A Calogero model for the non-Abelian quantum Hall effect

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APCTP Focus Program "Integrability, Duality, and Related Topics"

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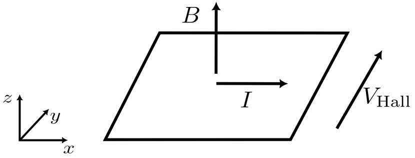
[Based on [JEB-Matsuo 2401.03087]]

Hall effect

- The **classical Hall effect** has been discovered by Edwin Hall in 1879. It is the observation of a transverse voltage for a conductor in a magnetic field. It is due to the Lorentz force acting on the electrons.



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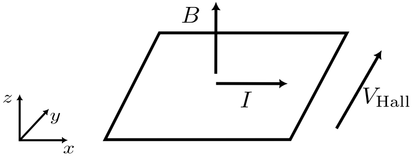
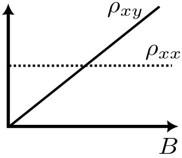


- As a result, the resistivity is expected to behave as follows

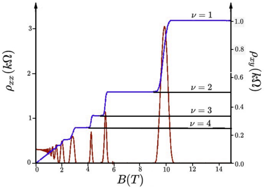


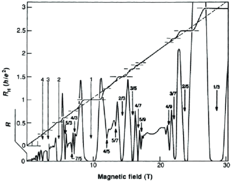
Quantum Hall effect

• The **quantum Hall effect** (QHE) refers to the observation of plateaux in the resistivity at low temperatures and strong magnetic field. The measured resistance is given by the formula

$$R_{xy} = \frac{V_{\text{Hall}}}{I} = \frac{h}{e^2 \nu}.$$

⇝ Plateaux are labeled by the **filling factor** ν, it takes integer or fractional values.





⇝ Integer QHE observed by Klaus von Klitzing in 1980 in MOSFET.

⇝ Fractional QHE discovered by Daniel Tsui and Horst Störmer in 1982 in GaAs samples.

Integer quantum Hall effect

- The **integer** QHE has a simple quantum mechanical explanation. It is due to the **impurities** in the material that lift the macroscopic degeneracy of the Landau levels. Impurities also turn extended states into localized states. Since these states cannot carry a current, they do not contribute to the conductivity.

$\Rightarrow$ Observe plateaux of length proportional to the amount of impurities!

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• The filling factor ν takes a specific set of rational values. The first explanation of the FQHE was proposed by Laughlin in 1983. He wrote the expression of an effective ground state wave function for the values filling factors $\nu = 1/k$ with k odd,

$$\psi(\boldsymbol{x}) = \prod_{a<b} (x_a - x_b)^k e^{-\frac{eB}{4\hbar}\sum_a |x_a|^2}.$$

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- The model exhibits excitations called **quasi-holes** with the wave function

$$\psi(\zeta_i, \boldsymbol{x}) = \prod_{i<j} (\zeta_i - \zeta_j)^{1/k} \times \prod_i \prod_a (\zeta_i - x_a) \times \prod_{a<b} (x_a - x_b)^k e^{-\frac{eB}{4\hbar} \sum_a |x_a|^2},$$

They carry a fraction of the charge of the electrons. They also obeys *a fractional statistics*, namely they are **(Abelian) anyons**!

- Model of the fractional QHE with more general filling factors $\nu = p/(k+pn)$ involve **non-Abelian anyons**. The statistics of these anyons (braiding) is given in terms of a non-trivial matrix transformation.

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• In fact, anyons are **topologically protected quantum states**. They are very good candidates to realize fault-tolerant quantum processors!

• So far, there have been no observation of non-Abelian anyons in condensed matter systems. But they have been realized recently on quantum processors.

[G. Q. AI and Collaborators, Nature 618 (2023) 264–269]

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  - Field theory (IR) description using 3d Chern-Simons theory

    ⇝ Abelian FQHE: $U(1)$ Chern-Simons level k ($\nu = 1/k$)

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    $\rightsquigarrow$ Non-Abelian FQHE: $U(p)$ Chern-Simons level k ($\nu = p/(k+p)$).

$\Rightarrow$ So, what's new?

- In **[2401.03087]**, we introduce **a new quantum model** for the non-Abelian FQHE. It is obtained as the diagonalization of a matrix model proposed by **[Dorey, Tong, Turner 2016]**. This model describes the vortices of 3d $U(p)$ Chern-Simons theory.

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$\rightsquigarrow$ **It relates the Chern-Simons and CFT approaches!**

• The Hamiltonian of our quantum system is a spin version of the Calogero Hamiltonian, a well known **integrable system**. It is expected to be also integrable.

$\rightsquigarrow$ **Use the algebraic techniques of integrable systems to do exact calculations!**

Outline

2. Derivation of the model

A bit of history...

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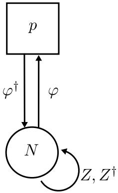
- Following this interpretation, Dorey Tong and Turner introduced in 2016 an extension of Polychronakos's matrix model with an additional $U(p)$ symmetry to render the dynamics of vortices in the non-Abelian Chern-Simons theory.

$\Rightarrow$ In **[JEB-Matsuo 2401.03087]**, we diagonalized the matrix model and obtained a spin version of the Calogero Hamiltonian involving order k symmetric representations of $U(p)$.

