ODE/IM correspondence for SUSY quantum mechanics

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ODE/IM correspondence

Deformed SUSY QM

Conclusion and Outlook

ODE/IM correspondence

ODE/IM correspondence

ODE: Ordinary Differential Equation
IM: quantum Integrable Model

a nontrivial relation between spectral analysis approach of ordinary differential equation (ODE), and the “functional relations” approach to 2d quantum integrable model (IM)

[Dorey-Tateo 9812211, Bazhanov-Lukyanov-Zamolodchikov 9812247, ...]

ODE/IM correspondence (2)

the Schrödinger equation with centrifugal potential term ($2M > 0$)

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^{2M} - E\right] y(x, E, \ell) = 0$$

XXZ spin chain in the continuum limit $N \to \infty$

$$H = \sum_{i=1}^{N} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \cos 2\eta \sigma_i^z \sigma_{i+1}^z)$$

$$\sigma_{N+1}^z = \sigma_1^z, \quad (\sigma_{N+1}^x \pm i\sigma_{N+1}^y) = e^{\pm 2i\phi}(\sigma_1^x \pm i\sigma_1^y)$$

CFT(minimal model $\mathcal{M}_{2,2M+2}$): $c = 1 - 6(\beta - \frac{1}{\beta})^2$, $\Delta_0 = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24}$

ODE	XXZ spin chain
order of the potential $2M$	anisotropy parameter $\eta = \frac{\pi}{2}(1-\beta^2) = \frac{\pi M}{2M+2}$
angular momentum ℓ	twist parameter $\phi = 2\pi p = \frac{\pi(2\ell+1)}{2M+2}$
energy E	spectral parameter $\theta = \frac{M+1}{2M} \log E$

ODE/IM correspondence (3)

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^{2M} - E\right] y(x,E,\ell) = 0$$

- $x=0$: regular singularity $\chi_+ = x^{\ell+1} + \cdots$, $\chi_- = x^{-\ell} + \cdots$
- $x=\infty$: irregular singularity $y^{\pm}(x) \sim \exp\left(\pm \frac{x^{M+1}}{M+1}\right)$
- $y^- = Q_+\chi_+ + Q_-\chi_-$ connection coefficients $Q_\pm$
- Wronskian $W[y_k, y_{k'}]$ of two aysmpt. sols. : Stokes coefficients

ODE	IM
Connection coeffs between 0 and ∞	Q-functions (T-Q relation)
Stokes coefficients	T-functions
Voros symbols (exact WKB periods)	Y-functions

ODE/IM correspondence (4)

How to check the correspondence

- Q-functions

  zeros of the connection coefficients $\Longleftrightarrow$ Bethe roots (NLIE)

  Wronskians $\Longleftrightarrow$ T-Q relation
- T-functions

  Wronskian of WKB solutions $\Longleftrightarrow$ quantum integrals of motion

  Plücker relations for Wronskians $\Longleftrightarrow$ T-system
- Y-functions

  Voros symbols (exact WKB periods) $\Longleftrightarrow$ Y-functions

  Plücker relation for Wronskians $\Longleftrightarrow$ Y-system

  DDP discontunuity formula $\Longleftrightarrow$ TBA equations

exact WKB method

the Schrödinger equation ($m = 1/2$)

$$-\hbar^2\psi''(q) + (V(q) - E)\psi(q) = 0.$$

WKB solution: $\psi(q) = \exp\left[\frac{i}{\hbar}\int^q Q(q')dq'\right]$.

the Riccati equation:

$$Q^2(q) - i\hbar\frac{dQ(q)}{dq} = p^2(q), \quad p(q) = (E - V(q))^{1/2},$$

$$Q(q) = \sum_{k=0}^{\infty} Q_k(q)\hbar^k = Q_{\rm even} + Q_{\rm odd}$$

$$Q_{\rm even} = P(q) = \sum_{n\geq 0} p_n(q)\hbar^{2n}, \quad Q_{\rm odd} = \frac{i\hbar}{2}\frac{d}{dq}\log P(q)$$

$p_0(q) = p(q)$ and $p_n(q)$ are determined recursively.

WKB periods and Voros symbols

potential: polynomial in q

$$V(q) = q^{r+1} + u_1 q^r + \cdots + u_r q$$

the WKB curve: $\Sigma_{\mathrm{WKB}} : y^2 = E - V(q)$.

WKB periods (*quantum periods*)

$$\Pi_\gamma(\hbar) = \oint_\gamma P(q)dq = \sum_{n=0}^{\infty} \hbar^{2n} \Pi_\gamma^{(n)}, \quad \Pi_\gamma^{(n)} = \oint_\gamma p_n(q)dq \qquad \gamma \in H_1(\Sigma_{\mathrm{WKB}}).$$

- $\Pi_\gamma^{(n)} \sim (2n)!$, $\Pi_\gamma(\hbar)$: asymptotic series in $\hbar$.

