APCTP Focus Program
“Integrability, Duality, and Related Topics” 2024

CFT on $T\bar{T}$-deformed Space
&
Correlators from Dynamical Coordinate Transformations

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Why is the $T\bar{T}$ deformation potentially interesting?

Talks by Ki-Seok (yesterday), Changrim (tomorrow), Feng (tomorrow)

- Unfortunately, it is not about toponium ($t\bar{t}$ bound state) recently "detected" at 3σ level in CMS.

- $T\bar{T} =$ " (stress tensor)^2 " is an irrelevant operator and power-counting non-renormalizable. The corresponding coupling $\dim[\mu] = -d$ that introduces a fundamental short-distance scale.

- Even though the irrelevant deformations are often ill-defined and hard to tame, the $T\bar{T}$-deformation appears to yield well-defined controllable theories.

- The $T\bar{T}$ deformation is model-independent in the sense that it exists in any local QFTs.

- Moreover, if the undeformed theory is integrable, it remains integrable by the $T\bar{T}$-deformation: The $T\bar{T}$-deformation preserves integrability.

- The most well-known irrelevant coupling is G_N in 4d gravity:

  The Einstein gravity is non-renormalizable and best regarded as a low-energy effective theory at $E \ll M_P$ (or species scale?): It is not a UV-complete theory and presumably requires extra UV DoFs such as strings to be complete.

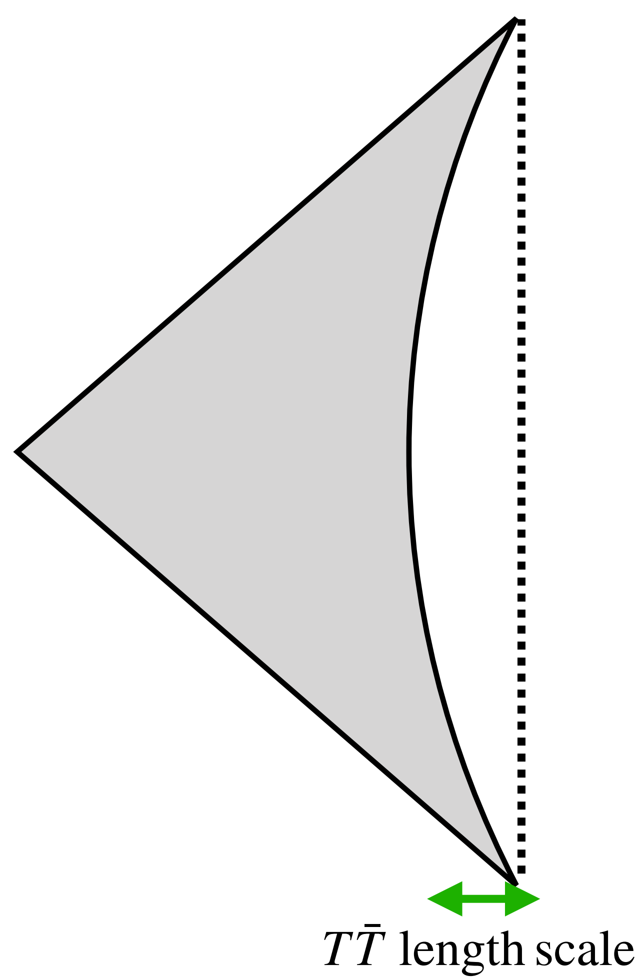
- In contrast, the $T\bar{T}$-deformed theories (at least in 2-dimensions) seem to be UV-complete, i.e., being valid at high energies beyond the energy scale $E \sim M_{T\bar{T}}$. They exhibit novel physics at short distance scales and the spacetime becomes somewhat dynamical, reminiscent of gravity:

  It provides a new class of quantum field theories that are presumably UV-complete despite irrelevant interactions. It is thus of great interest in the study of quantum field theory regardless of connections to string theory/holography.

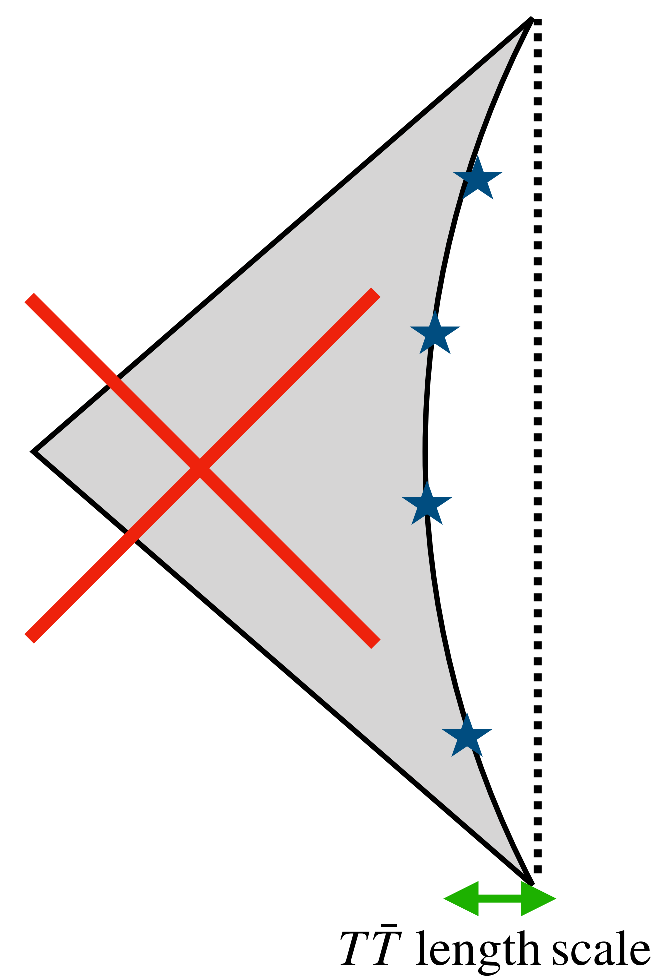
- From the viewpoint of AdS/CFT, the $T\bar{T}$-deformation of CFT is expected to be dual to some deformation near the boundary of AdS space since the field theory UV corresponds to the gravity IR.

  The most popular proposal is the cutoff AdS which only works for pure gravity in the absence of matter. It has the issue that GKPW with cutoff doesn't work, failing to reproduce the $T\bar{T}$-deformed correlators. McGough-Mezei-Verlinde Kraus-Liu-Marolf

  (I'll come back to this issue later in this talk.)



Cutoff (Poincare) AdS space



Cutoff AdS space w/ non-normalizable modes of matter

- Despite unresolved fundamental issues, there is a very appealing observation:

Kraus-Liu-Marolf, Caputa-Datta-Jiang-Kraus

The Brown-York tensor $T_{ij} = (K_{ij} - Kh_{ij} + h_{ij})/(4G)$ at the radial slice $\rho = \pi\mu/(4G)$ in the Fefferman-Graham gauge can be identified with the stress tensor $T_{ij}(x)$ of the $T\bar{T}$-deformed CFT.

$$ds^2_{AdS_3} = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\delta_{ij}dx^i dx^j$$

In other words, the $T\bar{T}$-deformed stress tensor $T_{ij}(x)$ reconstructs a bulk local field.

