Non-invertible symmetries and off-critical defects

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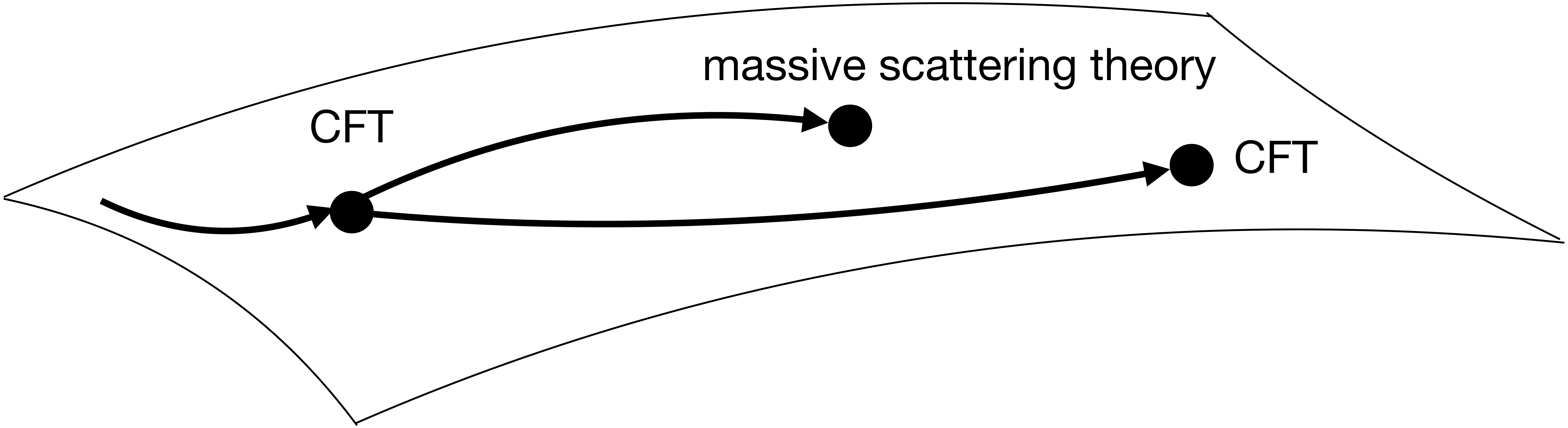
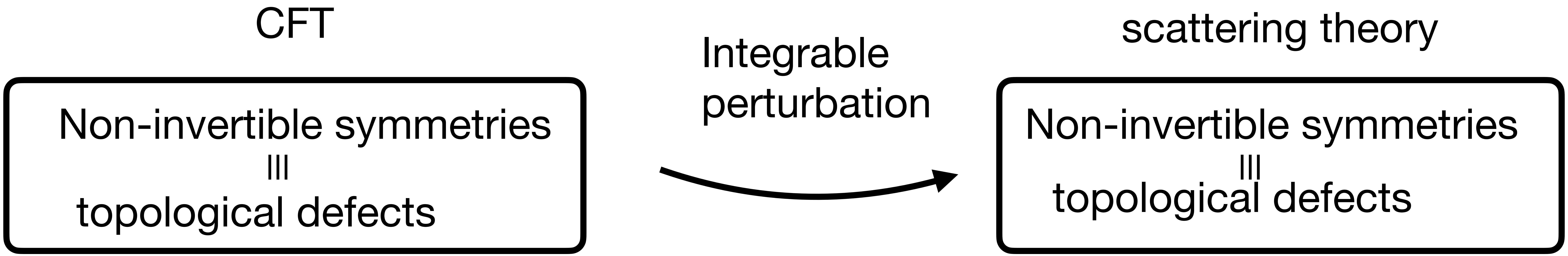
Motivation





How to maintain integrability of the defect? (Lee-Yang)

Conformal boundaries in CFT

local conservation laws $\bar{\partial} T = 0$ $\partial \bar{T} = 0$ and many more at $z = x + iy$

$x < 0$

$x = 0$

$$H_- = \int_{-\infty}^{0} dx (T_-(z) + \bar{T}_-(\bar{z}))$$

$$P_- = \int_{-\infty}^{0} dx (T_-(z) - \bar{T}_-(\bar{z}))$$

$T_-(z)$

energy is conserved

$\partial_y H_- = 0$

$(T_-(z) + \bar{T}_-(z))\,|_{x=0} = 0$

conformal boundary

momentum is not conserved

$\partial_y P_- \neq 0$

$(T_-(z) - \bar{T}_-(z))\,|_{x=0} \neq 0$

Topological defects in CFT

local conservation laws $\bar{\partial} T = 0$ $\partial \bar{T} = 0$ and many more at $z = x + iy$

$x = 0$

$$H_- = \int_{-\infty}^{0} dx (T_-(z) + \bar{T}_-(\bar{z}))$$

$$P_- = \int_{-\infty}^{0} dx (T_-(z) - \bar{T}_-(\bar{z}))$$

$T_-(z)$ $T_+(z)$

$$H_+ = \int_{0}^{\infty} dx (T_+(z) + \bar{T}_+(\bar{z}))$$

$$P_+ = \int_{0}^{\infty} dx (T_+(z) - \bar{T}_+(\bar{z}))$$

energy is conserved

$$\partial_y (H_- + H_+) = 0$$

$$T_-(z) + \bar{T}_-(z) = T_+(z) + \bar{T}_-(z)$$

at the defect

conformal defect

momentum is conserved

$$\partial_y (P_- + P_+) = 0$$

$$T_-(z) - \bar{T}_-(z) = T_+(z) - \bar{T}_-(z)$$

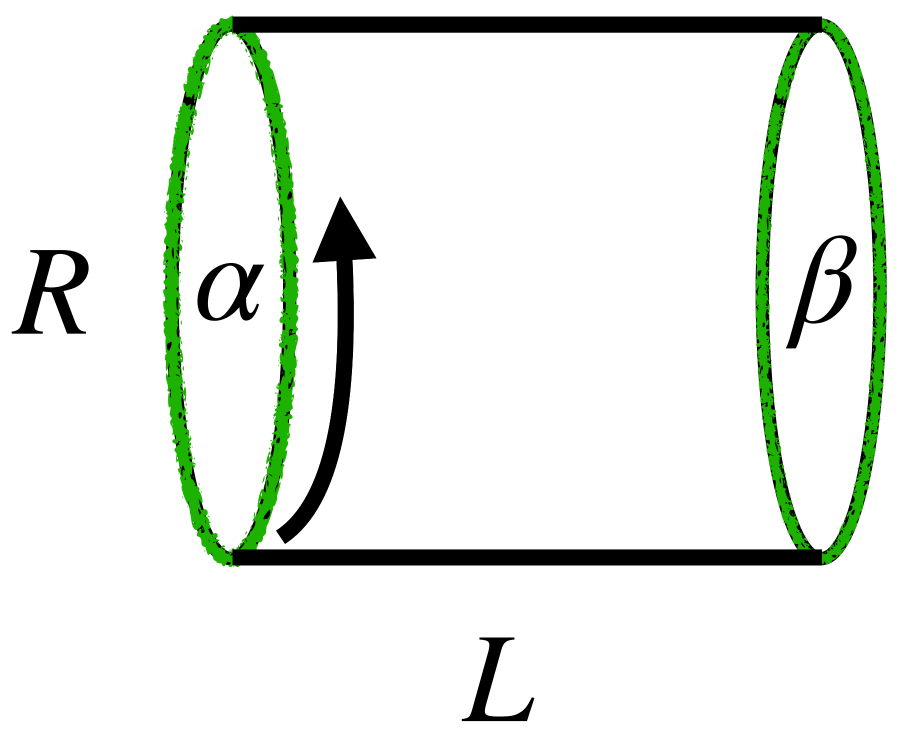
both are conserved

$$T_-(z) = T_+(z)$$
$$\bar{T}_-(z) = \bar{T}_+(z)$$

topological defect

Hilbert space with boundaries

[Cardy]

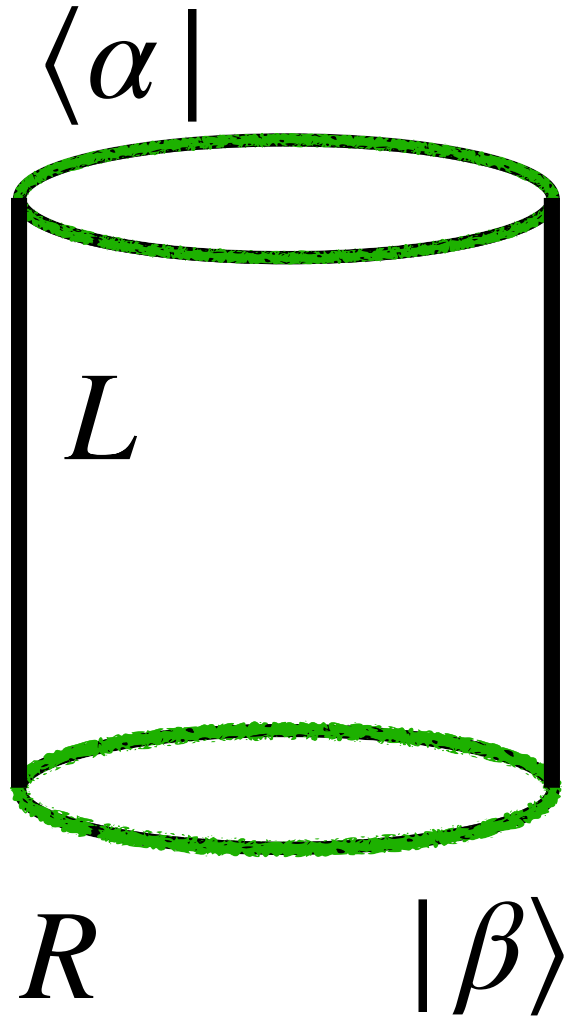


$$Z_{\alpha\beta} = \mathrm{Tr}(e^{-RH_{\alpha\beta}})$$

$$\mathscr{H}_{\alpha\beta} = \sum_{\gamma} N^{\gamma}_{\alpha\beta} V_{\gamma}$$