DTT matrix model

- The matrix model introduced by **[Dorey, Tong, Turner 2016]** describes the moduli space of vortices. It involves a $N \times N$ complex matrix $Z(t)$ and p N-dimensional vectors $\varphi_i(t)$ **[Hanany-Tong 1996]**



$$\mathcal{S} = \int dt \left[\frac{1}{2} iB\,\mathrm{tr}(Z^\dagger \mathcal{D}_t Z) + i \sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\,\mathrm{tr}\,\alpha - \omega\,\mathrm{tr}(Z^\dagger Z) \right]$$

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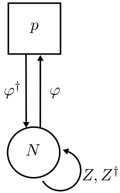

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- The model also contains a non-dynamical gauge field α entering in the covariant derivatives $\mathcal{D}_t Z = \partial_t Z - i[\alpha, Z]$, $\mathcal{D}_t \varphi_i = \partial_t \varphi_i - i\alpha\varphi_i$. This gauge field imposes the (classical) constraint

$$\frac{B}{2}[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = (k+p)\mathbb{1}_N.$$

• In **[Dorey, Tong, Turner 2016]**, the complex matrix Z is diagonalized. Instead, we followed Polychronakos's original approach and decomposed the matrix Z in terms of Hermitian matrices,

$$Z = X_1 + iX_2, \quad Z^\dagger = X_1 - iX_2,$$

As a result, the action takes the form

$$\mathcal{S} = \int dt \left(-\frac{B}{2} \operatorname{tr}\left(X_1 \dot{X}_2 - X_2 \dot{X}_1 \right) + i \sum_{i=1}^{p} \varphi_i^\dagger \dot{\varphi}_i - \omega \operatorname{tr}(X_1^2 + X_2^2) + \operatorname{tr}\left[\alpha G(X_1, X_2, \varphi_i, \varphi_i^\dagger) \right] \right),$$

$$\text{with} \quad G(X_1, X_2, \varphi_i, \varphi_i^\dagger) = -iB[X_1, X_2] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger - (k+p)\mathbb{1}_N,$$

Diagonalization

• The diagonalization is done following the usual method. Exploiting the $U(N)$ invariance, we decompose $X_1(t) = \Omega(t)^\dagger x(t)\Omega(t)$ with $(x)_{a,b} = x_a\delta_{a,b}$, $\Omega \in \mathrm{SU}(N)$. We derive the momenta and move to the Hamiltonian frame,

$$\mathcal{H} = \omega\,\mathrm{tr}(x^2 + (X_2)^2) = \omega \sum_{a=1}^{N} \left(x_a^2 + \frac{p_a^2}{B^2} \right) + 2\frac{\omega}{B^2} \sum_{a<b} \frac{(J_{a,b} - \Pi^+_{a,b})(J_{b,a} - \Pi^-_{a,b})}{(x_a - x_b)^2},$$

where $\Pi^\pm_{a,b}$ are momenta associated to angular degrees of freedom, and

$$J_{a,b} = \sum_{i=1}^{p} \varphi^\dagger_{i,a}\varphi_{i,b}, \quad p_a = \frac{\delta \mathcal{L}}{\delta \dot{x}_a} = BX_{2\ a,a},$$

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• Then, we impose the canonical quantization conditions. The gauge constraint imposed on physical states implies

$$\Pi^{\pm}_{a,b}\,|\mathrm{phys}\rangle = 0, \quad J_{a,a}\,|\mathrm{phys}\rangle = k\,|\mathrm{phys}\rangle\,.$$

$\rightsquigarrow$ **The parameter k identified to a mode number is quantized!**

- Due to the Vandermonde determinant in the measure coming from the diagonalization $dX_1 = d\Omega dx \Delta(\boldsymbol{x})$, the momentum p_a acts on wave functions as

$$p_a = -i\Delta(\boldsymbol{x})^{-1}\partial_a \Delta(\boldsymbol{x}), \quad \Delta(\boldsymbol{x}) = \prod_{\substack{a,b=1 \\ a<b}}^{N} (x_a - x_b).$$

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- Taking these facts into account, we find the Hamiltonian acting on physical states,

$$\mathcal{H} = \frac{\omega}{B^2}\left(\sum_{a=1}^{N}\left(-\Delta(\boldsymbol{x})^{-1}\partial_a^2\Delta(\boldsymbol{x}) + B^2x_a^2\right) + 2\sum_{a<b}\frac{J_{a,b}J_{b,a}}{(x_a - x_b)^2}\right),$$

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$\Rightarrow$ Let's examine this quantum system more closely!

3. Spin Calogero model

Definition of the model

- The model depends of the following parameters:
  - $N \in \mathbb{Z}^{\geq 0}$ the number of vortices in Chern-Simons theory
  - $p \in \mathbb{Z}^{>0}$ the rank of the non-Abelian $U(p)$ symmetry
  - $k \in \mathbb{Z}^{\geq 0}$ the level of the Chern-Simons theory (filling factor $\nu = p/(k+p)$).
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  - $B \in \mathbb{R}$ the magnetic field (strength of the confining potential)

- The model describes N particles of coordinates $x_a \in \mathbb{R}$ carrying a representation of the $U(p)$ symmetry. The spin structure is introduced using the bosonic oscillators $\varphi_{i,a}^\dagger$, $\varphi_{i,a}$,

$$[\varphi_{i,a}, \varphi_{j,b}^\dagger] = \delta_{i,j}\delta_{a,b}, \quad i,j = 1\cdots p, \quad a,b = 1\cdots N.$$

$\rightsquigarrow$ At level k, each particle carries a state built from k oscillators $\varphi_{i_1,a}^\dagger \cdots \varphi_{i_k,a}^\dagger |\emptyset\rangle$.

