c \mathrm{Tr}\left[-\frac{1}{4}F^2 - \frac{1}{2}D^\mu \phi_i^\dagger D_\mu \phi^i + i\overline{\psi}_A^{\dot\alpha} D_{\dot\alpha}^{\alpha} \psi_\alpha^A\right] + \mathcal{L}_{\mathrm{int}}$$

$$\mathcal{L}_{\mathrm{int}} = N_c \mathrm{Tr}\left[\frac{\lambda}{4}\{\phi_i^\dagger, \phi^i\}\{\phi_j^\dagger, \phi^j\} - \sum_{\{ijk\}} \lambda {q_k}^\sigma \phi_i^\dagger \phi_j^\dagger \phi^i \phi^j + \cdots\right]$$

- $\sigma = \pm 1$ for even and odd permutations of (123)
- $q_k = 1$: N=4 SYM
- $q_k = q$: β-deformed SYM

2. Strings moving in TsT transformed target space

Integrability of the γ-deformed SYM

1. Conjectured asymptotic Bethe ansatz [Beisert-Roiban]

$$Q_j \cdot U_j(x_{j,k}) \prod_{\substack{j'=1 \\ (j',k')\neq(j,k)}}^{7} \prod_{k'=1}^{K_{j'}} \frac{u_{j,k} - u_{j',k'} + \frac{i}{2} M_{j,j'}}{u_{j,k} - u_{j',k'} - \frac{i}{2} M_{j,j'}} = 1$$

   - Q_j: given by q_k, U_j given by the dressing phase
   - $Q_j = 1$: Beisert-Staudacher BAE for N=4 SYM

2. Exact S-matrix as Drinfeld-Reshetikhin twist [Bajnok-Bombadelli-Nepomechie-CA]

$$\mathbf{S}^{(q)}(p_1, p_2) \propto \mathcal{F}(q_k) \cdot \mathcal{S}_{su(2|2)} \otimes \mathcal{S}_{su(2|2)} \cdot \mathcal{F}(q_k)$$

3. Exactly solvable for **any value** of 'tHooft coupling constant

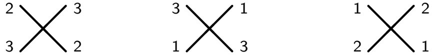
What do we mean by "exactly solvable"?

- Diagonalization of spin-chain (Hamiltonian) matrix
- Size of the matrix: N^L with L large
- Any existing supercomputer can not handle for $L > 100$
- Reduce this "impossible problem" to "solvable" one by "Integrability" (QISM)
- However, the Bethe-ansatz equations are not exactly solvable, but only numerically solvable

Strong twist Limit [Gürdogan-Kazakov]

1. Double scaling Limit
   - $\lambda \to 0$: weak coupling spin-chain
   - $q_k \to \infty$: strong twist
   - With $\lambda q_k = \xi_k$: finite
2. Lagrangian (non-unitary & chiral)

$$\mathcal{L}_{\text{int}} = N_c \mathrm{Tr}\left[\xi_1 \phi_2^\dagger \phi_3^\dagger \phi^2 \phi^3 + \xi_2 \phi_3^\dagger \phi_1^\dagger \phi^3 \phi^1 + \xi_3 \phi_1^\dagger \phi_2^\dagger \phi^1 \phi^2 + \cdots\right]$$



   - Toy model for exact correlation functions [Only a few Feynman diagrams for each order]
   - $\xi_1 = \xi_2 = 0$: related to "Fishnet model" [Zamolodchikov]
   - Spin-chain Hamiltonian move the excitations

$$\mathrm{Tr}[\cdots \phi_1 \phi_1 \cdots \phi_1 \overset{R}{\rightarrow}{\chi}\, \phi_1 \cdots \phi_1 \overset{L}{\leftarrow}{\chi'}\, \phi_1 \phi_1 \cdots]$$

$$\chi = \phi_2, \phi_3^\dagger \quad \mathrm{R-direction} \qquad \chi' = \phi_3, \phi_2^\dagger \quad \mathrm{L-direction}$$

Strongly twisted $su(3)$ Integrable spin-chain

- Composite operators made of ϕ_1, ϕ_2, ϕ_3
- R-matrix and Lax operator

$$\mathbf{R}(u) = \left(\begin{array}{ccc|ccc|ccc} 1 & & & & & & & & \\ & & & 1 & & & & & \\ & & \xi_2 u & & & & 1 & & \\ \hline & 1 & & \xi_3 u & & & & & \\ & & & & 1 & & & & \\ & & & & & & & 1 & \\ \hline & & 1 & & & & & & \\ & & & & & 1 & & \xi_1 u & \\ & & & & & & & & 1 \end{array}\right) \to \mathcal{L}_{n,a}(u)$$