$$Z_{\alpha\beta} = \sum_{\gamma} N^{\gamma}_{\alpha\beta} \chi_{\gamma}(q)$$

modular covariance



modular S-matrix

$$Z_{\alpha\beta} = \langle \alpha | e^{-HL} | \beta \rangle$$

$$(L_n - \bar{L}_{-n}) | \beta \rangle = 0$$

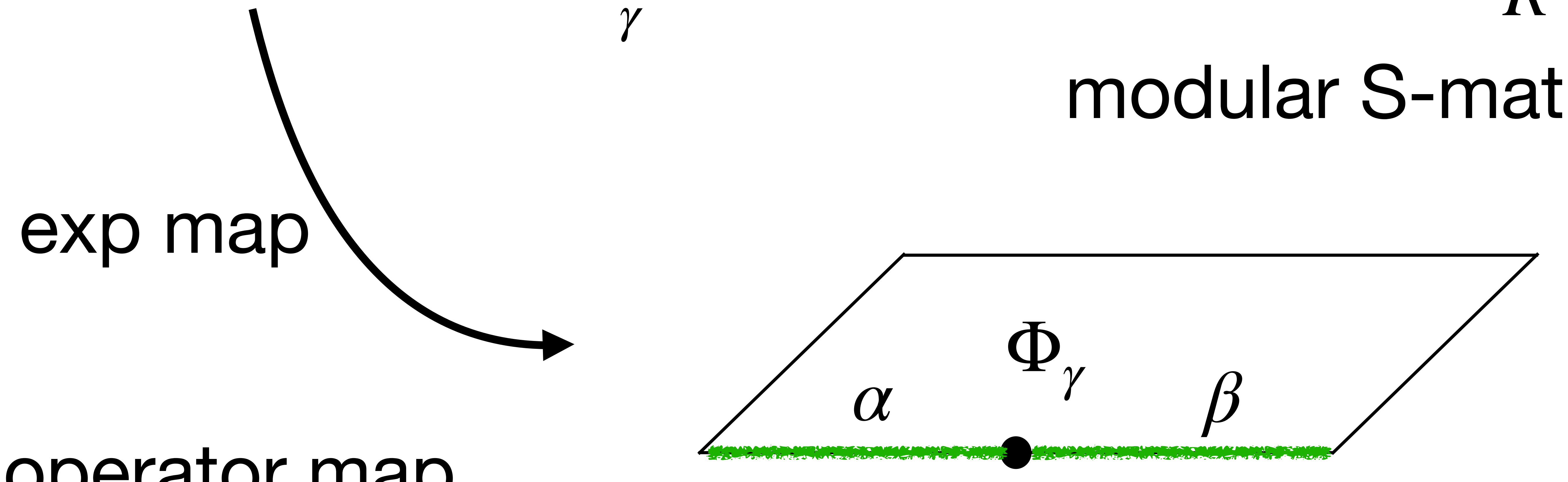
Ishibashi states

$$| h \rangle\rangle = G^{-1}_{ij} L_{-(i)} \bar{L}_{-(j)} | h \rangle$$

Carry states

$$| \alpha \rangle = g^{h}_{\alpha} | h \rangle\rangle$$

$$Z_{\alpha\beta} = \sum_{\alpha} g^{\gamma}_{\alpha} g^{\gamma}_{\beta} \chi_{\alpha}(\tilde{q})$$



state operator map

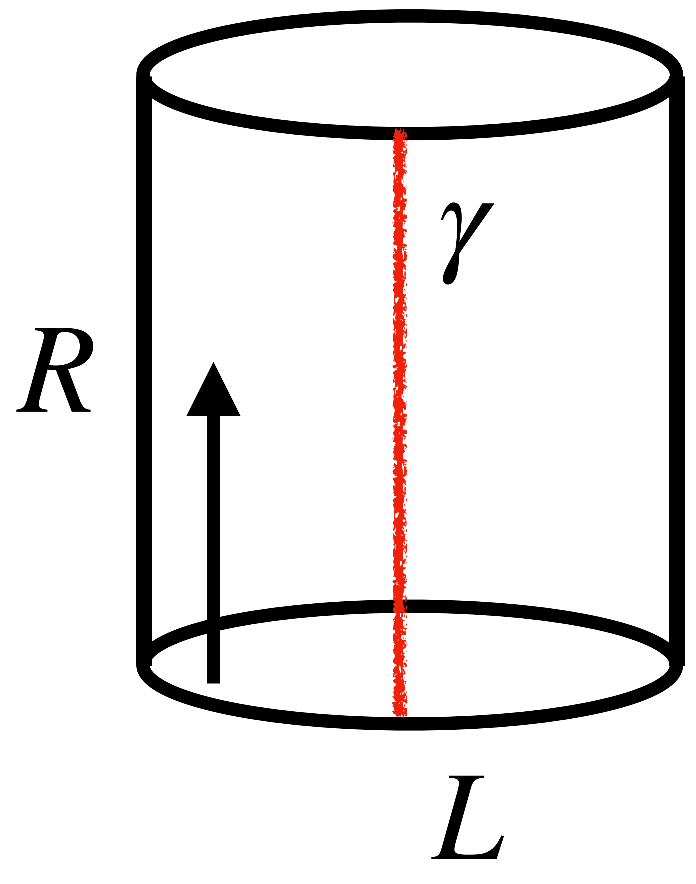
operators changing boundaries

Φ_{γ}

boundaries are labeled with Kac labels

Hilbert space with defects

[Petkova-Zuber]

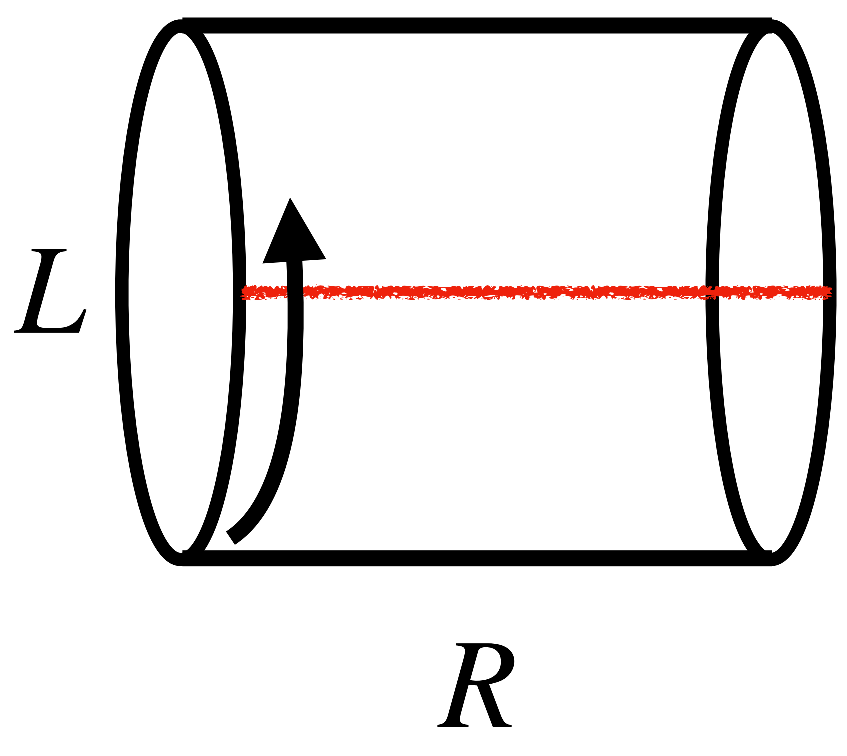


$$Z_\gamma = \mathrm{Tr}(e^{-RH_\gamma})$$

$$\mathscr{H}_\gamma = \sum_{\alpha\beta} N^\gamma_{\alpha\beta} V_\alpha \otimes \bar{V}_\beta$$

$$Z_\gamma = \sum_{\alpha,\beta} N^\gamma_{\alpha\beta} \chi_\alpha(q) \bar{\chi}_\beta(q)$$

modular
covariance



$$Z_\gamma = \mathrm{Tr}(e^{-HL} D_\gamma)$$

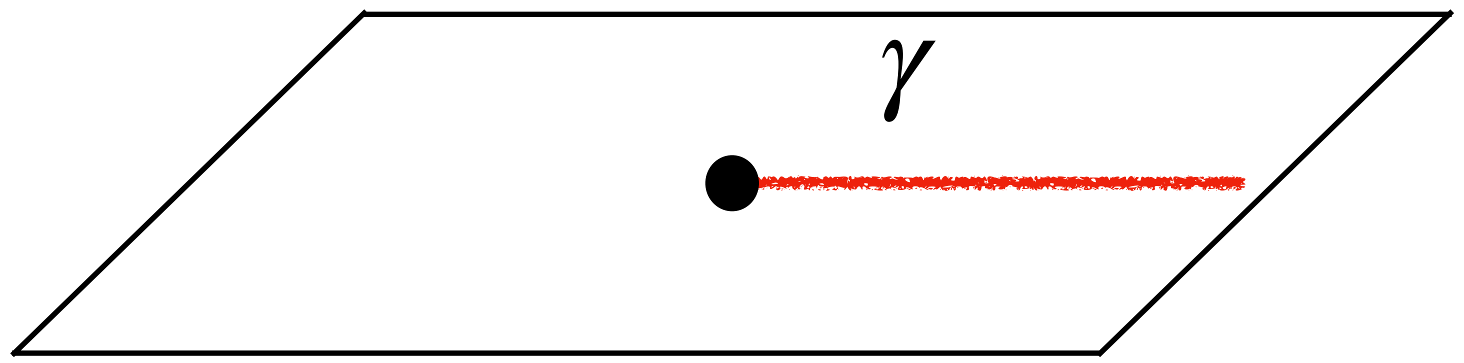
$$[D_\gamma, L_n] = 0 = [D_\gamma, \bar{L}_n]$$