$\Rightarrow$ **The model involves higher order symmetric representations of the $U(p)$ symmetry!**

- The Hamiltonian of the model reads

$$\mathcal{H} = \sum_{a=1}^{N} \left(-\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1 \\ a<b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2},$$

where $J_{a,b}$ are the generators of $\mathfrak{gl}(N)$ acting on the spin components,

$$J_{a,b} = \sum_{i=1}^{p} \varphi_{i,a}^{\dagger} \varphi_{i,b}, \quad [J_{a,b}, J_{c,d}] = \delta_{a,d} J_{c,b} - \delta_{b,c} J_{a,d}.$$

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$\Rightarrow$ Let's examine some specific values of p and k!

Specializations

- For $p = 1$, we recover the Calogero Hamiltonian describing the Abelian model

$$\mathcal{H}^{(p=1)} = \sum_{a=1}^{N} \left(-\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1 \\ a<b}}^{N} \frac{k(k+1)}{(x_a - x_b)^2}.$$

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- When $k = 0$, particles carry no spin degree of freedom and the interaction term vanishes.

$\rightsquigarrow$ We are left with N decoupled Harmonic oscillators!

Simplification

- To simplify our analysis, we also perform the following technical manipulations:
  - Conjugation of the Hamiltonian,

$$\tilde{\mathcal{H}} = e^{\frac{B}{2}\sum_a x_a^2}\mathcal{H}e^{-\frac{B}{2}\sum_a x_a^2} = \sum_a \left(-\frac{\partial^2}{\partial x_a^2} + 2Bx_a\frac{\partial}{\partial x_a}\right) + NB + \sum_{a\neq b}\frac{J_{a,b}J_{b,a}}{(x_a - x_b)^2}.$$

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  - Rescaling of the coordinates $x_a \to B^{-1/2}x_a$ and energies $\tilde{\mathcal{H}} \to B^{-1}\tilde{\mathcal{H}}$ to set $B = 1$.

Simplification

- To simplify our analysis, we also perform the following technical manipulations:
  - Conjugation of the Hamiltonian,

$$\tilde{\mathcal{H}} = e^{\frac{B}{2}\sum_a x_a^2}\mathcal{H}e^{-\frac{B}{2}\sum_a x_a^2} = \sum_a\left(-\frac{\partial^2}{\partial x_a^2} + 2Bx_a\frac{\partial}{\partial x_a}\right) + NB + \sum_{a\neq b}\frac{J_{a,b}J_{b,a}}{(x_a - x_b)^2}.$$

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  - Rescaling of the coordinates $x_a \to B^{-1/2}x_a$ and energies $\tilde{\mathcal{H}} \to B^{-1}\tilde{\mathcal{H}}$ to set $B = 1$.

$\Rightarrow$ Next step: diagonalization of the model!

4. Spectrum and eigenfunctions

Wedge states

- We use a polynomial representation for the spin states,

$$\varphi^\dagger_{i_1,a}\cdots\varphi^\dagger_{i_k,a}\,|\emptyset\rangle \to y_{i_1,a}\cdots y_{i_k,a}, \quad \varphi_{i,a} \to \partial_{i,a} = \frac{\partial}{\partial y_{i,a}}.$$

$\rightsquigarrow$ States are expressed as polynomial wave functions $\Psi(\boldsymbol{x},\boldsymbol{y})$.

They are homogeneous of total degree k in $y_{i,a}$ for each particle a.

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- To express the wave functions, we first define the wedge states as $N \times N$ determinants,

$$\mathcal{Y}_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{y}) = [y_{i_1} x^{n_1} \wedge \cdots \wedge y_{i_N} x^{n_N}] = \begin{vmatrix} y_{i_1,1}(x_1)^{n_1} & \cdots & y_{i_N,1}(x_1)^{n_N} \\ \vdots & & \vdots \\ y_{i_1,N}(x_N)^{n_1} & \cdots & y_{i_N,N}(x_N)^{n_N} \end{vmatrix}.$$

They are labeled by N-tuple integers $\boldsymbol{r} = (r_1, \cdots, r_N)$, ordered as $0 \leq r_1 < r_2 < \cdots < r_N$, which decompose as $r_a = n_a p + i_a - 1$ with $i_a \in [\![1, p]\!]$ under Euclidean division.

⇝ It vanishes trivially when two columns coincide (i.e. $r_a = r_b$ for $a \neq b$).

- For $k \in \mathbb{Z}^{>0}$ fixed, we introduce the following set of wave functions

$$\psi_{\mathfrak{r}}(\boldsymbol{x}, \boldsymbol{y}) = \Delta(\boldsymbol{x}) \prod_{\alpha=1}^{k} \mathcal{Y}_{\boldsymbol{r}^{(\alpha)}}(\boldsymbol{x}, \boldsymbol{y}), \quad \Delta(\boldsymbol{x}) = \prod_{a<b}(x_a - x_b).$$

They are labeled by k tuples of p integers (ordered as before), $\mathfrak{r} = \left\{\boldsymbol{r}^{(1)}, \cdots, \boldsymbol{r}^{(k)}\right\}$. We also assume that re-ordering of $\boldsymbol{r}^{(\alpha)}$ in $\mathfrak{r}$ does not give a new element.