- satisfy Yang-Baxter equation
- Monodromy and Transfer matrices

$$\mathcal{M}(u) \;=\; \mathcal{L}_L(u)\mathcal{L}_{L-1}(u)\cdots\mathcal{L}_1(u), \qquad \mathbf{T}(u) = \mathrm{Tr}_a \mathcal{M}_a(u)$$

Algebraic Bethe ansatz procedure

- Infinite # of conserved charges

$$[\mathbf{T}(u), \mathbf{T}(v)] = 0$$

- Hamiltonian acting on $\underline{\mathbf{3}}^{\otimes L}$ is constructed

$$\mathcal{H} = \sum_{n=1}^{L} \left[\xi_3 \overset{\cdots \mathbf{1} \otimes \mathbf{e}_{21} \otimes \mathbf{e}_{12} \otimes \mathbf{1} \cdots}{\overset{\downarrow}{\mathbf{e}_{21}^{n+1} \mathbf{e}_{12}^{n}}} + \xi_1 \mathbf{e}_{32}^{n+1} \mathbf{e}_{23}^{n} + \xi_2 \mathbf{e}_{13}^{n+1} \mathbf{e}_{31}^{n} \right]$$

- Fundamental Commutation Relation (FCR) between $\mathcal{M}$ gives the BAE

$$\mathbf{R}_{ab}(u-v)\mathcal{M}_a(u)\mathcal{M}_b(v) = \mathcal{M}_b(v)\mathcal{M}_a(u)\mathbf{R}_{ab}(u-v)$$

- However, **sensible FCR can not be obtained for this model**

Numerical diagonalization

- states belonged to (L, M, K) sector

$$|\overbrace{\phi_1 \cdots \phi_1}^{L-M}\overbrace{\phi_2 \cdots \phi_2}^{M-K}\overbrace{\phi_3 \cdots \phi_3}^{K}\rangle, \quad \& \quad \text{all permutations}$$

- Doable for small (L, M, K) since the matrix size is

$$\frac{L!}{(L-M)!(M-K)!K!}$$

$(L, M, K) = (4, 2, 1)$, size=12

```
-1,1,0,0,0,0,0,0,0,0,0,0
0,-1,1,0,0,0,0,0,0,0,0,0
0,0,-1,0,0,0,0,0,0,0,0,0
0,0,0,1,1,0,0,0,0,0,0,0
0,0,0,0,1,1,0,0,0,0,0,0
0,0,0,0,0,1,0,0,0,0,0,0
0,0,0,0,0,0,-0-1i,1,0,0,0,0
0,0,0,0,0,0,0,-0-1i,1,0,0,0
0,0,0,0,0,0,0,0,-0-1i,0,0,0
0,0,0,0,0,0,0,0,0,0+1i,1,0
0,0,0,0,0,0,0,0,0,0,0+1i,1
0,0,0,0,0,0,0,0,0,0,0,0+1i
```

$(L, M, K) = (5, 2, 1)$, size=20

```
1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,-0.81-0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,-0.81-0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,-0.81-0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,-0.81-0.59i,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,-0.81+0.59i,1,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,-0.81+0.59i,1,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,-0.81+0.59i,1,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,-0.81+0.59i,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0.31-0.95i,1,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0.31-0.95i,1,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31-0.95i,1,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31-0.95i,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31+0.95i,1,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31+0.95i,1,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31+0.95i,1
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31+0.95i
```

$(L, M, K) = (5, 3, 1)$, size=30

```
1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,-0.81-0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,-0.81-0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,-0.81-0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,-0.81-0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,-0.81-0.59i,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,-0.81+0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,-0.81+0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,-0.81+0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,-0.81+0.59i,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
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0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31-0.95i,0,0,0,0,0,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31+0.95i,1,0,0,0,0,0,0,0,0
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0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31+0.95i,1,0,0,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31+0.95i,0,0,0,0,0
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0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-0.81-0.59i,0,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-0.81+0.59i,0,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31-0.95i,0
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0.31+0.95i
```

$$\left(\begin{array}{c|c|c|c} [1] & & & \\ \hline & [\omega] & & \\ \hline & & \ddots & \\ \hline & & & [\omega^{L-1}] \end{array}\right), \quad \omega = e^{\frac{2\pi i}{L}}$$