- In the absence of matter, this ID follows from the "Hamiltonian constraint" $G^{\rho}{}_{\rho} = 0$ which is identical to the $T\bar{T}$ flow equation $\Theta = -(\mu/\pi)(T\bar{T} - \Theta^2)$.

- In the presence of matter, $G^{\rho}{}_{\rho} = (T_{\mathrm{matter}})^{\rho}{}_{\rho}$ and the RHS spoils the $T\bar{T}$ flow equation, whereas the flow equation holds true in the presence of (primary) operators, dual to bulk matter, in the $T\bar{T}$-deformed CFT. This indicates the problem of the cutoff AdS proposal in the presence of matter.

This, I believe, is an outstanding issue to be understood.

- In this talk, I discuss the equivalence of the $T\bar{T}$-deformed CFT and the "undeformed" CFT which are mapped to each other by a dynamical coordinate transformation and use it to study some aspects of the $T\bar{T}$ deformation. Conti-Negro-Tateo Cardy

- The dynamical coordinate transformation deforms the space in which CFT lives rather than the theory (CFT) itself and makes the coordinates operator-valued.

$T\bar{T}$-deformed CFT on undeformed space	=	undeformed CFT on $T\bar{T}$-deformed space

- The topics to be discussed are the deformation of the stress tensor, primary operators, and their correlators, and the short-distance property of the $T\bar{T}$-deformed space. Along the way, I will also discuss holographic description(s).

- This description in terms of the dynamical space is reminiscent of gravity and somewhat related to the JT gravity/massive gravity representation of the $T\bar{T}$ deformation. Dubovsky-Gorbenko et al Tolley

$T\bar{T}$ deformation as a coordinate transformation

- **Definition of "$T\bar{T}$" operator**

$$\mathcal{O}_{T\bar{T}} \equiv T\bar{T} - \Theta^2 = -\det T_{ij}$$

- **Definition of $T\bar{T}$ deformation**
  - ✦ Naively, one might think that the $T\bar{T}$-deformed CFT is defined by the action

$$S_{T\bar{T}} \overset{?}{=} S_{\text{CFT}} + \mu \int d^2x \mathcal{O}_{T\bar{T}}$$

  - ✦ However, this isn't true. It must be defined as an infinitesimal deformation from the $T\bar{T}$ deformed theory $\mathcal{T}[\mu]$ to $\mathcal{T}[\mu + \delta\mu]$ ∵ $T_{ij} = T_{ij}^{(\mu)}$ is the stress tensor of the $\mathcal{T}[\mu]$ theory rather than that of the undeformed $\mathcal{T}[0]$ = CFT theory.

$$S[\mu + \delta\mu] = S[\mu] + \frac{\delta\mu}{\pi^2} \int d^2x \mathcal{O}_{T\bar{T}}^{(\mu)} \equiv S[\mu] + \delta S$$

$T\bar{T}$ as a coordinate transformation — cont'd

- **Hubbard-Stratonovich transformation (linearizing $T\bar{T}$)**

$$\exp\left(-\delta S\right) \propto \int [dh] \exp\left[-\frac{1}{8\delta\mu}\int d^2x \epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl} - \frac{1}{4\pi}\int d^2x h_{ij}T^{ij}\right]$$

(1) h-integrals dominated by a saddle point for an infinitesimal $\delta\mu$

(2) Since T_{ij} by definition is a response to a small change of the metric, the $T\bar{T}$ deformation can be interpreted as the change of the background metric $g_{ij} \mapsto g_{ij} + h_{ij}$.

- **Saddle point of h integral**

$$h^*_{ij} = -\frac{\delta\mu}{\pi}\epsilon_{ik}\epsilon_{jl}T^{kl}$$

$T\bar{T}$ as a coordinate transformation — cont'd

- **A key observation of Cardy**

  ✦ The stress tensor is conserved (w/o local operator singularities)

$$\partial_i T^{ij} = 0$$

  - This imposes constraints on the saddle point h:

$$h_{ij} = \partial_i \alpha_j + \partial_j \alpha_i = -\frac{\delta\mu}{\pi}\epsilon_{ik}\epsilon_{jl}T^{kl} \qquad \text{with} \qquad \epsilon^{ij}\partial_i \alpha_j = 0$$

  ✦ The $T\bar{T}$ deformation merely amounts to curl-free diffeomorphisms

$$x_i \mapsto x_i + \alpha_i$$

- Introducing complex coordinates $(z, \bar{z}) \mapsto (z + \alpha_z, \bar{z} + \alpha_{\bar{z}})$

$$\boxed{\alpha_z = \frac{\delta\mu}{2\pi^2}\int_{\mathbb{R}^2} d^2x \frac{\bar{T}^{(\mu)}(x, \bar{x})}{z - x}}$$

Note: $\bar{\partial}\frac{1}{z-x} = 2\pi\delta^2(z-x)$

$\partial\bar{T} + \bar{\partial}\Theta = 0$

CFT on $T\bar{T}$-deformed space

The $T\bar{T}$-deformed CFT is equivalent to the undeformed CFT on the $T\bar{T}$-deformed space:

$$\mathscr{T}[\mu] \text{ on } \mathbb{R}^2 = \mathscr{T}[0] \text{ on } \mathbb{R}^2_{(\mu)}$$

mapped by a dynamical coordinate transformation $(z, \bar{z}) \mapsto \left(Z^{(\mu)}, \bar{Z}^{(\mu)}\right)$

The infinitesimal transformation, from $\mathscr{T}[\mu + \delta\mu]$ on $\mathbb{R}^2$ to $\mathscr{T}[\mu]$ on $\mathbb{R}^2_{(\delta\mu)}$, is given by

$$z \quad \mapsto \quad Z^{(\mu|\delta\mu)} = z + \frac{\delta\mu}{2\pi^2}\int_{\mathbb{R}^2} d^2x \frac{\bar{T}^{(\mu)}(x,\bar{x})}{z - x}$$

Cardy

The finite version, from $\mathscr{T}[\mu]$ on $\mathbb{R}^2$ to $\mathscr{T}[0]$ on $\mathbb{R}^2_{(\mu)}$, is known to be

$$z \quad \mapsto \quad \underbrace{Z^{(\mu)}}_{\equiv Z^{(0|\mu)}} = z + \frac{\mu}{2\pi^2}\int_{\mathbb{R}^2} d^2x \frac{\bar{T}^{(\mu)}(x,\bar{x})}{z - x}$$

Conti-Negro-Tateo

CFT on $T\bar{T}$-deformed space — cont'd

finite version — conceptually illuminating

- The map between operators in the $T\bar{T}$-deformed CFT and those in CFT on the $T\bar{T}$-deformed space, denoting $(Z,\bar{Z}) = (Z^{(\mu)},\bar{Z}^{(\mu)})$

proposal $$\mathcal{O}^{(\mu)}_{\Delta,\bar{\Delta}}(z,\bar{z}) = \left(1-\frac{\mu^2}{\pi^2}\bar{T}(\bar{Z})T(Z)\right)^{-\frac{\Delta+\bar{\Delta}}{2}} \mathcal{O}_{\Delta}(Z)\mathcal{O}_{\bar{\Delta}}(\bar{Z})$$

where the prefactor is the Jacobian of the (*non-holomorphic*) coordinate transformation

$$\begin{pmatrix} dZ \\ d\bar{Z} \end{pmatrix} = \begin{pmatrix} 1-\frac{\mu}{\pi}\Theta^{(\mu)}(z,\bar{z}) & \frac{\mu}{\pi}\bar{T}^{(\mu)}(z,\bar{z}) \\ \frac{\mu}{\pi}T^{(\mu)}(z,\bar{z}) & 1-\frac{\mu}{\pi}\Theta^{(\mu)}(z,\bar{z}) \end{pmatrix} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix}$$