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- There is a natural grading obtained by acting with the operator $\mathcal{D} = \sum_{a=1}^{N} x_a \partial_a$,

$$\mathcal{D}\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) = |\mathfrak{r}|\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}), \quad |\mathfrak{r}| = \sum_{\alpha=1}^{k}\sum_{a=1}^{N} \lfloor r_a^{(\alpha)}/p \rfloor = \sum_{\alpha=1}^{k}\sum_{a=1}^{N} n_a^{(\alpha)}.$$

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⚠ **These wave functions are not linearly independent!**

This can be seen e.g. from the existence of Plücker relations between determinants

$$\sum_{a=0}^{N} (-1)^a [z^{r_1} \wedge \cdots \wedge z^{r_{N-1}} \wedge z^{s_a}][z^{s_0} \wedge \cdots \wedge \cancel{z^{s_a}} \wedge \cdots \wedge z^{s_N}] = 0,$$

for arbitrary pair $\boldsymbol{r} = (r_1, \cdots, r_{N-1})$ and $\boldsymbol{s} = (s_0, \cdots, s_N)$, and Uglov's notation $z_a^{r_b} = x_a^{n_b} y_{i_b,a}$.

⇒ We will come back to this problem in the next section.

Main result

- We proved that the Hamiltonian $\mathcal{H}$ has a triangular action on functions $\psi_{\mathfrak{r}}(\boldsymbol{x}, \boldsymbol{y})$,

$$\tilde{\mathcal{H}}\psi_{\mathfrak{r}}(\boldsymbol{x}, \boldsymbol{y}) = E(\mathfrak{r})\psi_{\mathfrak{r}}(\boldsymbol{x}, \boldsymbol{y}) + \sum_{|\mathfrak{s}|=|\mathfrak{r}|-2} C(\mathfrak{r}, \mathfrak{s})\psi_{\mathfrak{s}}(\boldsymbol{x}, \boldsymbol{y}),$$

where $C(\mathfrak{r}, \mathfrak{s})$ is a coefficient. This can be inverted to write down the eigenfunctions,

$$\Psi_{\mathfrak{r}}(\boldsymbol{x}, \boldsymbol{y}) = \psi_{\mathfrak{r}}(\boldsymbol{x}, \boldsymbol{y}) + \sum_{|\mathfrak{s}|\leq|\mathfrak{r}|-2} D(\mathfrak{r}, \mathfrak{s})\psi_{\mathfrak{s}}(\boldsymbol{x}, \boldsymbol{y}) .$$

$\rightsquigarrow$ We deduce the energy spectrum $E(\mathfrak{r}) = 2|\mathfrak{r}| + N$.

Remark: $\mathfrak{sl}_2$-triplet

- **[This remark is due to Yehao Zhou.]** Recall the expression of our Hamiltonian,

$$\mathcal{H} = \sum_{a=1}^{N} \left(-\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + \sum_{\substack{a,b=1 \\ a \neq b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2}, \quad \mathcal{D} = \sum_a x_a \partial_a$$

$$\tilde{\mathcal{H}} = e^{\frac{B}{2}\sum_a x_a^2} \mathcal{H} e^{-\frac{B}{2}\sum_a x_a^2} = \sum_a -\frac{\partial^2}{\partial x_a^2} + \sum_{a \neq b} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2} + NB + 2B\mathcal{D}.$$

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- If we define

$$e = \frac{1}{2} \sum_a x_a^2, \quad f = -\frac{1}{2} \sum_a \frac{\partial^2}{\partial x_a^2} + \frac{1}{2} \sum_{\substack{a,b=1 \\ a \neq b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2}, \quad h = \mathcal{D} + \frac{N}{2}$$

our main result implies that **they form an $\mathfrak{sl}_2$-triplet!**

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

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$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

- In particular, we have

$$\mathcal{H} = 2f + 2B^2 e, \quad \tilde{\mathcal{H}} = e^{Be}\mathcal{H}e^{-Be} = 2f + 2Bh.$$

• This remark has interesting implications for the wave functions. Recall that the product of wedge states diagonalize the grading operator

$$\mathcal{D}\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) = |\mathfrak{r}|\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}), \quad \text{i.e.} \quad h\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) = \left(\frac{N}{2} + |\mathfrak{r}|\right)\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y})$$

$\rightsquigarrow$ Our Hamiltonian can be written as

$$\tilde{\mathcal{H}} = 2Be^{\frac{f}{2B}}he^{-\frac{f}{2B}}$$

and so its eigenfunctions have the form

$$\Psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) = e^{\frac{f}{2B}}\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y})$$

where f is a nilpotent operator. This is indeed what we observed...

Ground states

• We deduce the ground states wave function from the triangular action. When $k = 1$, the ground state is unique if p divides N and corresponds to $\boldsymbol{r}_0 = (0, 1, 2, \cdots, N-1)$,

$$\mathcal{Y}_{\boldsymbol{r}_0}(\boldsymbol{x}, \boldsymbol{y}) = [y_1 \wedge \cdots \wedge y_p \wedge y_1 x \wedge \cdots \wedge y_p x \wedge \cdots \wedge y_1 x^{m-1} \wedge \cdots \wedge y_p x^{m-1}].$$

⇝ The ground state is also a singlet for $k > 1$ if $p|N$, with the wave function $\psi_{\mathfrak{r}_0} = \Delta(\mathcal{Y}_{\boldsymbol{r}_0})^k$.

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$\rightsquigarrow$ The ground state is also a singlet for $k > 1$ if $p|N$, with the wave function $\psi_{\mathfrak{r}_0} = \Delta(\mathcal{Y}_{\boldsymbol{r}_0})^k$.