Borel resummation

[Ecallé, Voros, Delabaere-Pham, Delabaere-Dillinger-Pham, Aoki-Kawai-Takei]

asymptotic series		analytic function

Borel transf.

$$\boxed{\phi(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}} \quad \longrightarrow \quad \boxed{\tilde{\phi}(\xi) = \sum_{n=0}^{\infty} c_n \frac{\xi^n}{n!}}$$

Asym. exp. $\nwarrow$ $\swarrow$ Laplace transf.

$$\boxed{s[\phi](z) = \int_0^{\infty} d\xi e^{-\xi z} \tilde{\phi}(\xi)}$$

Borel resummation

Y-system and TBA equations

effective (IR) description of 2d integrable QFT

(pseudo) particles with mass m_a

interacting with S-matrices $S_{ab}(\theta)$ (θ: rapidity)

pseudo energy $\epsilon_a(\theta)$: TBA equation (in the kink limit)

$$\epsilon_a(\theta) = m_a e^{\theta} - \sum_b \int_{-\infty}^{\infty} \phi_{ab}(\theta - \theta') \log\Big(1 + e^{-\epsilon_b(\theta)}\Big) d\theta'$$

Kernel function $\phi_{ab}(\theta) = -i \frac{d}{d\theta} \log S_{ab}(\theta)$

kink limit of massive TBA $\cosh\theta \to e^{\theta}$

TBA eqs. $\Longleftrightarrow$Y-system [Al.B. Zamolodchikov]

$$Y_a(\theta + \frac{i\pi}{h}) Y_a(\theta - \frac{i\pi}{h}) = (1 + Y_{a+1}(\theta))(1 + Y_{a-1}(\theta))$$

Y-function $Y_a(\theta) = e^{\epsilon_a(\theta)}$

Voros symbols and Y-functions

$$\mathcal{V}_\gamma = \exp\left(\frac{i}{\hbar} s[\Pi_\gamma](\hbar)\right) \Leftrightarrow Y_\gamma(\theta) = \exp(\epsilon_\gamma(\theta))$$

exact WKB	IM
the Planck constant: $\hbar = e^{-\theta}$	rapidity: θ
the exact WKB period $s(\Pi_\gamma)$	pseudo-energy $\epsilon_\gamma(\theta)$
classical WKB period $\Pi_\gamma^{(0)}$	mass m_γ

Riemann-Hilbert problem

- classical limit $\hbar \to 0$, UV limit $\theta \to \infty$
  $s(\Pi_\gamma)(\hbar) \to \Pi_\gamma^{(0)}$ the classical period
  $\epsilon_\gamma(\theta) \to m_\gamma e^\theta$
- Asymptotic expansions
- Global analytic structure(singularity and discontinuity)

Wall-crossing of TBA equations

[Gaiotto-Moore-Neitzke, Alday-Maldacena-Sever-Vieira, ...]

- the Schrödinger equation $\Longleftrightarrow$ the quantum Seiberg-Witten curve
- the WKB period Π_γ $\Longleftrightarrow$ the SW period a_γ (BPS charge)
- Wall-crossing of TBA $\Longleftrightarrow$ $a_{\gamma+\gamma'} = a_\gamma + a_{\gamma'}$ $(a_\gamma / a_{\gamma'} \in \mathbf{R})$

  ϵ_γ, $\epsilon_{\gamma'}$, $\epsilon_{\gamma+\gamma'}$
- $V(x) = x^{2M} + u_1 x^{2M-1} + \cdots \to \cdots \to V(x) = x^{2M} - E$

  minimal chamber $\to \cdots \to$ maximal chamber

  TBA for Homogeneous Sine-Gordon model $\to \cdots \to$ A_{2M-1} TBA

  [Castro-Alvaredo-Fring-Korff-Miramontes] $\to \cdots \to$ [Al.B.Zamolodchikov]

Spectral problem of the Schrödinger equation

Exact Quantization Conditions + Quantum WKB periods solve the spectral problem of quantum mechanics exactly. [Voros, Silverstone, Delebaere-Dillinger-Pham ,...]

Pseudo energy obtained from the TBA system includes the full perturbative+non-perturbative information.

- monomial potential [Dorey-Dunning-Tateo, ...]
- polynomial potential [I-Marinõ-Shu, Emery, ...]

It would be interesting to explore more general spectral problems:

- higher order ODE
- 3D problem
  the Stark effect $V = -\frac{k}{r} + Fz$ [I-Yang, 2307.03504]
- SUSY quantum mechanics (fermionic dof)
- effective potential $V_{eff} = V_0 + \hbar V_1 + \hbar^2 V_2 + \cdots$

Deformed SUSY QM

Deformed SUSY QM

- $x(t)$ and N_f fermions $\psi_i(t)$ $(i = 1, \ldots, N_f)$

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(W')^2 + \psi_i^\dagger(\partial_t + W'')\psi_i$$

- Superpotential $W(x)$: a polynomial in x of order N
- Integrating out fermions yields the effective potential

$$V_{\text{eff}}(x) = \frac{1}{2}\big(W'(x)\big)^2 + m\hbar W''(x)$$

with $m = (2k - N_f/2)$ $(k = 0, \ldots, N_f)$.