$$[\omega^n] = \begin{array}{c|c|c|c} \ddots & \longleftarrow \ \ell_1 \ \longrightarrow & \longleftarrow \ \ell_2 \ \longrightarrow & \longleftarrow \ \ell_3 \ \longrightarrow \\ \hline & \begin{matrix} \omega^n & 1 & & \\ & \omega^n & 1 & \\ & & \ddots & 1 \\ & & & \omega^n \end{matrix} & & \\ \hline & & \begin{matrix} \omega^n & 1 & & \\ & \omega^n & 1 & \\ & & \ddots & 1 \\ & & & \omega^n \end{matrix} & \\ \hline & & & \begin{matrix} \omega^n & 1 & & \\ & \omega^n & 1 & \\ & & \ddots & 1 \\ & & & \omega^n \end{matrix} \end{array}$$

Sizes of Jordan Blocks

- For each eigenvalue, degenerate eigenvalues splits into Jordan Blocks of various sizes
- For $M = 2, K = 1$, only one JB for each eigenvalue
- For $M = 3, K = 1$ (within the limit of my desktop)

L	(L-1)(L-2)/2	Sizes of JBs
5	6	5+1
6	10	7+3
7	15	9+5+1
8	21	11+7+3
9	28	13+9+5+1

- Conjecture for general L

$$(2L-5)+(2L-9)+\cdots+3 \text{ or } 1$$

- How about $M \geq 4$?

Jordan blocks

- An example: $\mathbf{A} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ with only eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(\mathbf{A} - a)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow (\mathbf{A} - a)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

- An order N Jordan block: with unit column vectors $\mathbf{e}_n$

$$\mathbf{A} = \begin{pmatrix} a & 1 & & & \\ & a & 1 & & \\ & & \ddots & \ddots & \\ & & & a & 1 \\ & & & & a \end{pmatrix} \quad \rightarrow \quad (\mathbf{A} - a)\,\mathbf{e}_n = \mathbf{e}_{n-1}$$

$$\therefore \quad (\mathbf{A} - a)^N\,\mathbf{e}_N = 0$$

- $\mathbf{e}_1$: true eigenvector
- $\mathbf{e}_n, n = 2, \cdots, N$: generalized eigenvectors or Jordan descendents
- Each Jordan block has one true eigenvector

Goal

Understand the Jordan Block structures

in the context of Integrability

General twist of $su(3)$: algebraic Bethe ansatz

Lax operator

$$\mathcal{L}(u) = \left(\begin{array}{ccc|ccc|ccc}
u_+ & & & & & & & & \\
 & \frac{u}{q_3} & & 1 & & & & & \\
 & & q_2\,u & & & & 1 & & \\
\hline
 & 1 & & q_3\,u & & & & & \\
 & & & & u_+ & & & & \\
 & & & & & \frac{u}{q_1} & & 1 & \\
\hline
 & & 1 & & & & \frac{u}{q_2} & & \\
 & & & & & 1 & & q_1\,u & \\
 & & & & & & & & u_+
\end{array}\right)$$

$$u_+ \equiv u + 1$$

Algebraic Bethe ansatz

- Monodromy and Transfer matrices

$$
\begin{aligned}
\mathcal{M}(u) &= \mathcal{L}_L(u)\mathcal{L}_{L-1}(u)\cdots\mathcal{L}_1(u) = \left(\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{array}\right), \\
\mathbf{T}(u) &= \mathrm{Tr}_a \mathcal{M}_a(u)
\end{aligned}
$$

- FCR from

$$
\mathbf{R}_{ab}(u-v)\mathcal{M}_a(u)\mathcal{M}_b(v) = \mathcal{M}_b(v)\mathcal{M}_a(u)\mathbf{R}_{ab}(u-v)
$$

- Eigenvalues and eigenvectors from Bethe ansatz equations

Strong twist limit

$$
\left(\begin{array}{ccc|ccc|ccc}
u_+ & & & 1 & & & & & \\
 & \frac{u}{q_3} & & & & & & & \\
 & & q_2\,u & & & & 1 & & \\
\hline
 & 1 & & q_3\,u & & & & & \\
 & & & & u_+ & & & & \\
 & & & & & \frac{u}{q_1} & & 1 & \\
\hline
 & & 1 & & & & \frac{u}{q_2} & & \\
 & & & & & 1 & & q_1\,u & \\
 & & & & & & & & u_+
\end{array}\right)
$$

$$
\to \left(\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & & & 1 & & & & & \\
 & & \xi_2\,u & & & & 1 & & \\
\hline
 & 1 & & \xi_3\,u & & & & & \\
 & & & & 1 & & & & \\
 & & & & & & & 1 & \\
\hline
 & & 1 & & & & & & \\
 & & & & & 1 & & \xi_1\,u & \\
 & & & & & & & & 1
\end{array}\right)
$$