Example: for $N = 4$, $p = 2$ and $k = 1$ we have

$$\mathcal{Y}_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{y}) = [y_1 \wedge y_2 \wedge y_1 x \wedge y_2 x] = \begin{vmatrix} y_{1,1} & y_{2,1} & y_{1,1}x_1 & y_{2,1}x_1 \\ y_{1,2} & y_{2,2} & y_{1,2}x_2 & y_{2,2}x_2 \\ y_{1,3} & y_{2,3} & y_{1,3}x_3 & y_{2,3}x_3 \\ y_{1,4} & y_{2,4} & y_{1,4}x_4 & y_{2,4}x_4 \end{vmatrix}.$$

- When $N = mp + q$ with $0 < q \leq p - 1$, the ground state for $k = 1$ is $\binom{p}{q}$-fold degenerate.

It corresponds to a choice of spin for q extra particles,

$$\mathcal{Y}_{\boldsymbol{r}_0}(\boldsymbol{x}, \boldsymbol{y}) = [y_1 \wedge \cdots \wedge y_p \wedge y_1 x \wedge \cdots \wedge y_p x \wedge \cdots \wedge y_1 x^{m-1} \wedge \cdots \wedge y_p x^{m-1} \wedge y_{i_1} x^m \wedge \cdots \wedge y_{i_q} x^m].$$

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- When $k > 1$, ground states wave functions are products of k determinants,

$$\psi_{\mathfrak{r}_0}(\boldsymbol{x}, \boldsymbol{y}) = \Delta(\boldsymbol{x}) \prod_{\alpha=1}^{k} \mathcal{Y}_{\boldsymbol{r}_0^{(\alpha)}}(\boldsymbol{x}, \boldsymbol{y}), \quad \mathfrak{r}_0 = (\boldsymbol{r}_0^{(1)}, \cdots, \boldsymbol{r}_0^{(k)}).$$

⇝ The corresponding energy is simply $E(\mathfrak{r}_0) = kE(\boldsymbol{r}_0)$.

Knizhnik-Zamolodchikov equation

- We show that the ground state wave functions obey the KZ equation

$$(p+k)\partial_a \Phi_{\mathfrak{r}_0}(\boldsymbol{x},\boldsymbol{y}) = \sum_{b\neq a} \frac{J_{a,b}J_{b,a}}{x_a - x_b} \Phi_{\mathfrak{r}_0}(\boldsymbol{x},\boldsymbol{y}).$$

with $\Phi_{\mathfrak{r}_0}(\boldsymbol{x},\boldsymbol{y}) = \Delta(\boldsymbol{x})^{-1}\psi_{\mathfrak{r}_0}(\boldsymbol{x},\boldsymbol{y})$.

$\rightsquigarrow$ Indicates a connection with $\mathfrak{sl}(p)$ Kac-Moody symmetry!

Generalized statistics

- In general, the wedge states $\mathcal{Y}_{\boldsymbol{r}_0}(\boldsymbol{x},\boldsymbol{y})$ do not vanish when two coordinates coincide $x_a \to x_b$ due to the presence of the spin variables $y_{i,a}$. But they do vanish when $p+1$ particles approach each other since at least two particles will have the same spin.

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⇝ **Wedge states describe particles obeying a generalized exclusion principle!**

• Using a similar argument, we can introduce wave functions for holes of coordinate ζ by combining p particles of different spins. We find the $(N+p) \times (N+p)$ determinant

$$\begin{vmatrix} y_{1,1} & \cdots & y_{p,1} & \cdots & y_{i_1,1}x_1^{m+1} & \cdots & y_{i_q,1}x_1^{m+1} \\ \vdots & & & \ddots & & & \vdots \\ y_{1,N} & \cdots & y_{p,N} & \cdots & y_{i_1,N}x_N^{m+1} & \cdots & y_{i_q,N}x_N^{m+1} \\ y_{1,N+1} & \cdots & y_{p,N+1} & \cdots & y_{i_1,N+1}\zeta^{m+1} & \cdots & y_{i_q,N+1}\zeta^{m+1} \\ \vdots & & & \ddots & & & \vdots \\ y_{1,N+p} & \cdots & y_{p,N+p} & \cdots & y_{i_1,N+p}\zeta^{m+1} & \cdots & y_{i_q,N+p}\zeta^{m} \end{vmatrix} \propto \prod_{a=1}^{N}(\zeta - x_a).$$

Example

- Many known FQHE wave functions can be recovered from $\psi_{\mathfrak{r}}(\boldsymbol{x}, \boldsymbol{y})$.

For instance, taking $k = 2$, $p = 2$, $N = 2m$, we find a singlet ground state $\mathfrak{r}_0 = (\boldsymbol{r}_0, \boldsymbol{r}_0)$ with $\boldsymbol{r}_0 = \{0, 1, \cdots, 2m - 1\}$.

⇝ The corresponding wave function reproduces the Moore-Read wave function,

$$\Phi_{\mathfrak{r}_0}(\boldsymbol{x}, \boldsymbol{y}) = (\mathcal{Y}_{\boldsymbol{r}_0}(\boldsymbol{x}, \boldsymbol{y}))^2 \propto \mathrm{Pf}\left(\frac{(y_{1a}y_{2b} - y_{2a}y_{1b})^2}{x_a - x_b}\right) \prod_{a<b}(x_a - x_b).$$

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- Two main remaining questions at this stage:
  - How to get rid of extra states due to Plücker-like relations between determinants?
  - Can we observe the emergence of Kac-Moody symmetry at large N?

⇒ We will use a free fermion formalism to address these questions.