$$Z_\gamma = \sum_\alpha \frac{S_{\gamma\alpha}}{S_{0\alpha}} \chi_\alpha(\tilde{q}) \bar{\chi}_\alpha(\tilde{q})$$

modular S-matrix

defects are labeled
with Kac labels

exp map

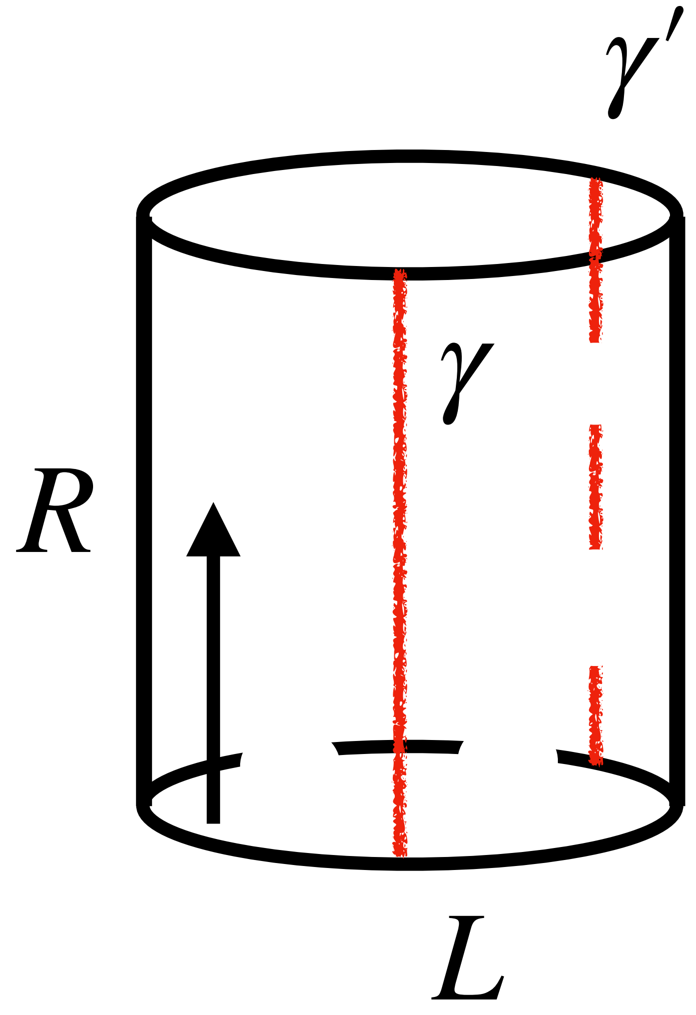


state operator map
operators creating the defect

$$\Phi_{\alpha\beta}$$

Hilbert space with two defects

[Petkova-Zuber]

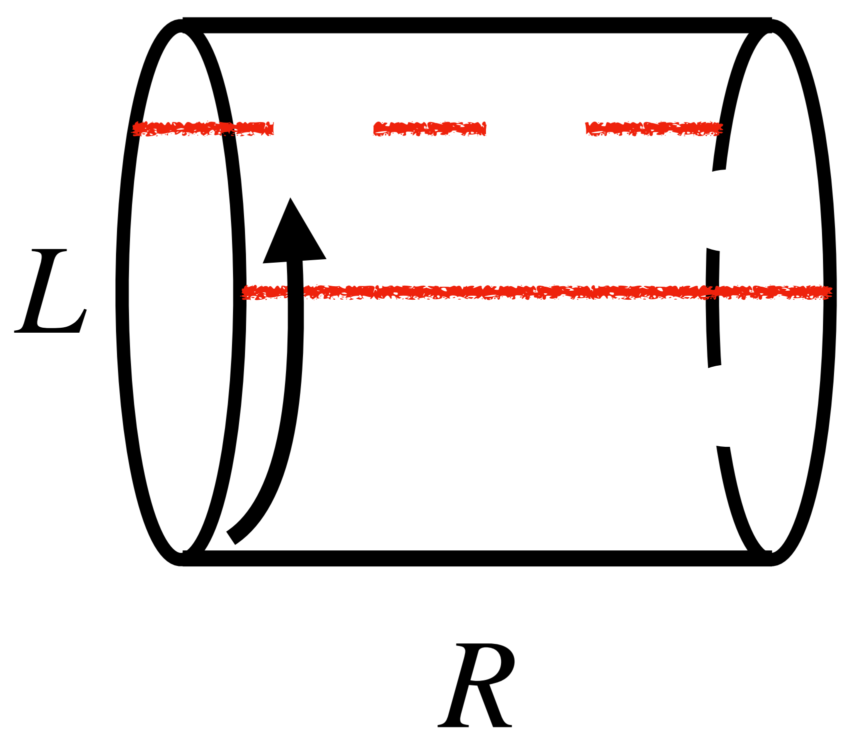


$$Z_{\gamma\gamma'} = \mathrm{Tr}(e^{-RH_{\gamma\gamma'}})$$

$$\mathscr{H}_{\gamma\gamma'} = \sum_{\alpha\beta} N_{\alpha\beta}^{\gamma\gamma'} V_\alpha \otimes \bar{V}_\beta$$

$$Z_{\gamma\gamma'} = \sum_{\alpha,\beta} N_{\alpha\beta}^{\gamma\gamma'} \chi_\alpha \bar{\chi}_\beta$$

modular covariance

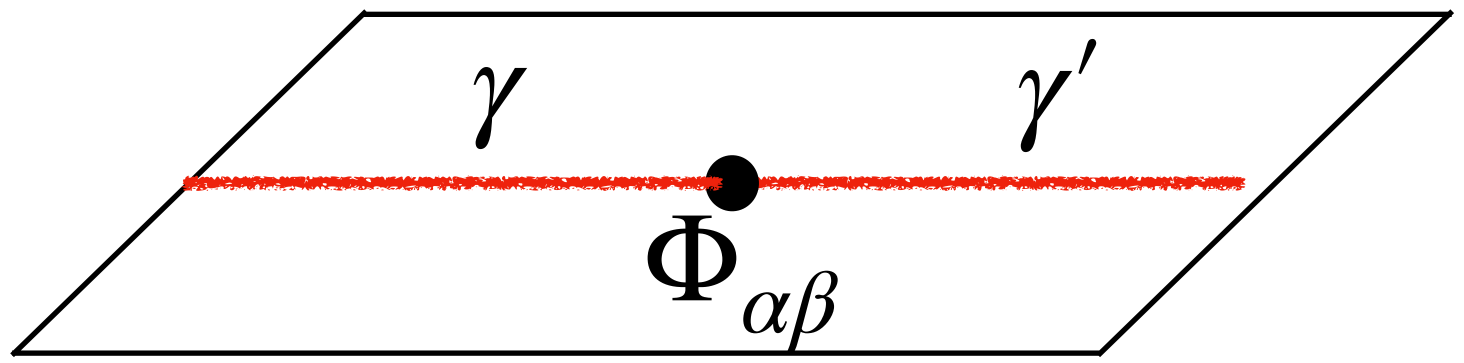


$$Z_\gamma = \mathrm{Tr}(e^{-HL} D_\gamma D_{\gamma'})$$

$$[D_\gamma, L_n] = 0 = [D_\gamma, \bar{L}_n]$$

$$Z_{\gamma\gamma'} = \sum_\alpha \frac{S_{\gamma'\alpha}}{S_{0\alpha}} \frac{S_{\gamma\alpha}}{S_{0\alpha}} \chi_\alpha \bar{\chi}_\alpha$$

exp map



state-operator map

operators living on the defect $\quad \gamma = \gamma'$

$\Phi_{\alpha\beta}$

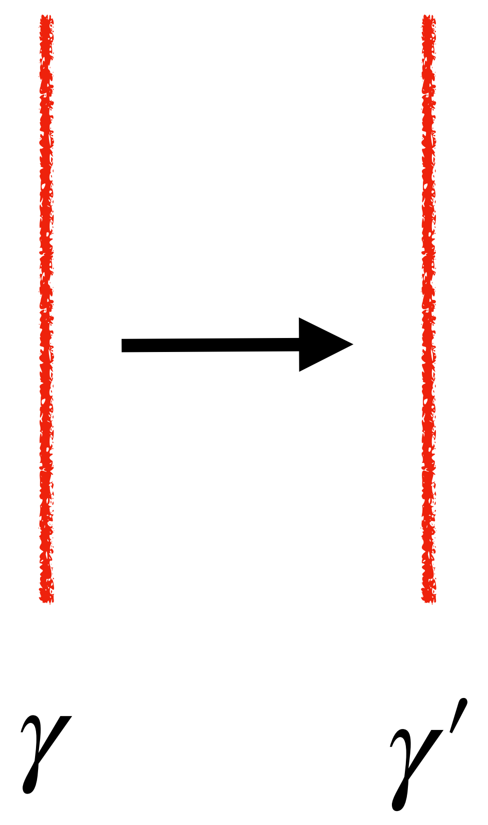
defect fusion

$$D_\gamma D_{\gamma'} = \sum_\delta N_{\gamma\gamma'}^\delta D_\delta$$

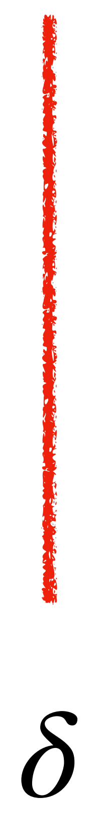
$$N_{\alpha\beta}^{\gamma\gamma'} = \sum_\delta N_{\alpha\beta}^\delta N_{\gamma\gamma'}^\delta$$