- by taking $u \to \varepsilon u, \;\; q_k \equiv \frac{\xi_k}{\varepsilon} \quad \text{with} \quad \varepsilon \to 0 \qquad (\varepsilon \propto \lambda)$

Strong twist limit of the BAE

- Taking the limit $q_k \equiv \frac{\xi_k}{\varepsilon}$ with $\varepsilon \to 0$

$$\left(\frac{u_m+1}{u_m}\right)^L = \frac{\varepsilon^{3K-L}\cdot {\xi_3}^L}{(\xi_1\xi_2\xi_3)^K}\prod_{\substack{n=1\\n\neq m}}^{M}\frac{u_m-u_n+1}{u_m-u_n-1}\prod_{j=1}^{K}\frac{u_m-v_j-1}{u_m-v_j}$$

$$1 = \frac{\varepsilon^{3M-2L}\cdot(\xi_2\xi_3)^L}{(\xi_1\xi_2\xi_3)^M}\prod_{n=1}^{M}\frac{v_k-u_n+1}{v_k-u_n}\prod_{\substack{j=1\\j\neq k}}^{K}\frac{v_k-v_j-1}{v_k-v_j+1}$$

Explicit solutions

- Bethe roots can be found **exactly**

$$
\begin{aligned}
\phi_2 \quad &: \quad u_n = \quad 0 + \varepsilon^{\alpha} \hat{u}_n, \qquad n = 1, \cdots, M_1 (\equiv M - K) \\
\phi_3 \quad &: \quad u_{M_1+k} = -1 + \varepsilon^{\beta} \hat{w}_k, \qquad k = 1, \cdots, K \\
&\quad\quad v_k = -2 + \varepsilon^{\beta} \hat{w}_k + \varepsilon^{\gamma} \hat{v}_k, \qquad k = 1, \cdots, K
\end{aligned}
$$

with

$$
\alpha = \frac{L - M - K}{L - M + K}, \quad \beta = \frac{L - 3(M - K)}{L - M + K}, \quad \gamma = 2L - 3M - \beta(K - 1)
$$

$$
\begin{aligned}
\hat{w}_k \quad &= \quad -\frac{(\xi_1 \xi_3)^{\frac{M_1}{L - M_1}}}{\xi_2} \, \omega_{L-M_1}^{n_k + \frac{K-1}{2}}, \quad n_k = \{1, \cdots, L - M_1\} \\
\hat{u}_n \quad &= \quad \left(\frac{(\xi_1 \xi_2 \xi_3)^K}{\xi_3{}^L} (-1)^{M-1} \prod_{k=1}^{K} \hat{w}_k \right)^{\frac{1}{L}} \omega_L^{i_n}, \quad i_n = \{1, \cdots, L\}
\end{aligned}
$$

Eigenvalues of Transfer matrix

$$
\begin{aligned}
T(u) &= \exp \frac{2\pi i}{L}\left[\sum_{k=1}^{K} n_k - \sum_{m=1}^{M_1} i_m + \frac{1}{2}M_1(M_1-1) + \frac{1}{2}K(K-1)\right] \\
&\equiv \omega_L^{\mathcal{N}(\{n_k\},\{i_m\})}
\end{aligned}
$$

- They are L-th roots of unity
- **Cyclic state** is obtained by $\mathcal{N}(\{n_k\},\{i_m\}) = 0 \mod L$

$$
\sum_{k=1}^{K} n_k - \sum_{n=1}^{M_1} i_n + \frac{1}{2}M_1(M_1-1) + \frac{1}{2}K(K-1) = 0 \mod L
$$

with

$$
n_k = \{1, \cdots, L - M_1\}, \qquad i_n = \{1, \cdots, L\}
$$

- Related to a famous Mathematics problem:

Pólya's counting theory

G. Pólya's Counting Theory

of inequivalent necklaces with L beads of several colors

- Naive guess:

$$\frac{1}{L} \cdot \frac{L!}{(L-M)!(M-K)!K!} = \frac{(L-1)!}{(L-M)!(M-K)!K!}$$

  is NOT always true

- Generating function (for three colors) [Pólya]

$$\begin{aligned} Z(x,y,z) &= -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \ln\left[1 - x^n - y^n - z^n\right] \\ &= \sum_{L,M,K} d(L,M,K) \cdot x^{L-M} y^{M-K} z^K \end{aligned}$$

  with EulerPhi function $\phi(n)$

- One-loop partition function of $N = 4$ SYM [Spradlin-Volovich]

Comparison for nontrivial cases only

except trivial cases

L	M	K	naive counting	Pólya counting	Bethe ansatz
14	6	2	6435/2	3225	3225
16	6	2	15015/2	7518	7518
18	6	2	15470	15484	15484
20	6	2	29070	29088	29088
20	8	2	176358	176400	176400
20	10	4	1939938	1940064	1940064
21	9	3	1175720	1175730	1175730
22	6	2	101745/2	50895	50895
22	8	2	406980	407040	407040
22	10	4	6172530	6172740	6172740
24	6	2	168245/2	84150	84150
24	8	2	1716099/2	858132	858132
24	9	3	4576264	4576278	4576278
24	10	4	17160990	17161320	17161320

Eigenvalues and total sizes are understood

How about sizes of Jordan subcells?

Fishnet model

A special case $\xi_1 = \xi_2 = 0, \xi_3 = 1$

- Hamiltonian

$$\mathcal{H} = \sum_{n=1}^{L} \mathbf{e}_{21}^{n+1} \mathbf{e}_{12}^{n}$$

- Cyclic eigenstates can be written as

$$\sum_{n=1}^{L} |1 \cdots 121 \cdots 12 \cdots 21 \cdots 121 \cdots 1 \overset{[n]}{\overset{\downarrow}{3}} 1 \cdots 1\rangle$$

- The Hamiltonian $\mathcal{H}$ moves each 2 one step Right

$$\mathcal{H} : \sum_{n=1}^{L} | \cdots 1 \overset{\rightarrow}{2} 1 \cdots 1 \overset{\rightarrow}{2} \cdots \overset{\rightarrow}{2} 1 \cdots 1 \overset{\rightarrow}{2} 1 \cdots 1 \overset{[n]}{\overset{\downarrow}{3}} \cdots \rangle$$

- True eigenvector

$$| \overbrace{11\cdots11}^{L-M}\overbrace{22\cdots2}^{M-1}\,3\rangle$$

- Lowest Jordan descendent

$$| \overbrace{22\cdots2}^{M-1}\overbrace{11\cdots11}^{L-M}\,3\rangle$$

- By acting $\mathcal{H}$ on the lowest descendent repeatedly

$$\mathcal{H}^N| \overbrace{22\cdots2}^{M-1}\overbrace{11\cdots11}^{L-M}\,3\rangle = | \overbrace{11\cdots11}^{L-M}\overbrace{22\cdots2}^{M-1}\,3\rangle, \; N = (M-1)(L-M)$$

Hence, the size of JB $= N + 1$

- In this way, one can find all true eigenvectors and their orders $\rightarrow$ sizes of JB subcells
- JB structure is very rich for higher M (Ex) $M = 5$

L	JB x1	JB x2	JB x3	JB x4	JB x5	JB x6
8	1,5,7,9,13					
9	1,11,13,17	5,9				
10	1,7,11,15,17,21	5,9,13				
11	7,11,15,19,21,25	1,5,17	9,13			
12	1,7,19,23,25,29	11,15,21	5,9,13,17			
13	7,23,27,29,33	1,11,15,19,25	5,21	9,13,17		
14	27,31,33,37	1,7,11,23,29	5,15,19,25	9,17,21	13	
15	7,31,35,37,41	1,27,33	11,15,19,23,29	5,25	9,13,21	17
16	35,39,41,45	1,7,31,37	11,27,33	5,15,19,23,29	9,25	13,17,21

Summary and Conclusion

- Certain integrable models are not Bethe-ansatz solvable
- They may develop Jordan Blocks which show deep mysterious structures
- Our approach based on algebraic Bethe ansatz explains eigenvalues and size of JBs
- Jordan subcell structure can be understood with analyzing the Hamiltonian matrix

Take home message:

- Strongly twisted SYM is not simple: **integrable but not Bethe solvable**
- **Jordan Block** structure provides **new challenge to integrability**

Thanks for attention!