5. Free fermion formalism

Fermions at finite N

- The symmetries of the wedge states for $k > 1$ can be understood by introducing a set of (non-relativistic) free fermion oscillators,

$$\left\{\psi_n^{i,\alpha}, \bar{\psi}_m^{j,\beta}\right\} = \delta_{i,j}\delta_{\alpha,\beta}\delta_{n,m}, \quad \left\{\psi_n^{i,\alpha}, \psi_m^{j,\beta}\right\} = \left\{\bar{\psi}_n^{i,\alpha}, \bar{\psi}_m^{j,\beta}\right\} = 0\,.$$

The fermionic modes carry three types of indices,

- a spin label $i, j \in [\![1, p]\!]$,
- a Chern-Simons level $\alpha, \beta \in [\![1, k]\!]$,
- a mode number $n \in \mathbb{Z}^{\geq 0}$.

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- a spin label $i, j \in [\![1, p]\!]$,
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- a mode number $n \in \mathbb{Z}^{\geq 0}$.

- The Fock space is built from the action of $\bar{\psi}_n^{i,\alpha}$ on vacuum $|0\rangle$ annihilated by modes $\psi_n^{i,\alpha}$. It is graded by $\hat{N}_\alpha = \sum_{n=0}^{\infty}\sum_{i=1}^{p} \bar{\psi}_n^{i,\alpha}\psi_n^{i,\alpha}$ which counts the modes $\bar{\psi}_n^{i,\alpha}$ for each α. It decomposes accordingly into

$$\mathcal{F}^{(k)}_{(N_1,\cdots,N_k)} = \left\{\prod_{\alpha=1}^{k}\left(\bar{\psi}_{n_1^{(\alpha)}}^{i_1^{(\alpha)},\alpha}\cdots\bar{\psi}_{n_{N_\alpha}^{(\alpha)}}^{i_{N_\alpha}^{(\alpha)},\alpha}\right)|0\rangle, \quad n_{a_\alpha}^{(\alpha)} \in \mathbb{Z}^{\geq 0},\ i_{a_\alpha}^{(\alpha)} \in [\![1, p]\!], \quad a_\alpha = 1\cdots N_\alpha\right\}$$

$\rightsquigarrow$ We also introduce the dual state such that $\langle 0|\bar{\psi}_n^{i,\alpha} = 0$.

- These fermionic modes can be used to represent loop algebras with positive modes:
  - $\widehat{\mathfrak{u}}(k)_+$ (or $\mathfrak{gl}(k)[z]$) generated by $L_n^{\alpha\beta}$, $\alpha, \beta \in [\![1, k]\!]$, $n \geq 0$,

$$L_n^{\alpha\beta} = \sum_{\ell=0}^{\infty} \sum_{i=1}^{p} \bar{\psi}_{n+\ell}^{i,\alpha} \psi_{\ell}^{i,\beta}, \quad \left[L_n^{\alpha\beta}, L_m^{\gamma\delta}\right] = \delta_{\beta,\gamma} L_{n+m}^{\alpha\delta} - \delta_{\alpha,\delta} L_{n+m}^{\gamma\beta}.$$

  - $\widehat{\mathfrak{u}}(p)_+$ generated by $K_n^{ij} = \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^{k} \bar{\psi}_{n+\ell}^{i,\alpha} \psi_n^{j,\alpha},$, $i, j \in [\![1, p]\!]$, $n \geq 0$,
  - $\widehat{\mathfrak{u}}(pk)_+$ generated by $M_n^{i+p(\alpha-1), j+p(\beta-1)} = \sum_{\ell=0}^{\infty} \bar{\psi}_{n+\ell}^{i,\alpha} \psi_{\ell}^{j,\beta}$,
  - $\widehat{\mathfrak{u}}(1)_+$ generated by $J_n = \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^{k} \sum_{i=1}^{p} \bar{\psi}_{n+\ell}^{i,\alpha} \psi_n^{i,\alpha} = \sum_{\alpha=1}^{k} L_n^{\alpha\alpha} = \sum_{i=1}^{p} K_n^{ii}$

$\rightsquigarrow$ We note that $\left[L_n^{\alpha\beta}, K_m^{ij}\right] = 0$.

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⇝ We note that $\left[L_n^{\alpha\beta}, K_m^{ij}\right] = 0$.

- We have the decomposition **[T. Nakanishi, A. Tsuchiya 1992]**

$$\widehat{\mathfrak{u}}(pk)_+ \supset \widehat{\mathfrak{su}}(p)_+ \oplus \widehat{\mathfrak{su}}(k)_+ \oplus \widehat{\mathfrak{u}}(1)_+$$

where $\widehat{\mathfrak{su}}(k)_+$ is defined using traceless generators $\bar{L}_n^{\alpha\beta} = L_n^{\alpha\beta} - \frac{\delta_{\alpha,\beta}}{k}\sum_{\gamma} L_n^{\gamma\gamma}$.