Defects as non-local symmetries

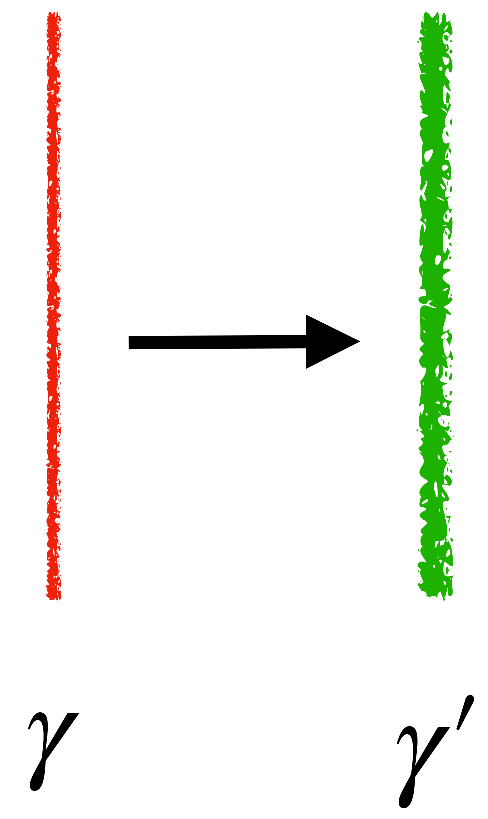
defects can fuse



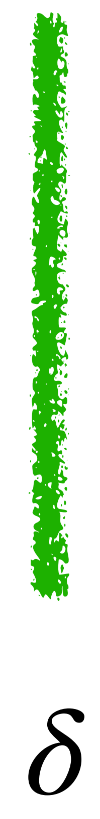
$$D_\gamma D_{\gamma'} = \sum_\delta N^\delta_{\gamma\gamma'} D_\delta$$



defects can act on boundaries



$$D_\gamma |\gamma'\rangle = \sum_\delta N^\delta_{\gamma\gamma'} |\delta\rangle$$



Lee-Yang model

simplest CFT   non-unitary minimal model $\mathscr{M}_{2,5}$ $c = -\frac{22}{5}$

$h_{11} = h_0 = 0$ $h_{12} = h_1 = -\frac{1}{5}$

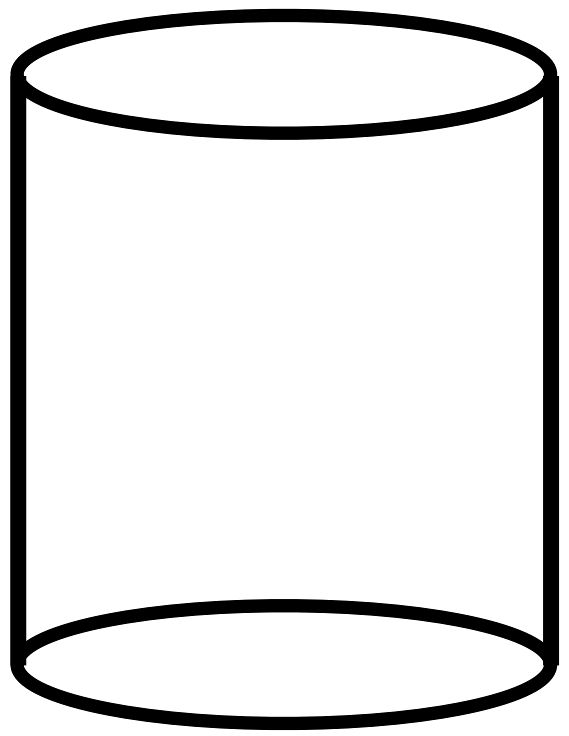
fusion rules $0 \times 0 = 0$ $1 \times 1 = 0 + 1$

$0 \times 1 = 1 \times 0 = 1$

modular S-matrix $S = -\frac{1}{\sqrt{d+2}} \begin{pmatrix} 1 & d \\ d & -1 \end{pmatrix}$ $d = \frac{1-\sqrt{5}}{2}$

identity defect
no defect

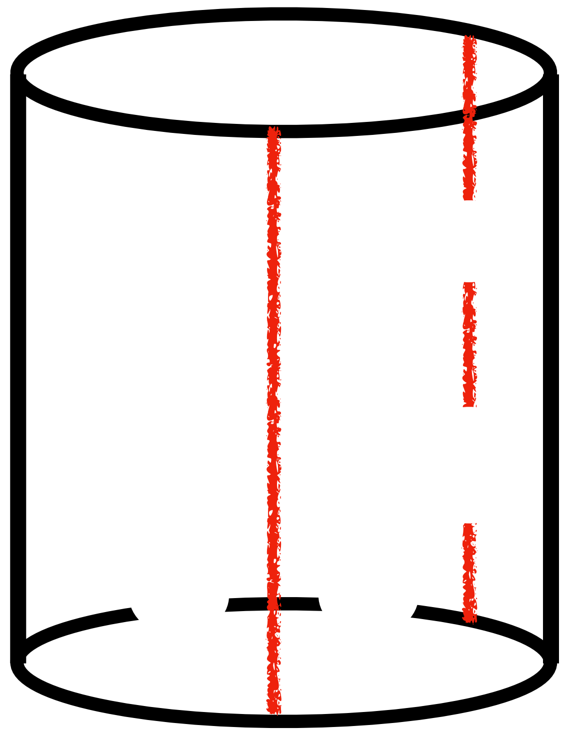
non-trivial defect
ϕ defect



Hilbert space

$$\mathscr{H} = V_0 \otimes \bar{V}_0 + V_1 \otimes \bar{V}_1$$

$[\mathbb{1}]$ $[\Phi]$



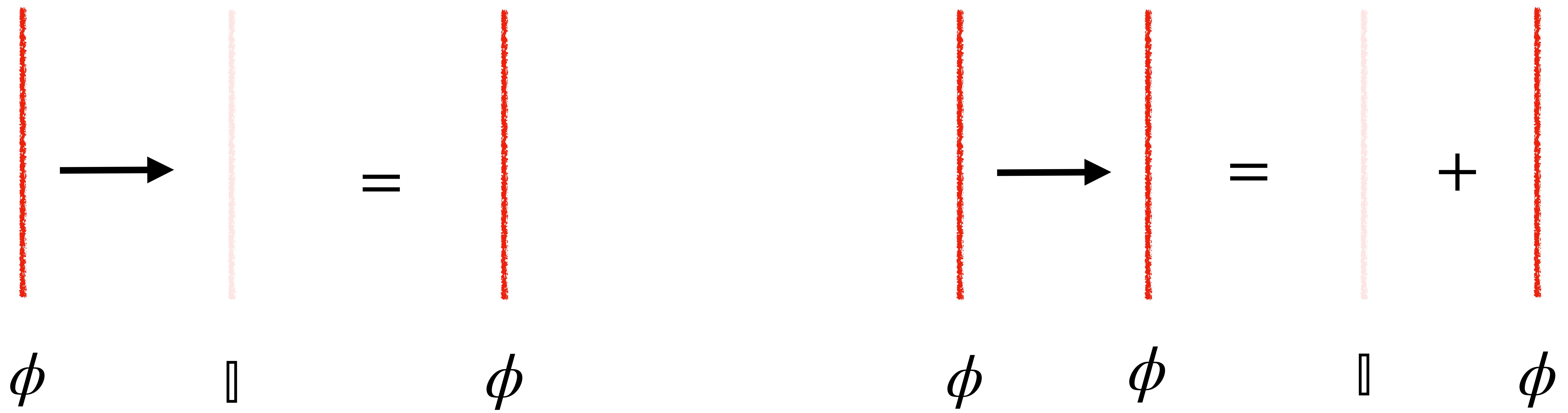
Hilbert space

$$\mathscr{H} = V_0 \otimes \bar{V}_1 + V_1 \otimes \bar{V}_0 + 2 V_1 \otimes \bar{V}_1$$

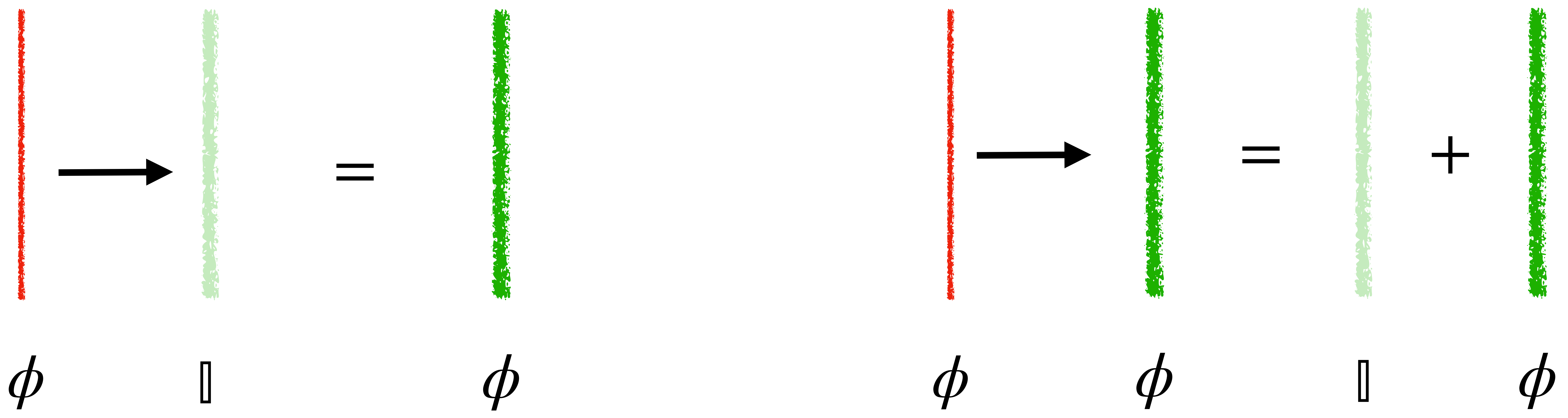
$[\bar{\varphi}]$ $[\varphi]$ $[\Phi_\pm]$