✦ In the case of the stress tensor, comparing the two conservation laws $0 = \frac{\partial}{\partial x^a} T^{(\mu)\,a}{}_b(x) = \frac{\partial}{\partial X^a} T^{(0)\,a}{}_b(X)$, one finds

$$T^{(\mu)}(z,\bar{z}) = \frac{T(Z)}{1-\frac{\mu^2}{\pi^2}\bar{T}(\bar{Z})T(Z)} \quad \text{with} \quad (\Delta,\bar{\Delta}) = (2,0)$$

$$\Theta^{(\mu)}(z,\bar{z}) = \frac{-\frac{\mu}{\pi}\bar{T}(\bar{Z})T(Z)}{1-\frac{\mu^2}{\pi^2}\bar{T}(\bar{Z})T(Z)} \quad \text{with} \quad (\Delta,\bar{\Delta}) = (1,1)$$

$\dim[\mu] = (-1,-1)$

CFT on $T\bar{T}$-deformed space — cont'd

infinitesimal version — practically useful

- The flow equations for the stress tensor from the map:

$$0 = \underbrace{\tilde{\partial}_a T^{(\mu)\,a}{}_b(\tilde{x})}_{\text{“undeformed” } T \text{ on deformed space}} = \underbrace{\partial_a T^{(\mu+\delta\mu)\,a}{}_b(x)}_{\text{deformed } T \text{ on undeformed space}}$$

This yields the following recursion relations and we can systematically find the deformed stress tensor in terms of the CFT stress tensor:

from infinitesimal map

$$T^{(\mu)}(z,\bar{z}) = T(z) + \int_0^\mu \frac{d\lambda}{2\pi^2}\Bigg[2\int_{\mathbb{R}^2} d^2x \frac{\partial\bar{T}^{(\lambda)}(x,\bar{x})T^{(\lambda)}(z,\bar{z})}{z-x} + \int_{\mathbb{R}^2} d^2x \frac{\bar{T}^{(\lambda)}(x,\bar{x})\partial T^{(\lambda)}(z,\bar{z})}{z-x} + \int_{\mathbb{R}^2} d^2x \frac{T^{(\lambda)}(x,\bar{x})\bar{\partial}T^{(\lambda)}(z,\bar{z})}{\bar{z}-\bar{x}}\Bigg]$$

from infinitesimal map

$$\Theta^{(\mu)}(z,\bar{z}) = \int_0^\mu \frac{d\lambda}{2\pi^2}\Bigg[-2\pi\left(\bar{T}^{(\lambda)}(z,\bar{z})T^{(\lambda)}(z,\bar{z}) - \Theta^{(\lambda)}(z,\bar{z})^2\right) + \partial_z\int_{\mathbb{R}^2} d^2x \frac{\bar{T}^{(\lambda)}(x,\bar{x})\Theta^{(\lambda)}(z,\bar{z})}{z-x} + \partial_{\bar{z}}\int_{\mathbb{R}^2} d^2x \frac{T^{(\lambda)}(x,\bar{x})\Theta^{(\lambda)}(z,\bar{z})}{\bar{z}-\bar{x}}\Bigg]$$

equivalent

from finite map

$$\Theta^{(\mu)}(z,\bar{z}) = -\frac{\mu}{\pi}\left[T^{(\mu)}\bar{T}^{(\mu)}(z,\bar{z}) - \Theta^{(\mu)}(z,\bar{z})^2\right]$$

$T\bar{T}$-deformed space

- The metric on the $T\bar{T}$-deformed space reads

$$ds^2 = dzd\bar{z} = \left(1 + \frac{\mu^2}{\pi^2} T(Z)\bar{T}(\bar{Z})\right) dZd\bar{Z} - \frac{\mu}{\pi} T(Z)dZ^2 - \frac{\mu}{\pi}\bar{T}(\bar{Z})d\bar{Z}^2$$

  ✦ This is operator-valued and the explicit expression depends on the states.

- This can be uplifted to 3d, which suggests the holographic description:

$$ds^2_{3D} = \frac{d\mu^2}{4\mu^2} + \frac{4\pi G}{\mu}\left[\left(1 + \frac{\mu^2}{\pi^2} T(Z)\bar{T}(\bar{Z})\right) dZd\bar{Z} - \frac{\mu}{\pi} T(Z)dZ^2 - \frac{\mu}{\pi}\bar{T}(Z)d\bar{Z}^2\right]$$

  ✦ AdS_3 (Banados space) in the Fefferman-Graham coordinates in which the $T\bar{T}$ coupling μ is identified with the radial coordinate. The dynamical coordinates $(Z, \bar{Z})$ is more natural for holography.

$T\bar{T}$-deformed space — cont'd

- The $T\bar{T}$-deformed space is dynamical in the sense that it backreacts to insertions of operators.