$\widehat{\mathfrak{su}}(k)_+$-invariance

- Wedge states can be realized as projection of states in the fermionic Fock space $\mathcal{F}^{(k)}_{(N,\cdots,N)}$,

$$\prod_{\alpha=1}^{k}[x^{n_1^{(\alpha)}} y_{i_1^{(\alpha)}} \wedge \cdots \wedge x^{n_N^{(\alpha)}} y_{i_N^{(\alpha)}}] = \langle \boldsymbol{x}, \boldsymbol{y}| \prod_{\alpha=1}^{k} \left(\bar{\psi}^{i_1^{(\alpha)},\alpha}_{n_1^{(\alpha)}} \cdots \bar{\psi}^{i_N^{(\alpha)},\alpha}_{n_N^{(\alpha)}} \right) |0\rangle$$

with the projector $\langle \boldsymbol{x}, \boldsymbol{y}|$ constructed as follows,

$$\langle \boldsymbol{x}, \boldsymbol{y}| = \langle 0|\Psi(x_N, y_N) \cdots \Psi(x_1, y_1)$$

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$\widehat{\mathfrak{su}}(k)_+$-invariance

- Wedge states can be realized as projection of states in the fermionic Fock space $\mathcal{F}^{(k)}_{(N,\cdots,N)}$,

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$\Rightarrow$ Resolve problem of state overcounting by taking the quotient of $\mathcal{F}^{(k)}_{\boldsymbol{N}}$ under $\widehat{\mathfrak{su}}(k)_+$-symmetry!

Example: For $k = 2$, we have three series of generators L_n^{12}, L_n^{21} and $L_n^{11} - L_n^{22}$ for $\widehat{\mathfrak{su}}(2)_+$.

Fermionic states have the form $\bar{\psi}_{n_1}^{i_1,1} \cdots \bar{\psi}_{n_{N_1}}^{i_{N_1},1} \bar{\psi}_{m_1}^{j_1,2} \cdots \bar{\psi}_{m_{N_2}}^{j_{N_2},2} |0\rangle$.

$\rightsquigarrow$ Consider the action of L_n^{21} on the following states

$$\begin{aligned} &L_n^{21} \bar{\psi}_{n_1}^{i_1,1} \cdots \bar{\psi}_{n_{N+1}}^{i_{N+1},1} \bar{\psi}_{m_1}^{j_1,2} \cdots \bar{\psi}_{m_{N-1}}^{j_{N-1},2} |0\rangle \\ &\qquad = (-1)^N \sum_{a=1}^{N+1} (-1)^{a-1} \bar{\psi}_{n_1}^{i_1,1} \cdots \cancel{\bar{\psi}_{n_a}^{i_a,1}} \cdots \bar{\psi}_{n_{N+1}}^{i_{N+1},1} \bar{\psi}_{n_a+n}^{i_a,2} \bar{\psi}_{m_1}^{j_1,2} \cdots \bar{\psi}_{m_{N-1}}^{j_{N-1},2} |0\rangle \end{aligned}$$

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⇝ After projection on $\langle \boldsymbol{x}, \boldsymbol{y}|$, we find a generalization of Plücker relations,

$$\sum_{a=1}^{N+1} (-1)^a [y_{i_1} x^{n_1} \wedge \cdots \wedge \cancel{y_{i_a} x^{n_a}} \wedge \cdots \wedge y_{i_{N+1}} x^{n_{N+1}}] \cdot [y_{i_a} x^{n_a+n} \wedge y_{j_1} x^{m_1} \wedge \cdots \wedge y_{j_{N-1}} x^{m_{N-1}}] = 0.$$

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Fermions in large N

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• We can use this fermionic description to see the emergence of the Kac-Moody symmetry in the large N limit (recall $N = \#$ number of vortices).

• The symmetries of the model in the large N limit are described using the fermionic modes

$$\left\{\psi_r^{i,\alpha}, \bar{\psi}_s^{j,\beta}\right\} = \delta_{i,j}\delta_{\alpha,\beta}\delta_{r+s,0}, \quad \left\{\psi_r^{i,\alpha}, \psi_s^{j,\beta}\right\} = \left\{\bar{\psi}_r^{i,\alpha}, \bar{\psi}_s^{j,\beta}\right\} = 0 .$$

$\rightsquigarrow$ Still carry spin i, j and CS α, β indices, but now have half-integer mode indices $r, s \in \mathbb{Z} + \frac{1}{2}$.

$\rightsquigarrow$ The Fock space is obtained by the action of negative modes $\psi_{-r}^{i,\alpha}$, $\bar{\psi}_{-r}^{i,\alpha}$ on the vacuum $|\emptyset\rangle$, with $\psi_r^{i,\alpha}|\emptyset\rangle = \bar{\psi}_r^{i,\alpha}|\emptyset\rangle = 0$, $(r > 0)$.

- The fermionic Fock space $\mathcal{F}_{\infty}^{(p,k)}$ admits the action of the following Kac-Moody algebra,
  - $\widehat{\mathfrak{u}}(pk)_1$ generated by $M_n^{i+(\alpha-1)p;j+(\beta-1)p} = \sum_{s\in\mathbb{Z}+1/2} :\bar{\psi}_s^{i\alpha}\psi_{n-s}^{j,\beta}:$,
  - $\widehat{\mathfrak{u}}(p)_k$ generated by $K_n^{ij} = \sum_{s\in\mathbb{Z}+1/2}\sum_{\alpha=1}^{k} :\bar{\psi}_s^{i\alpha}\psi_{n-s}^{j,\alpha}:$
  - $\widehat{\mathfrak{u}}(k)_p$ generated by $L_n^{\alpha\beta} = \sum_{s\in\mathbb{Z}+1/2}\sum_{i=1}^{p} :\bar{\psi}_s^{i\alpha}\psi_{n-s}^{i,\beta}:$.
  - $\widehat{\mathfrak{u}}(1)_{pk}$ generated by $J_n = \sum_i K_n^{ii} = \sum_\alpha L_n^{\alpha\alpha}$.