Defects as non-local symmetries

defects can fuse



defects can act on boundaries

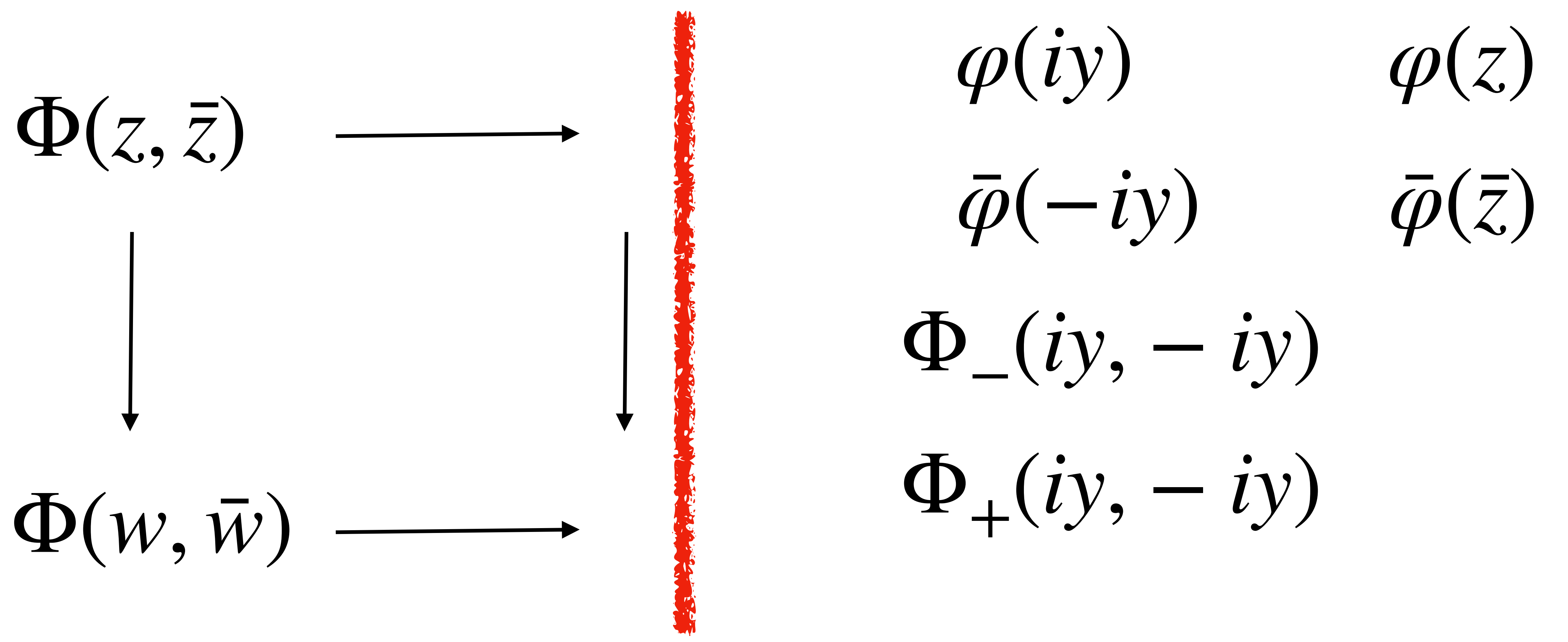


Lee-Yang model structure constants

fusion rules $0\times 0=0$ $0\times 1=1\times 0=1$ $1\times 1=0+1$

bulk OPE

$$\Phi(z,\bar z)\Phi(w,\bar w)=C^{\mathbb{I}}_{\Phi\Phi}|z-w|^{4h}+\dots$$
$$+C^{\Phi}_{\Phi\Phi}\Phi(w,\bar w)|z-w|^{2h}+\dots$$



bulk defect OPE

$$\lim_{x\pm 0}\Phi(z,\bar z)=\Phi_\pm(iy,-iy)+\dots$$

defect defect OPE

$$\varphi(z)\varphi(w)=C^{\mathbb{I}}_{\varphi\varphi}|z-w|^{2h}+C^{\varphi}_{\varphi\varphi}\varphi(w)|z-w|^{h}+\dots$$

$$\varphi(z)\bar\varphi(\bar z)=C^{\Phi_+}_{\varphi\bar\varphi}\Phi_+(z,\bar z)+C^{\Phi_-}_{\varphi\bar\varphi}\Phi_-(z,\bar z)+\dots$$

singular vectors + associativity $(L_{-1}^2-\frac{2}{5}L_{-2})|h\rangle=0$ complete solution [Bajnok-Hollo-Watts]

Integrable perturbations

[A.B. Zamolodchikov]

relevant perturbation $S = S_{CFT} + \lambda \int \Phi(z, \bar{z}) d^2 z$ $[\lambda] = (1-h, 1-h)$

conservation laws are deformed $\bar{\partial}(:\partial^{n_1}T \ldots \partial^{n_k}T:) = \bar{\partial}T_s = 0 + \lambda \mathcal{O}_1 + \lambda^2 \mathcal{O}_2 + \ldots$

compare dimensions $(s,1)$ $[\mathcal{O}_1] = (h+s-1, h)$

~~$[\mathcal{O}_2] = (2h+s-2, 2h-1)$~~

perturbation theory is first order exact

if $\mathcal{O}_1 = \partial \Theta_{s-1}$ then offcritical conservation law $\bar{\partial}T_s - \partial \Theta_{s-1} = 0$

Counting argument $[\mathbb{I}]/[\partial \mathbb{I}]$ $\xrightarrow{\bar{\partial}}$ $[\Phi]/[\partial\Phi]$

level s T_s $\longrightarrow$ Θ_{s-1} level s-1

Energy $\bar{\partial}T = \pi(1-h)\partial\Phi$ $\partial\bar{T} = \pi(1-h)\bar{\partial}\Phi$ $H = T + \bar{T} + 2\pi(1-h)\Phi$ $P = T - \bar{T}$

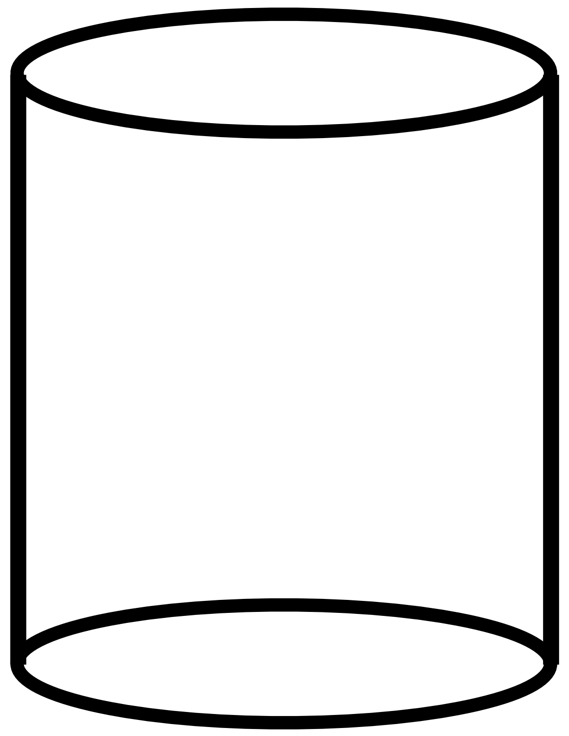
singular vectors are needed: $\Phi_{2,1}$ $\Phi_{1,3}$ $\Phi_{1,5}$ are always integrable

Truncated Conformal Space Approach

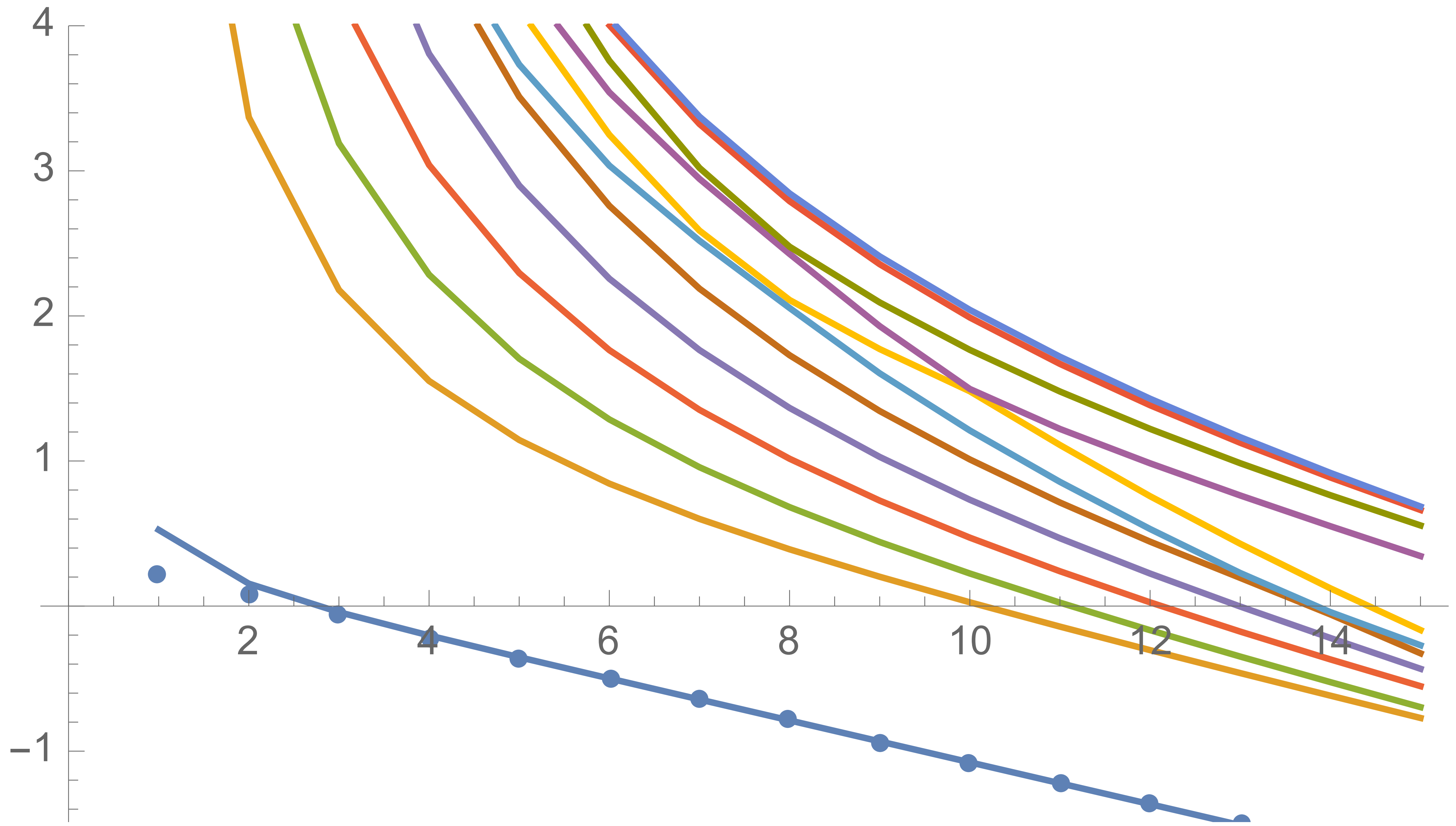
[Yurov-Zamolodchikov]

relevant perturbation on the cylinder

$$H = \frac{2\pi}{L}\left[L_0 + \bar{L}_0 - \frac{c}{12} + \lambda\left(\frac{2\pi}{L}\right)^{2-2h}\int_0^{2\pi}\Phi(e^{i\theta}, e^{-i\theta})d\theta\right]$$