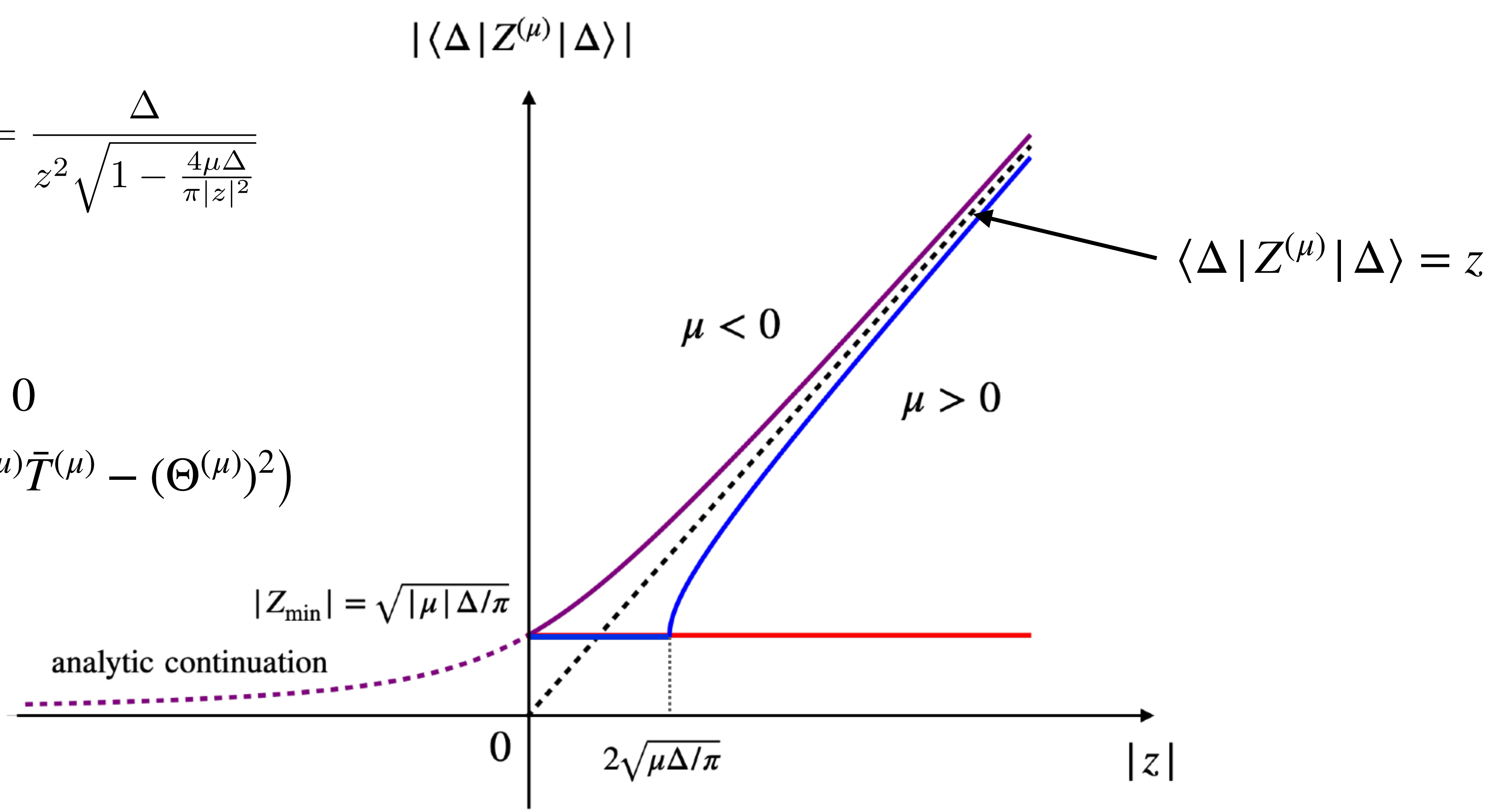
$$\langle\Delta|Z^{(\mu)}|\Delta\rangle \quad = z + \frac{\mu}{2\pi^2}\int d^2x \frac{\langle\Delta|\bar{T}^{(\mu)}(x,\bar{x})|\Delta\rangle}{z-x} = \frac{z}{2}\left[\sqrt{1-\frac{4\mu\Delta}{\pi|z|^2}}+1\right]$$

$$\langle\Delta|T^{(\mu)}(z,\bar{z})|\Delta\rangle = \frac{\Delta}{z^2\sqrt{1-\frac{4\mu\Delta}{\pi|z|^2}}}$$

following from

$$\partial\bar{T}^{(\mu)} + \bar{\partial}\Theta^{(\mu)} = 0$$

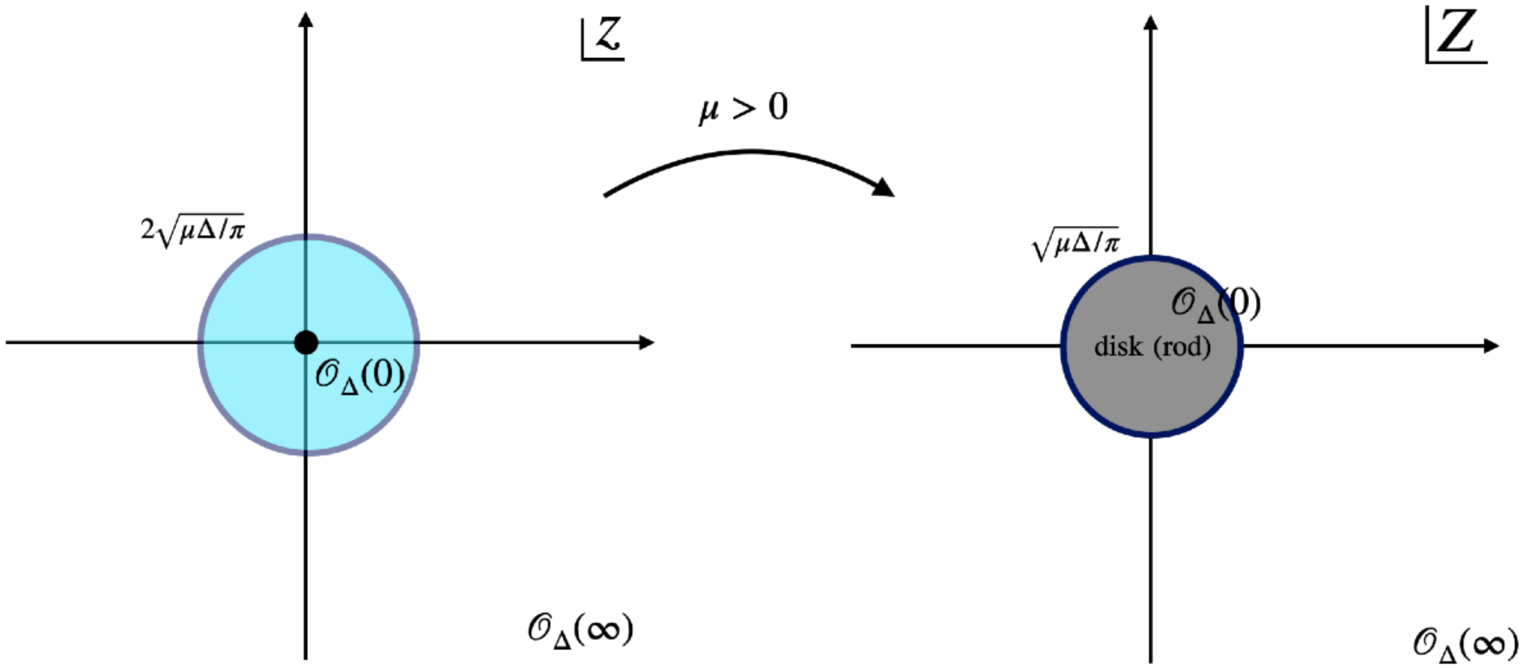
$$\Theta^{(\mu)} = -\frac{\mu}{\pi}\left(T^{(\mu)}\bar{T}^{(\mu)} - (\Theta^{(\mu)})^2\right)$$



$T\bar{T}$-deformed space — cont'd

Doyon-Cardy, Jiang

- The back-reaction of an operator insertion to the $T\bar{T}$-deformed space:



A free space opens up inside the operator in Z-space for $\mu < 0$

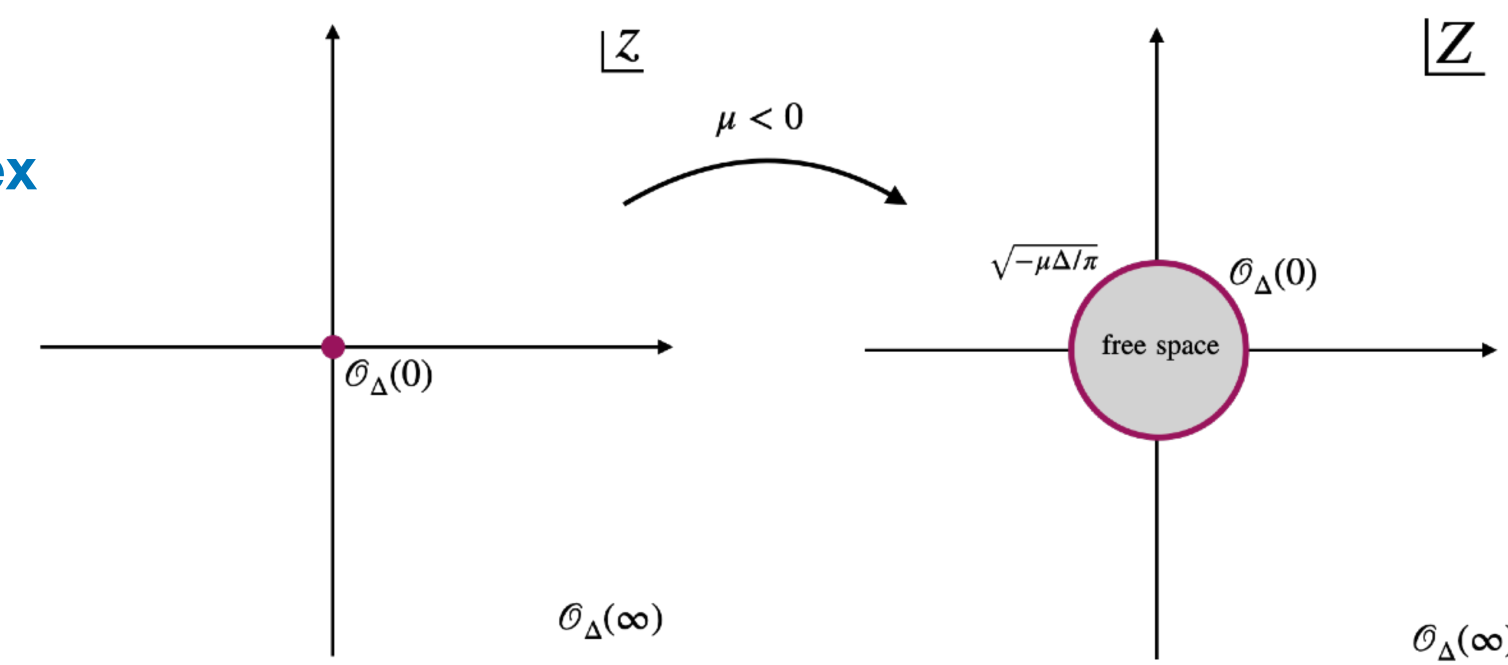
Hagedorn phase $\mu < 0$

(there is a limiting high temperature $T_H \le \sqrt{6\pi/(c|\mu|)}$)

UV cutoff phase $\mu > 0$

(the energy spectrum becomes complex at high energies, i.e. for large $(\Delta, \bar{\Delta})$)

The operator puffs up to the size $\sqrt{\mu\Delta/\pi}$ in Z-space for $\mu > 0$



$T\bar{T}$-deformed space — cont'd

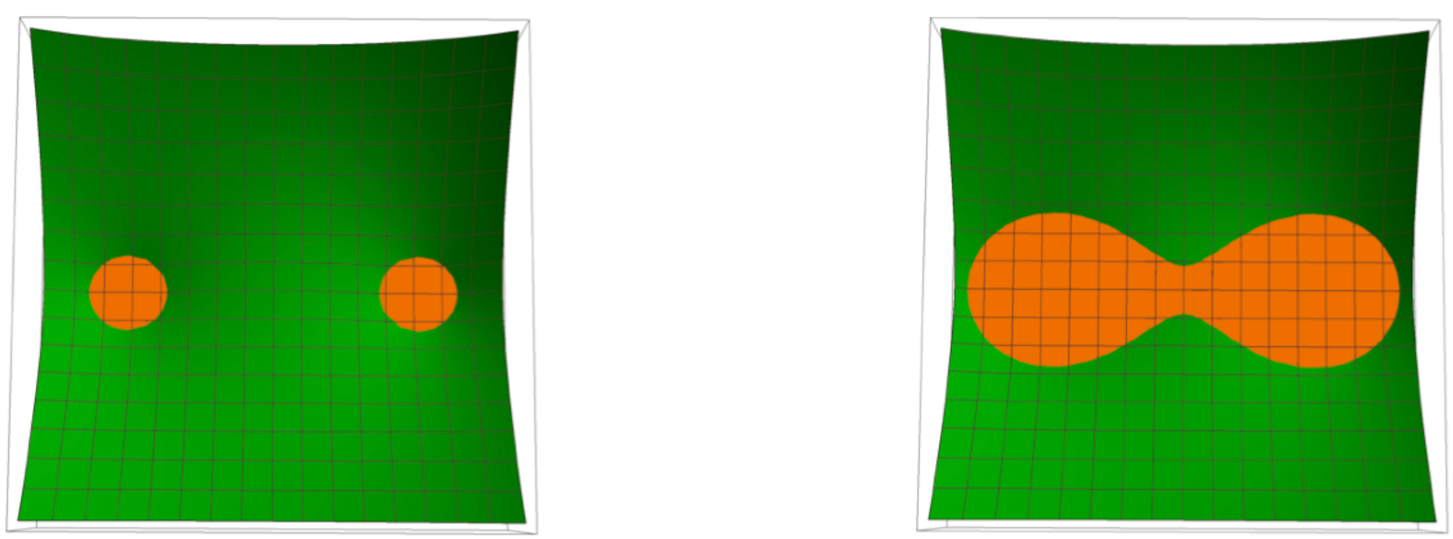
- A few observations for the property of the dynamical coordinate map $(z,\bar{z}) \mapsto (Z^{(\mu)}, \bar{Z}^{(\mu)})$:

(1) $$\langle \Delta | T(Z) | \Delta \rangle = \frac{\Delta}{\left(\langle \Delta | Z | \Delta \rangle\right)^2}$$ as may be expected from CFT

(2) The circle of radius $\sqrt{|\mu|\Delta/\pi}$ corresponds to the coordinate singularity of the 2d metric of the $T\bar{T}$-deformed space:

$$\det g_{ab} = -(1/4)(1 - \mu^2/\pi^2 T(Z)\bar{T}(\bar{Z})) = 0 \quad \text{on state } |\Delta\rangle$$

(3) In the case of two operators at $Z = \pm a$, the CFT stress tensor vev is $T(Z) = (2a)^2\Delta/((Z-a)^2(Z+a)^2)$, we get the following picture:



Correlators from CFT on $T\bar{T}$-deformed space

- The basic idea is to compute the correlators in the "Heisenberg" picture:

$$\underbrace{\langle \mathcal{O}^{(\mu)}_{\Delta_1,\bar{\Delta}_1}(z_1,\bar{z}_1)\mathcal{O}^{(\mu)}_{\Delta_2,\bar{\Delta}_2}(z_2,\bar{z}_2)\cdots\mathcal{O}^{(\mu)}_{\Delta_n,\bar{\Delta}_n}(z_n,\bar{z}_n)\rangle_0}_{\text{Heisenberg}} = \underbrace{\langle \mathcal{O}^{(0)}_{\Delta_1,\bar{\Delta}_1}(z_1,\bar{z}_1)\mathcal{O}^{(0)}_{\Delta_2,\bar{\Delta}_2}(z_2,\bar{z}_2)\cdots\mathcal{O}^{(0)}_{\Delta_n,\bar{\Delta}_n}(z_n,\bar{z}_n)\rangle_\mu}_{\text{Schrodinger}}$$

in which the operators in the "Heisenberg" picture are given by the map

$$\mathcal{O}^{(\mu)}_{\Delta,\bar{\Delta}}(z,\bar{z}) = \left(1 - \frac{\mu^2}{\pi^2}\bar{T}(\bar{Z})T(Z)\right)^{-\frac{\Delta+\bar{\Delta}}{2}} \mathcal{O}_\Delta(Z)\mathcal{O}_{\bar{\Delta}}(\bar{Z})$$

✦ However, the holomorphic and anti-holomorphic parts do not factorize in the deformed space:

$$\begin{aligned}\langle \mathcal{O}_\Delta(Z_1)\mathcal{O}_{\bar{\Delta}}(\bar{Z}_1)\mathcal{O}_\Delta(Z_2)\mathcal{O}_{\bar{\Delta}}(\bar{Z}_2)\rangle &\neq \langle \mathcal{O}_\Delta(Z_1)\mathcal{O}_\Delta(Z_2)\rangle\langle \mathcal{O}_{\bar{\Delta}}(\bar{Z}_1)\mathcal{O}_{\bar{\Delta}}(\bar{Z}_2)\rangle \\ &= \frac{1}{\langle \Delta | Z_1 - Z_2 | \Delta\rangle^{2\Delta}\langle \bar{\Delta} | \bar{Z}_1 - \bar{Z}_2 | \bar{\Delta}\rangle^{2\bar{\Delta}}}\end{aligned}$$

- Practically, this is not an issue. We can use (1) the finite map from $z \mapsto Z^{(\mu)}$, (2) the recursion to find the stress tensor $T^{(\mu)}$ and $\Theta^{(\mu)}$ in terms of the CFT stress tensor, and (3) expand the expressions in μ.

Correlators from CFT on $T\bar{T}$-deformed space — cont'd

- For example, using the proposed map of the deformed operators, we find

$$\langle \mathcal{O}^{(\mu)}_{\Delta,\Delta}(z_1,\bar{z}_1)\mathcal{O}^{(\mu)}_{\Delta,\Delta}(z_2,\bar{z}_2)\rangle_0 = \langle J_1^{-\Delta}J_2^{-\Delta}\mathcal{O}_\Delta(Z_1)\mathcal{O}_\Delta(\bar{Z}_1)\mathcal{O}_\Delta(Z_2)\mathcal{O}_\Delta(\bar{Z}_2)\rangle_0$$
$$= \frac{1}{|z_{12}|^{4\Delta}} - \frac{\mu\Delta}{\pi^2 z_{12}^{2\Delta+1}}\int d^2x \underbrace{\frac{\langle \bar{T}(\bar{x})\mathcal{O}_\Delta(\bar{z}_1)\mathcal{O}_\Delta(\bar{z}_2)\rangle_0}{z_1 - x}}_{\text{hol−antihol coupling}} + (z_1 \leftrightarrow z_2) + \text{c.c.} + \mathcal{O}(\mu^2)$$
$$= \frac{1}{|z_{12}|^{4\Delta}} - \frac{8\mu\Delta^2}{\pi|z_{12}|^{4\Delta}}\left(\frac{\ln|z_{12}/\epsilon|^2}{|z_{12}|^2} - \frac{1}{|z_{12}|^2} - \frac{1}{2\epsilon}\left(\frac{1}{z_{12}} + \frac{1}{\bar{z}_{12}}\right)\right) + \mathcal{O}(\mu^2)$$

where the point-splitting regularization was used. We have also checked that this works to the second order for a few known terms in the literature.

- This computation works order by order in perturbation theory in the $T\bar{T}$ coupling μ.

Correlators from CFT on $T\bar{T}$-deformed space — cont'd

- However, as it turns out, a very intuitive and simple calculation works in the semiclassical limit, $\Delta \gg \sqrt{c}$ (or the double scaling limit $\lim_{\mu\to 0,\Delta\to\infty} \mu\Delta^2 = \text{fixed}$) for which the hol-antihol factorization occurs:

$$\lim_{\mu\to 0,\Delta\to\infty} \left\langle \prod_{i=1}^{n} \mathcal{O}_{\Delta_n}(Z_i)\mathcal{O}_{\bar{\Delta}_i}(\bar{Z}_i) \right\rangle_{\text{CFT}} \underbrace{=}_{\text{factorization}} \left\langle \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(Z_i) \right\rangle_{\text{CFT}} \left\langle \prod_{i=1}^{n} \mathcal{O}_{\bar{\Delta}_i}(\bar{Z}_i) \right\rangle_{\text{CFT}}$$

$$\underbrace{=}_{\text{classicalization}} \left\langle \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(Z_i^{cl}) \right\rangle_{\text{CFT}} \left\langle \prod_{i=1}^{n} \mathcal{O}_{\bar{\Delta}_i}(\bar{Z}_i^{cl}) \right\rangle_{\text{CFT}}$$

where the "classical" coordinate Z^{cl} is defined by

$$dz = dZ^{cl} - \frac{\mu}{\pi} \frac{\left\langle \bar{T}(\bar{Z}^{cl}) \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(\bar{Z}_i^{cl}) \right\rangle_{\text{CFT}}}{\langle \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(\bar{Z}_i^{cl}) \rangle_{\text{CFT}}} d\bar{Z}^{cl}$$

✦ This is a consequence of the following property for all semi-heavy operators

$$\frac{\langle T(x)T(y) \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(w_i) \rangle}{\langle \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(w_i) \rangle} \approx \prod_{z=x,y} \frac{\langle T(z) \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(w_i) \rangle}{\langle \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(w_i) \rangle}$$

Correlators from CFT on $T\bar{T}$-deformed space — cont'd

- Heavy operator 2pt functions as an illustration:

$$\langle \mathcal{O}^{(\mu)}_{\Delta,\Delta}(z_1,\bar{z}_1)\mathcal{O}^{(\mu)}_{\Delta,\Delta}(z_2,\bar{z}_2)\rangle \approx \frac{1}{|Z_1^{cl}-Z_2^{cl}|^{4\Delta}} = \frac{1}{(2a)^{4\Delta}}$$

where the deformed and undeformed coordinates are mapped by

$$z = \int \left(dZ^{cl} - \frac{\mu}{\pi}\langle \bar{T}(\bar{Z}^{cl})\rangle d\bar{Z}^{cl} \right) = Z^{cl} + \frac{\mu\Delta}{\pi}\left(\frac{1}{\bar{Z}^{cl}-a} + \frac{1}{\bar{Z}^{cl}+a} + \frac{1}{a}\ln\frac{\bar{Z}^{cl}-a}{\bar{Z}^{cl}+a} \right)$$

with the regularization $(Z_1^{cl}, Z_2^{cl}) = (a-\epsilon, -a+\epsilon)$. For $(z_1,z_2)=(b,-b)$, we find