$\Rightarrow$ We have the following decomposition as a conformal embedding,

$$\widehat{\mathfrak{su}}(pk)_1 \supset \widehat{\mathfrak{su}}(p)_k \oplus \widehat{\mathfrak{su}}(k)_p.$$

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- Accordingly, the fermionic Fock space $\mathcal{F}_\infty^{(p,k)}$ can be decomposed as

$$\mathcal{F}_\infty^{(p,k)} = \bigoplus_{\sigma\in\mathbb{Z}} \mathcal{F}_{\infty,\sigma}^{(p,k)}, \quad \mathcal{F}_\sigma^{p,k} \cong \bigoplus_{\boldsymbol{\lambda}} \mathcal{W}_{\boldsymbol{\lambda}}^{\widehat{\mathfrak{su}}(p)_k} \otimes \mathcal{W}_{\boldsymbol{\lambda}'}^{\widehat{\mathfrak{su}}(k)_p} \otimes \mathcal{W}_\sigma^{\widehat{\mathfrak{u}}(1)}.$$

where $\sigma\in\mathbb{Z}$ is the $\widehat{\mathfrak{u}}(1)_{pk}$ charge, $\mathcal{W}_{\boldsymbol{\lambda}}^{\hat{\mathfrak{g}}_k}$ is an irreducible representation of $\hat{\mathfrak{g}}_k$ labeled by certain partitions λ, and λ' is the transposed of λ.

- The large N limit of wedge states can be obtained by

1. Factoring out a monomial in x_a (and then discarding it), eg for the vacua at $k=1$,

$$\mathcal{Y}_{\boldsymbol{r}_0}(\boldsymbol{x},\boldsymbol{y}) = \left(\prod_{a=1}^{N} x_a^{m-1}\right)[y_1 x^{-m+1} \wedge \cdots \wedge y_p x^{-m+1} \wedge \cdots \wedge y_1 \wedge \cdots \wedge y_p \wedge x y_{i_1} \wedge \cdots \wedge x y_{i_q}]$$

2. Re-ordering the wedges starting from the higher exponent for x,

$$\mathcal{Y}_{\boldsymbol{r}_0}(\boldsymbol{x},\boldsymbol{y}) \propto [(y_{i_1}x)\wedge\cdots\wedge(y_{i_q}x)\wedge y_1\wedge\cdots\wedge y_p\wedge y_1 x^{-1}\wedge\cdots\wedge y_p x^{-1}\cdots\wedge y_1 x^{-m+1}\wedge\cdots\wedge y_p x^{-m+1}].$$

⇝ The large N limit, obtained as $m\to\infty$ for $N=mp+q$, produces an infinite wedge product,

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- Formally, these wedge products can be obtained as

$$\prod_{\alpha=1}^{k}\mathcal{Y}^{\infty}_{\boldsymbol{r}_0^{(\alpha)}}(\boldsymbol{x},\boldsymbol{y}) = \langle\boldsymbol{x},\boldsymbol{y}|\prod_{\alpha=1}^{k}\prod_{r=1}^{q}\bar{\psi}^{i_r^{(\alpha)},\alpha}_{-1/2}|\emptyset\rangle\,.$$

with the projector $\langle\boldsymbol{x},\boldsymbol{y}|$ defined as a certain limit of the previous one.

- In large N limit, the $\widehat{\mathfrak{su}}(k)_+$-invariance of the projector $\langle \boldsymbol{x}, \boldsymbol{y}|$ is expected to be replaced by

$$\langle \boldsymbol{x}, \boldsymbol{y}| \bar{L}_n^{\alpha\beta} = 0, \quad n \leq 0 \,,$$

If so, the decomposition of $\widehat{\mathfrak{su}}(pk)_1$ implies that linearly-independent eigenfunctions are spanned by $\widehat{\mathfrak{su}}(p)_k \oplus \widehat{\mathfrak{u}}(1)_{pk}$. The extra $\widehat{\mathfrak{u}}(1)_{pk}$ factor can be introduced by an extra free boson.

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Remark: The fields $\Psi(x_a, y_a)$ satisfy the $\widehat{\mathfrak{u}}(p)_k$ primary field condition,

$$[K_n^{ij}, \Psi(x_a, y_a)] = -x_a^{-n} y_{i,a} \partial_{j,a} \Psi(x_a, y_a).$$

$\rightsquigarrow$ It would be nice to relate it to the KZ equation!

6. Discussion

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- We discussed the large N limit of the fermionic description and observed the emergence of the $\widehat{\mathfrak{su}(p)}_k$ Kac-Moody symmetry.

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$\rightsquigarrow$ Spectrum of the Hamiltonian (labeled by Gelfand-Tsetlin patterns).

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Open questions

Holes: How do we include the holes in this description? In the abelian case,

$$\mathcal{H}_{\text{total}} = \mathcal{H}_g(\boldsymbol{x}) - g\mathcal{H}_{\frac{1}{g}}(\boldsymbol{y}), \quad \mathcal{H}_g(\boldsymbol{x}) = \sum_a \left(-\frac{\partial^2}{\partial x_a^2} + x_a^2\right) + 2\sum_{\substack{a,b=1\\a<b}}^N \frac{g(g-1)}{(x_a - x_b)^2}.$$

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Deformations: It is possible to introduce trigonometric/elliptic deformations, relativistic deformations, and β-deformations of the coupling. Generalization of Uglov's construction of Calogero-Sutherland wave functions?

Algebras: This type of quantum models have also connections with the AGT correspondence (e.g. **[Estienne, Pasquier, Santachiara, Serban]**). Toroidal quantum groups are known to play an interesting role in this context!

⇒ **It would be interesting to investigate the interplay between these algebraic structures and the braiding of non-Abelian anyons!**

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Thank you !!!