Calculate the matrix elements of H and diagonalise on a truncated space



bulk energy $\quad E_0(L) = \epsilon_b L + O(e^{-mL})$

gap $\quad E_1(L) = \epsilon_b L + m + O(e^{-mL})$

free one particle states

$$E_2(L) = \epsilon_b L + \sqrt{m^2 + (2\pi/L)^2} + O(e^{-mL})$$

interacting two particle states $\quad E_*(L) = \epsilon_b L + \sqrt{m^2 + (2\pi n_1/L)^2} + \sqrt{m^2 + (2\pi n_2/L)^2} + O(1/mL)$

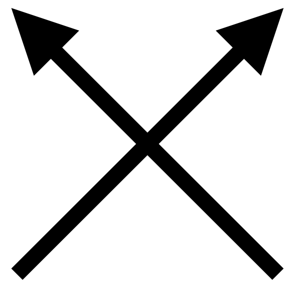
Scaling Lee-Yang model

one particle type of mass m $E = m\cosh\theta \qquad P = m\sinh\theta$

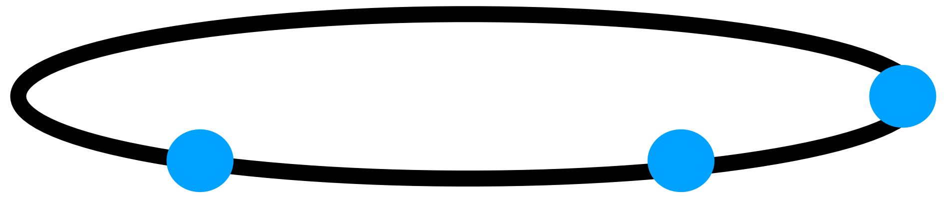
[Cardy-Mussardo]

factorised scattering

$$S(\theta_1 - \theta_2) = S(\theta) = \frac{\sinh\theta + i\sin\frac{\pi}{3}}{\sinh\theta - i\sin\frac{\pi}{3}}$$



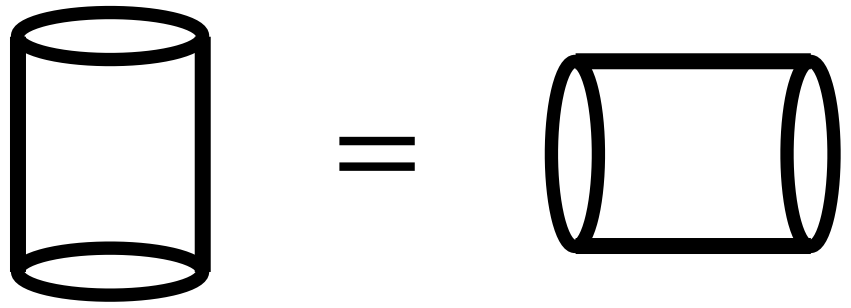
$\theta_1 \quad \theta_2$



L

large volume spectrum on the circle $E_n = \sum_i m\cosh\theta_i \qquad e^{im\sinh\theta_j L}\prod_{k:k\neq j} S(\theta_j - \theta_k) = 1$

exact ground-state energy



$$E_0(L) = \epsilon_b L - m\int \cosh\theta \log(1 + e^{-\epsilon(\theta)}) \qquad \epsilon(\theta) = mL\cosh\theta - \int \phi(\theta - v)\log(1 + e^{-\epsilon(v)})dv$$

$$\epsilon_b = -\frac{1}{4\sqrt{3}}m^2$$

[Al.B. Zamolodchikov]

$$\phi(\theta) = -i\partial\log S(\theta)$$

Integrable defect perturbations

relevant defect perturbation $S = S_{CFT} + \mu \int \varphi(z)dz$ $[\mu] = (1-h,0)$

defect conditions are deformed $T_-(iy) - T_+(iy) = \mu \mathcal{O}_1 + \mu^2 \mathcal{O}_2 + \dots$ $T_-(z)$ $T_+(z)$

compare dimensions $(2,0)$ $[\mathcal{O}_1] = (h+1,0)$

$[\mathcal{O}_2] = (2h,0)$

...

perturbation theory is first order exact

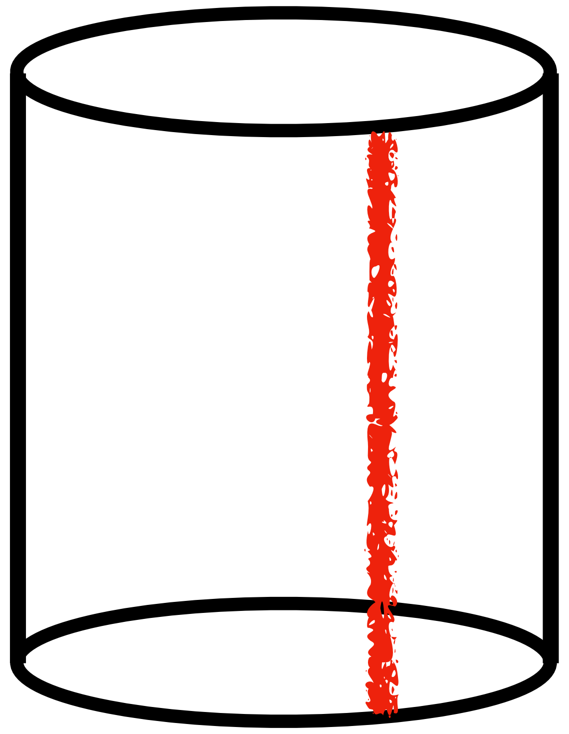
The only operator $\mathcal{O}_1 = 2\pi\mu(1-h)\partial\varphi$

Conserved energy $H = H_- + H_+ + 2\pi(1-h)\mu\partial\varphi$ defect remains topological

Conserved momentum $P = P_- + P_+ + 2\pi(1-h)\mu\partial\varphi$ but the theory is not a CFT

[Bajnok-Hollo-Watts]