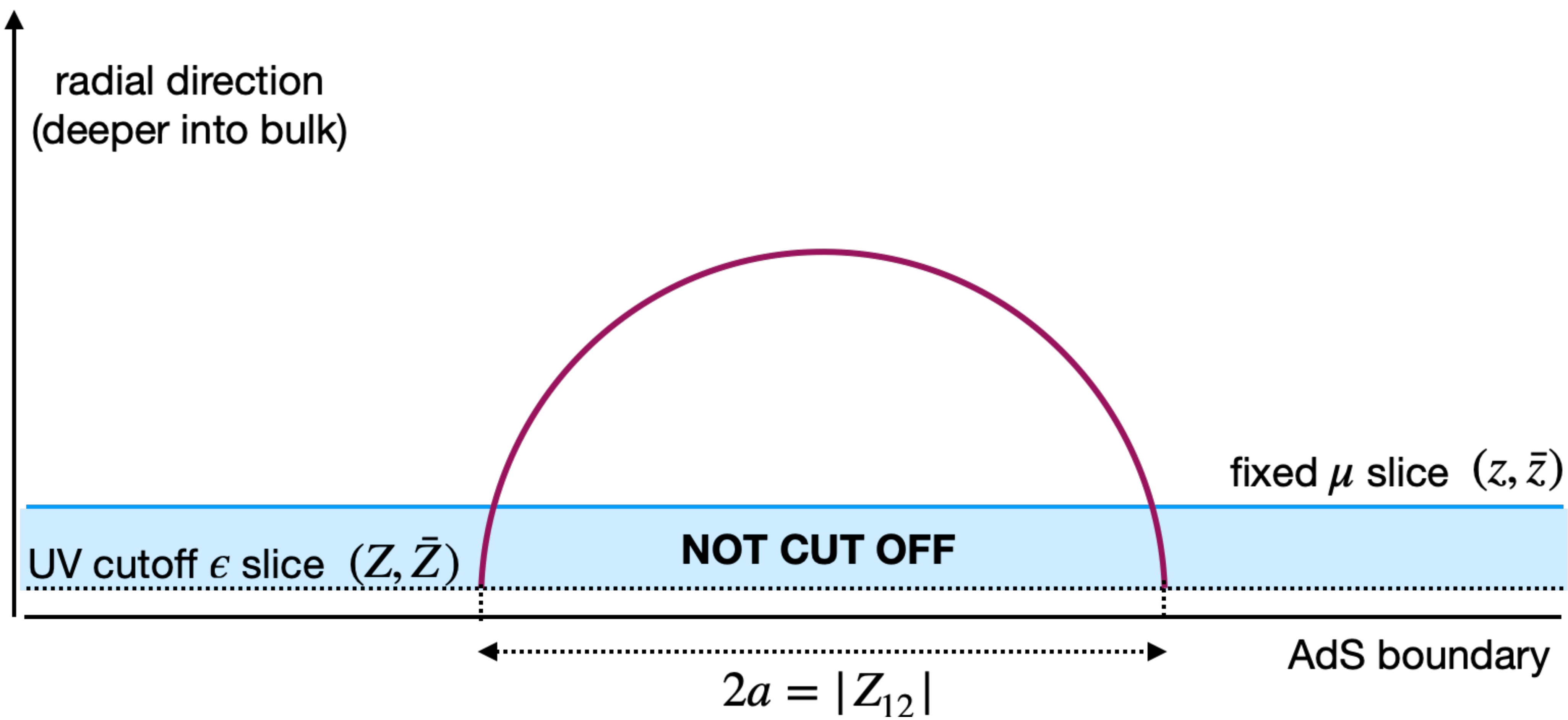
$$a = b + \epsilon - \frac{\mu\Delta}{\pi}\left(-\frac{1}{\epsilon} + \frac{1}{2b+\epsilon} - \frac{1}{b+\epsilon}\ln\frac{2b+\epsilon}{-\epsilon} \right) + \cdots$$

✦ The renormalised answer agrees with the known result by Cardy at large Δ:

$$\langle \mathcal{O}^{(\mu)}_{\Delta,\Delta}(b)\mathcal{O}^{(\mu)}_{\Delta,\Delta}(-b)\rangle \approx \sum_{n=0}^{\infty} \frac{\mu^n}{n!\pi^n}(-1)^n 2^{2n} \prod_{k=0}^{n-1}(2\Delta+k)^2 \frac{\ln^n(2b/\epsilon)}{(2b)^{4\Delta+2n}}$$

Towards holography with matter

- We can simply rephrase the 2pt correlator results in the AdS language. (We do not know how to systematically include $1/\Delta$ corrections.)



The parts of the geodesic in the light blue strip between the constant μ and ϵ slices *do* contribute to the 2pt correlator. So the constant μ slice is *not* a cutoff surface. Instead, the distance $2a = |Z^{cl}_{12}|$ is remeasured in terms of $2b = |z_{12}|$ in the flat coordinates $(z, \bar{z})$ on the fixed μ slice determined by the dynamical coordinate map.

Towards holography with matter — cont'd

An ensemble of AdS₃
with
"Gaussian" average over boundary metric (diffeomorphisms)

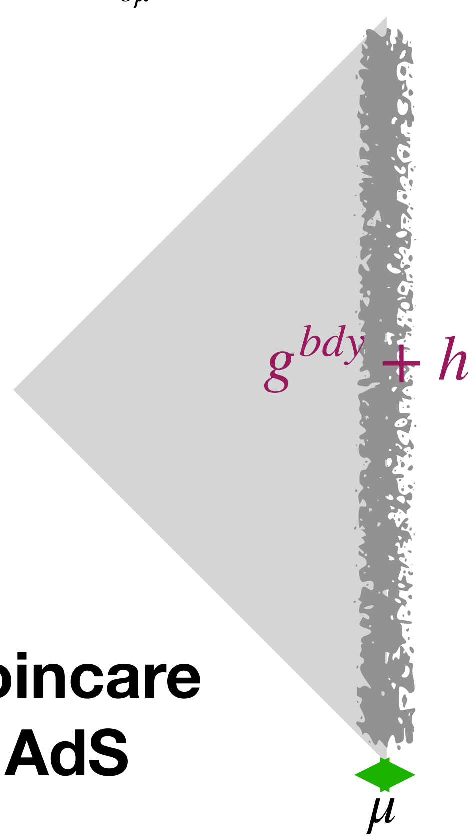
Hirano-Shigemori 2020

- Translating the Hubbard-Stratonovich representation of the $T\bar{T}$ deformation into AdS/CFT