Truncated defect Conformal Space Approach I

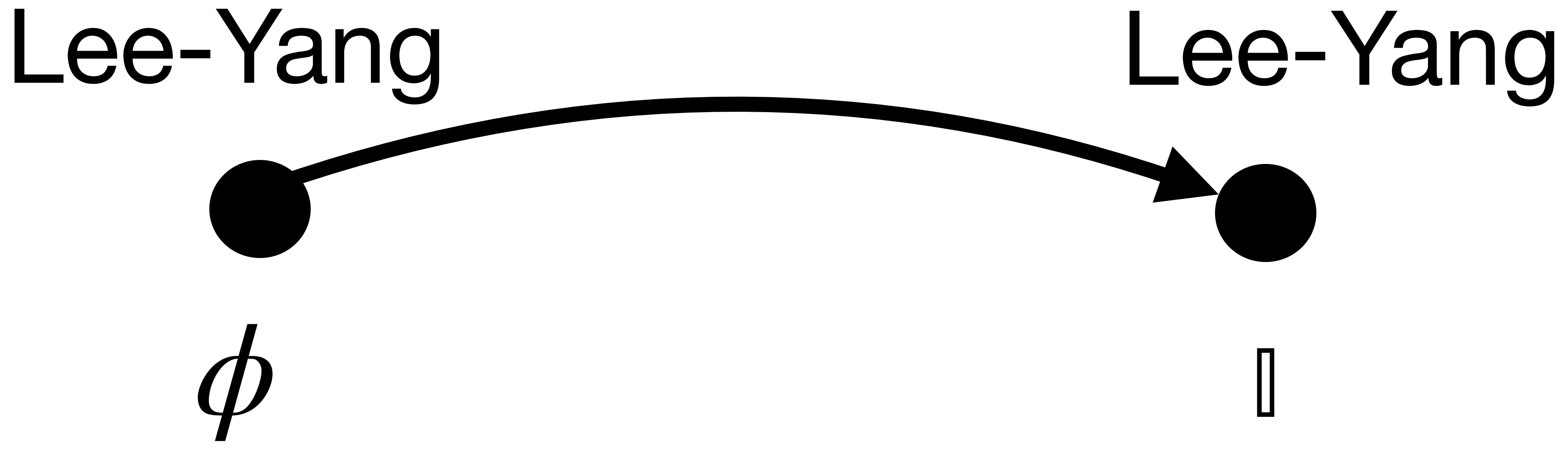


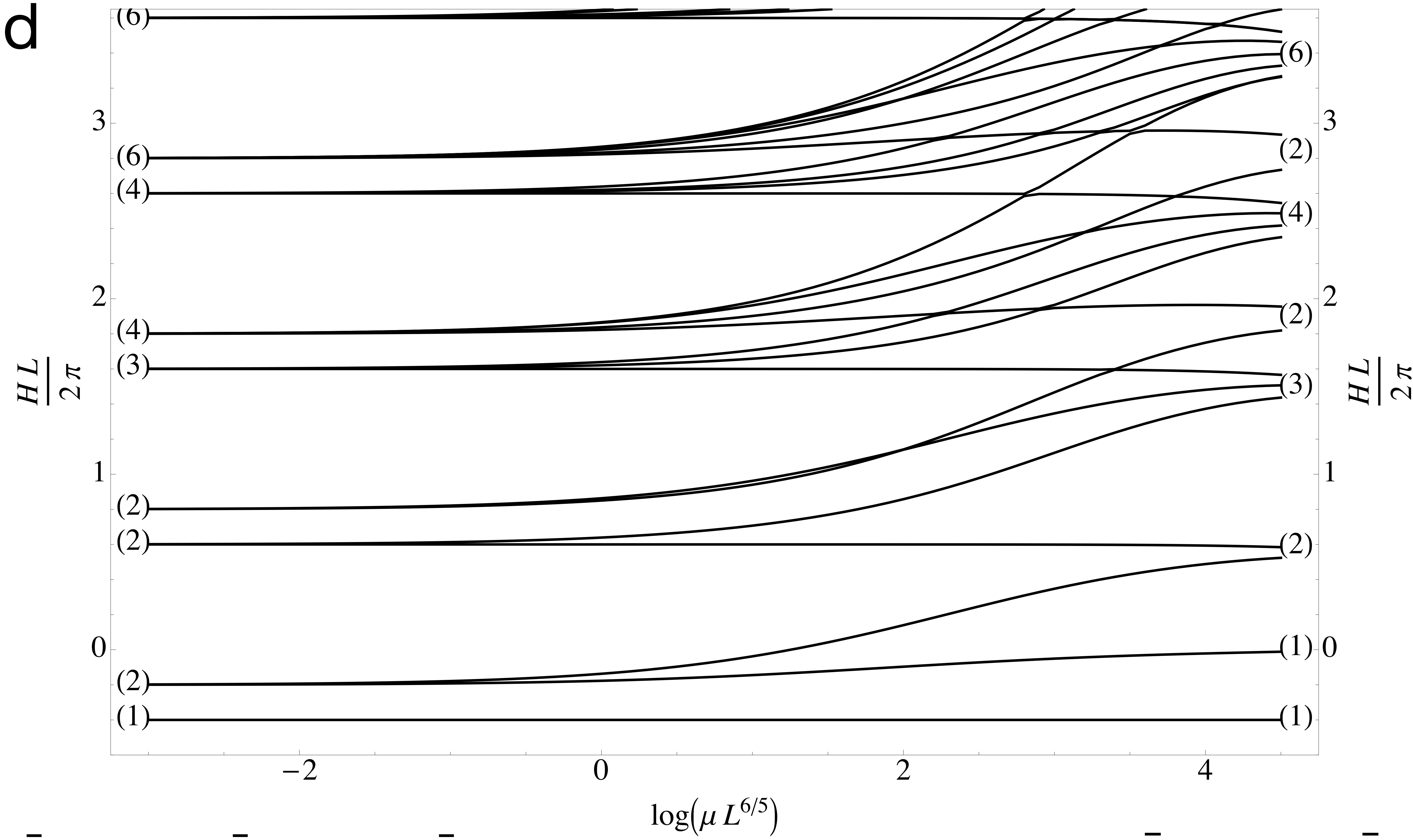
relevant perturbation only on the defect

$$H = \frac{2\pi}{L}\left[L_0 + \bar{L}_0 - \frac{c}{12} - \mu\left(\frac{L}{2\pi}\right)^{1-h}\varphi(1)\right]$$

Calculate the matrix elements of H and diagonalise on a truncated space

defect flow: from Lee-Yang with ϕ to Lee-Yang with $\mathbb{1}$





$V_0 \otimes \bar{V}_1 + V_1 \otimes \bar{V}_0 + V_1 \otimes \bar{V}_1$ $\qquad$ $V_0 \otimes \bar{V}_0 + V_1 \otimes \bar{V}_1$

Integrable bulk and defect perturbations

bulk and defect perturbation

$$S = S_{CFT} + \lambda \int \Phi(z,\bar{z}) d^2 z + \mu \int \varphi(z) dz + \bar{\mu} \int \bar{\varphi}(\bar{z}) d\bar{z}$$

$$\mathscr{E} = T + \bar{T} + 2\pi(1-h)\Phi \qquad \mathscr{P} = T - \bar{T}$$

$$T_-(z) \qquad T_+(z)$$

defect conditions are deformed

$$T_-(iy) - T_+(iy) = \mu\pi(1-h)\partial\varphi + \pi(2\mu\bar{\mu}C^+_{\varphi\bar{\varphi}} - \lambda h)(\Phi_- - \Phi_+) + \dots$$

$$\bar{T}_-(iy) - \bar{T}_+(iy) = -\bar{\mu}\pi(1-h)\bar{\partial}\bar{\varphi} + \pi(2\mu\bar{\mu}C^+_{\varphi\bar{\varphi}} - \lambda h)(\Phi_- - \Phi_+) + \dots$$

Conserved energy

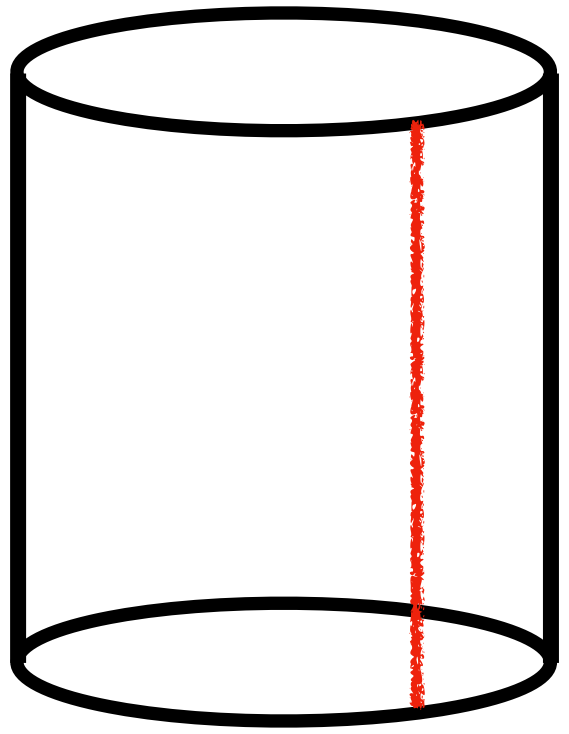
$$H = H_- + H_+ + 2\pi(1-h)(\mu\partial\varphi + \bar{\mu}\bar{\partial}\bar{\varphi})$$

Conserved momentum requires $\boxed{(2\mu\bar{\mu}C^+_{\varphi\bar{\varphi}} - \lambda h) = 0}$ defect remains topological

$$P = P_- + P_+ + 2\pi(1-h)(\mu\partial\varphi - \bar{\mu}\bar{\partial}\bar{\varphi})$$

[Bajnok-Hollo-Watts]

Truncated defect Conformal Space Approach



relevant bulk and defect perturbation

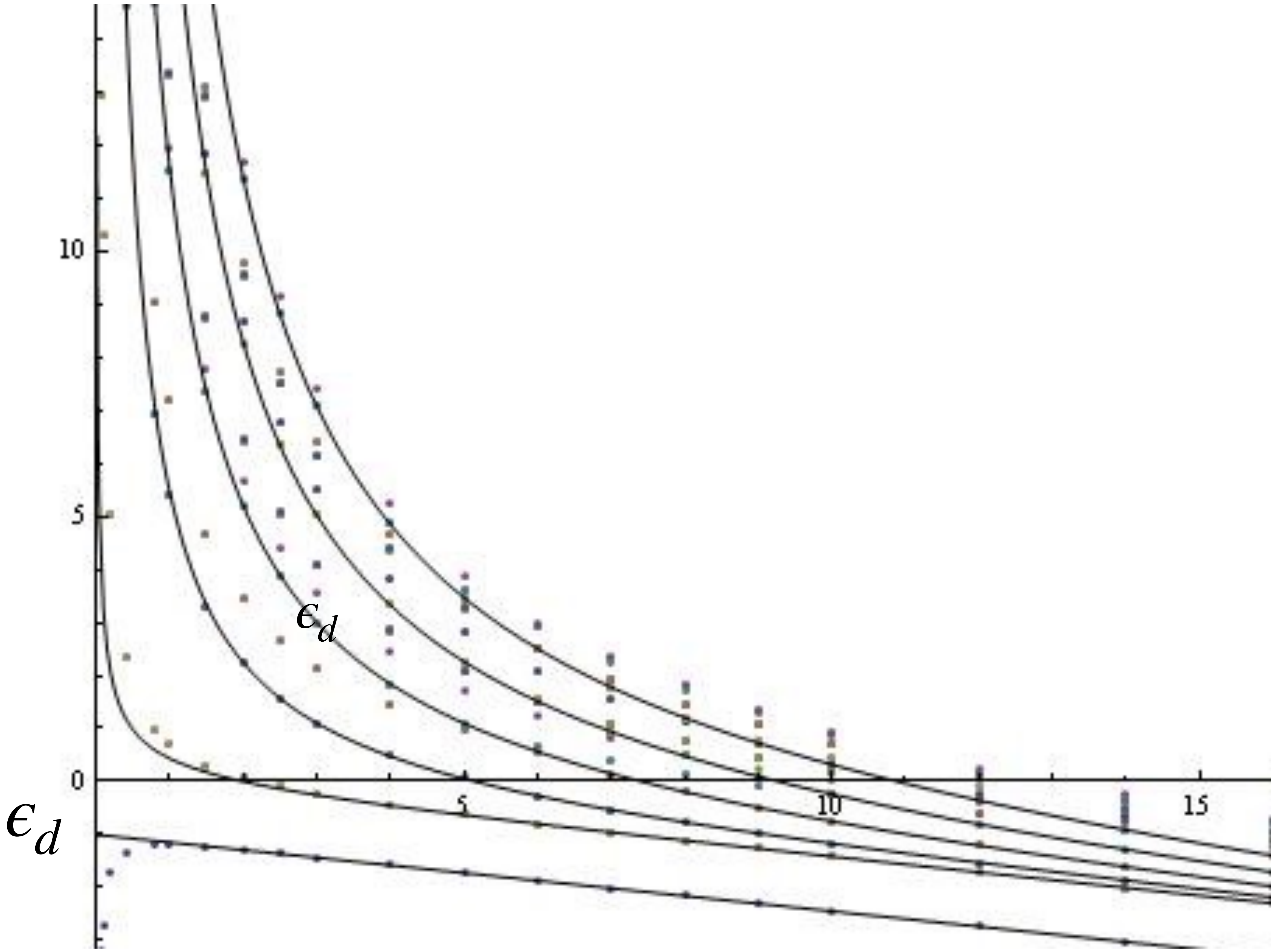
$$H = \frac{2\pi}{L}\left[L_0 + \bar{L}_0 - \frac{c}{12} + \lambda\left(\frac{2\pi}{L}\right)^{2-2h}\int_0^{2\pi}\Phi(e^{i\theta}, e^{-i\theta})d\theta - \mu\left(\frac{L}{2\pi}\right)^{1-h}\varphi(1) - \bar{\mu}\left(\frac{L}{2\pi}\right)^{1-h}\bar{\varphi}(1)\right]$$

Calculate the matrix elements of H and diagonalise on a truncated space

bulk energy $E_0(L) = \epsilon_b L + \epsilon_d + O(e^{-mL})$

gap $E_1(L) = \epsilon_b L + \epsilon_d + m + O(1/mL)$

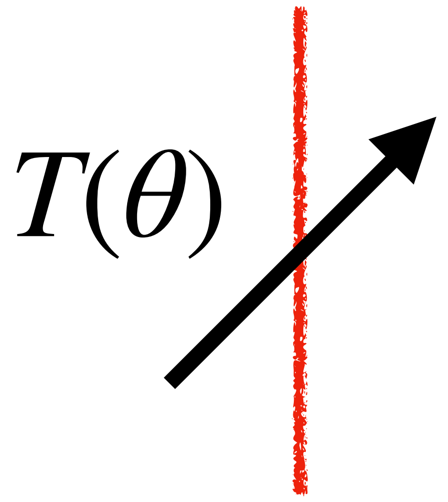
interacting one particle states

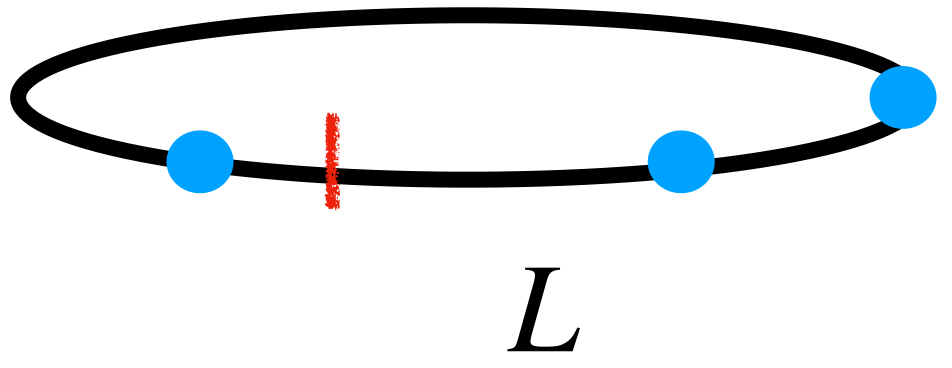


Defect scaling Lee-Yang model

factorised scattering, purely transmitting defects

$$S(\theta) = \frac{\sinh\theta + i\sin\frac{\pi}{3}}{\sinh\theta - i\sin\frac{\pi}{3}} \qquad T(\theta) = S(\theta - i\pi(1/2 - b/6))$$

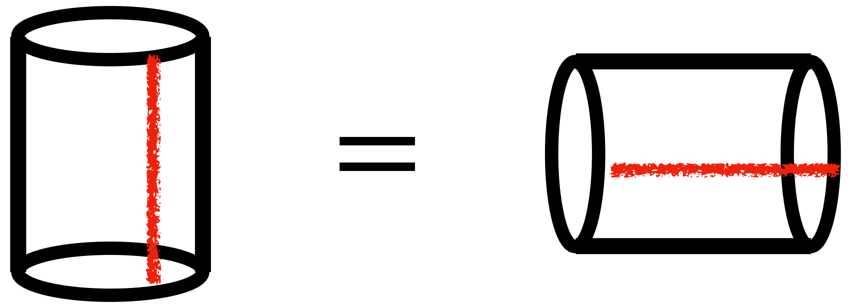




large volume spectrum

$$E_n = \sum_i m\cosh\theta_i \qquad e^{im\sinh\theta_j L}T(\theta_j)\prod_{k:k\neq j} S(\theta_j - \theta_k) = 1$$

exact ground-state energy



$$\epsilon(\theta) = mL\cosh\theta + \log T(\theta - i\pi/2) - \int \phi(\theta - v)\log(1 + e^{-\epsilon(v)})dv$$

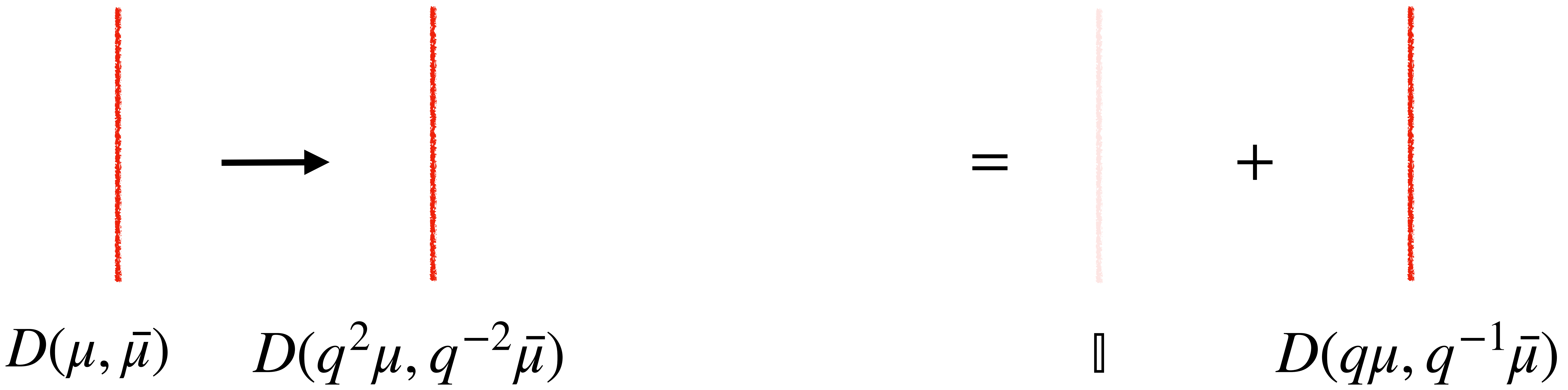
$$E_0(L) = \epsilon_b L - m\int \cosh\theta \log(1 + e^{-\epsilon(\theta)}) \qquad \epsilon_d = m\sin(\pi b/6) \qquad \epsilon_b = -\frac{1}{4\sqrt{3}}m^2$$

$$\mu = |h_c|/2m^{6/5}e^{i(b+3)\pi/5}$$

Offcritical defects as non-local symmetries

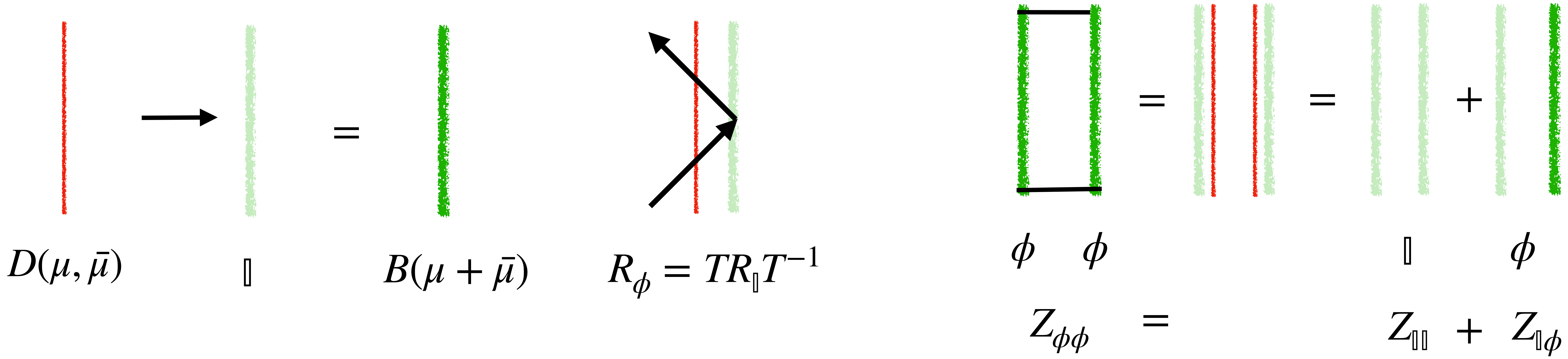
defects can fuse

[Runkel]



defects can act on boundaries

[Bajnok]



Conclusions, outlook

- Topological defects can survive offcritical deformations if both the bulk and the defect with left/right holomorphic fields are perturbed, but there is a constraint between the parameters

- In the integrable theory defects are represented by transmission factors

- defects generate non-invertible symmetries, which can act on boundaries and imply identities between reflection and transmission factors and partition functions

- we presented the results for the Lee-Yang model

- [Runkel] analyzed (1,s) type defects for the $\Phi_{1,3}$ perturbations of minimal models. They form T-system fusion hierarchies.

- [in many papers including Nakayama, Komatsu et al, …] they analysed (r,1) type defects, which commute with $\Phi_{1,3}$ and implement non-invertible symmetries in the offcritical theory. They restrict RG flows