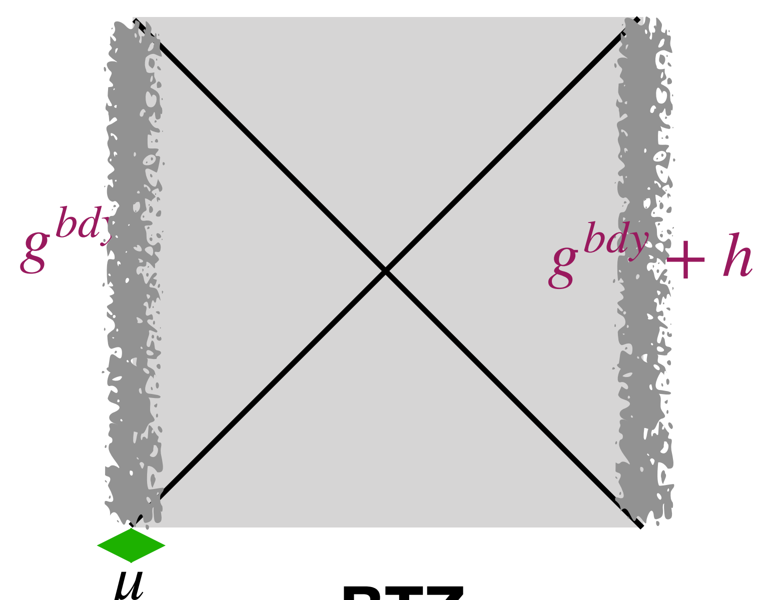
$$\underbrace{\mathcal{O}_{T\bar{T}_{\delta\mu}\mathrm{AdS}}\left[\phi^{bdy}(x)\right]}_{\text{observable in } T\bar{T}_{\delta\mu}\mathrm{AdS}} = \mathcal{N}^{-1}\int [dh] \exp\left[-\frac{1}{8\delta\mu}\int d^2x \epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl}\right] \underbrace{\mathcal{O}_{\mathrm{AdS}}\left[\phi^{bdy}(x+\alpha)\right]}_{\text{observable in AdS}}$$

with $h_{ij} = \partial_i\alpha_j + \partial_j\alpha_i$

$$\int [dh] e^{\frac{1}{\delta\mu}\int d^2x h^2}$$



Poincare AdS



BTZ

Variance μ (over the scale of deformation) by iteration

Towards holography with matter — cont'd

- A weakness of this proposal is that it requires an infinite iterations of the infinitesimal Gaussian averaging to reach a finite $T\bar{T}$ coupling μ.

- This can be improved to a one-go finite-coupling averaging by adopting the 2d topological gravity (flat space JT gravity) or the massive gravity description of the $T\bar{T}$-deformed CFT:

$$\underbrace{\mathcal{O}_{T\bar{T}_\mu \mathrm{AdS}}\left[\phi^{bdy}(x)\right]}_{\text{observable in } T\bar{T}_\mu\mathrm{AdS}} = \int \frac{[de][dX]}{V_{\text{diff}}} \exp\left[\frac{\pi}{\mu}\int d^2x \epsilon^{\alpha\beta}\epsilon_{ab}(\partial_\alpha X^a - e^a_\alpha)(\partial_\beta X^b - e^b_\beta)\right] \underbrace{\mathcal{O}_{\mathrm{AdS}}\left[\phi^{bdy}(x); h^{bdy}_{\alpha\beta}\right]}_{\text{observable in AdS}}$$

with $h^{bdy}_{\alpha\beta} = e^a_\alpha e^b_\beta \delta_{ab}$

This follows from the fact that the $T\bar{T}$-deformed CFT is equivalent to CFT coupled to the 2d massive gravity (or flat space JT gravity with CC).

Tolley Dubovsky-Gorbenko et al

I claim that this should be equivalent to the cutoff AdS in the absence of matter but differs from it in the presence.

$T\bar{T}$-deformed BCFT

not firm-footing yet
work in progress?

- The $T\bar{T}$ deformation of BCFT has not been much studied. The CFT-based approach developed here may have an advantage in this direction:

(1) From the recursion relations for $T^{(\mu)}(z,\bar{z})$, one can show that

$$T^{(\mu)}(z,\bar{z}) - \bar{T}^{(\mu)}(z,\bar{z}) = e^{-\frac{2}{\pi}\int_0^\mu d\lambda \Theta^{(\lambda)}(z,\bar{z})}(T(z) - \bar{T}(\bar{z}))$$

This implies that the CFT boundary condition $T(z) = \bar{T}(\bar{z})$ at $z = \bar{z}$ on UHP is preserved under the $T\bar{T}$ deformation. In other words, the boundary condition in the $T\bar{T}$-deformed CFT can be represented by

$$T(Z) = \bar{T}(\bar{Z}) \quad \text{at} \quad Z = \bar{Z}$$

(2) In the doubling trick, the $T\bar{T}$-deformed bulk BCFT operators on UHP may be given by

$$\mathcal{O}^{(\mu)}_{\Delta,\bar{\Delta}}(z,\bar{z}) = \left(1 - \frac{\mu^2}{\pi^2} T(Z) \underbrace{T}_{\text{hol}}(\bar{Z})\right)^{-\frac{\Delta+\bar{\Delta}}{2}} \mathcal{O}_\Delta(Z) \underbrace{\mathcal{O}_{\bar{\Delta}}}_{\text{hol}}(\bar{Z})$$

$T\bar{T}$-deformed BCFT — cont'd

- The boundary (Cardy) states are determined by the equivalence of the annulus and cylinder partition functions related by the S transformation, $(\ell, \mu/R^2) \leftrightarrow (1/\ell, \mu/(R^2\ell^2))$ with $q = e^{-2\pi\ell}$ and S^1 radius R:

$$\underbrace{|B_a\rangle}_{\text{Cardy}} = \sum_{\Delta} C_{\Delta a}(\mu/R^2) \underbrace{|\Delta\rangle_I}_{\text{Ishibashi}} \qquad \text{with} \qquad C_{\Delta a}(\mu/R^2) = \sum_{p=0}^{\infty} (\pi\mu/R^2)^p C_{\Delta a}^{(p)}$$

The S invariance requires

$$n_{ab}^{\Delta} = \sum_{\Delta'} C_{a\Delta'}^{(0)} S_{\Delta\Delta'} C_{\Delta' b}^{(0)}$$

$$0 = \sum_{\Delta'} \left(C_{a\Delta'}^{(1)} S_{\Delta\Delta'} C_{\Delta' b}^{(0)} + C_{a\Delta'}^{(0)} S_{\Delta\Delta'} C_{\Delta' b}^{(1)} \right) + \frac{1}{2}(2\pi x) \sum_{\Delta'} C_{a\Delta'}^{(0)} S_{\Delta\Delta'} C_{\Delta' b}^{(0)}$$

$$\cdots$$

This determines the $T\bar{T}$ deformation of the Cardy states

$$C_{\Delta a}^{(1)} = \frac{2\pi x}{4} C_{\Delta a}^{(0)}, \quad C_{\Delta a}^{(2)} = -\frac{3(2\pi x)^2}{32} C_{\Delta a}^{(0)}, \quad C_{\Delta a}^{(3)} = -\frac{7(2\pi x)^3}{128} C_{\Delta a}^{(0)}, \quad \cdots$$

where $2\pi x = \pi(L_0 + \bar{L}_0 - c/12)$. (However, not clear if $(\mathscr{L}_n^{(\mu)} - \overline{\mathscr{L}}_{-n}^{(\mu)})|B_a\rangle = 0$?)

Thank you!