Here we regard m as a continuous parameter. Deformed SUSY QM

- Exact WKB analysis [Behtash-Dunne-Schaefer-Sulejmanpasic-Ünsal, Fujimori-Kamata-Mizumi-Nitta-Sakai, Kamata-Misumi-Sueishi-Ünsal]
- odd $\hbar$-power terms appear in the quantum periods
- The ground state $E = 0$ (degenerate of curve and divergence in quantum corrections)

WKB expansion

Schrödinger eq. with effective potential:

$$\Big(-\hbar^2 \frac{d^2}{dx^2} + Q_0(x) + \hbar Q_1(x) \Big) \psi(x) = 0,$$

WKB solution:

$$\psi(x) = \exp \Big(\frac{1}{\hbar} \int P(x') dx' \Big), \quad P(x) = \sum_{n=0}^{\infty} \hbar^n p_n(x)$$

$$\begin{aligned}
p_0 &= \pm \sqrt{Q_0}, \\
p_1 &= \frac{Q_1}{2p_0} - \frac{1}{2} \frac{d}{dx} \log p_0, \\
p_2 &= -\frac{Q_1}{8Q_0^{3/2}} + \frac{Q_0''}{48Q_0^{3/2}} + d(*), \\
p_3 &= \frac{Q_1^3}{16Q_0^{5/2}} - \frac{Q_1 Q_0''}{32Q_0^{5/2}} + \frac{Q_1''}{48Q_0^{3/2}} + d(*),
\end{aligned}$$

Quantum periods

WKB curve $y^2 = Q_0(x)$

$$\Pi_\gamma^{(n)} = \oint_\gamma p_n(x)dx, \quad p_n dx = \sum_{a=0}^{2N-4} B_a^{(n)} \frac{x^a}{y} dx + d(*)$$

cubic superpotential $W(x) = x^3/3 - \frac{u_2}{2}x$

$$Q_0(x) = x^4 - u_2 x^2 + \frac{u_2^2}{4} - 2E, \quad Q_1(x) = 4mx.$$

turning points $\pm a$, $\pm b$: $a = \sqrt{\frac{u_2}{2} + \sqrt{2E}}, \quad b = \sqrt{\frac{u_2}{2} - \sqrt{2E}}.$

$$\Pi_\gamma^{(0)} = \frac{2a^2b^2}{3}\Pi_{\gamma,0} - \frac{a^2+b^2}{3}\Pi_{\gamma,2},$$

$$\Pi_\gamma^{(1)} = 2m\Pi_{\gamma,1},$$

$$\Pi_\gamma^{(2)} = \frac{(a^2+b^2)(-1+12m^2)}{6(a^2-b^2)^2}\Pi_{\gamma,0} - \frac{a^4+b^4+2a^2b^2(-5+48m^2)}{24a^2b^2(a^2-b^2)^2}\Pi_{\gamma,2},$$

Elliptic integral: $\Pi_{\gamma,a} = \oint_\gamma \frac{x^a}{y}dx$

ODE/IM correspondence and deformed TBA

ODE

$$\Big(-\frac{d^2}{dz^2}+\big(\hat{W}'(z)\big)^2-2\hat{E}+2\hat{m}\hat{W}''(z)\Big)\hat{\psi}(z)=0,\ \hat{W}(z)=\sum_{a=0}^{N}b_a z^a.$$

- invariant under the rotation

$$(z,b_a,\hat{E},\hat{m})\to(\omega z,\omega^{N-a}b_a,\omega^{2N-2}\hat{E},\omega^N\hat{m}),\quad \omega=e^{\frac{2\pi i}{2N}}$$

- $\hat{y}(z,b_a,\hat{E},\hat{m})$: the subdominant solution along the positive real axis

$$\hat{y}(z,b_a,\hat{E},\hat{m})\sim\frac{1}{2i}z^{n_N}\exp\Big(-\frac{z^{2N}}{N}\Big),$$

$$\hat{y}_k(z,b_a,\hat{E},\hat{m})=\omega^{\frac{k}{2}}\hat{y}(\omega^{-k}z,\omega^{-k(N-a)}b_a,\omega^{-k(2N-2)}\hat{E},\omega^{-kN}\hat{m}).$$

the subdominant solution in the sector $\mathcal{S}_k:|\arg(z)-\frac{2k\pi}{2N}|<\frac{\pi}{2N}\ (k\in\mathbb{Z})$

Y-system for SUSY QM

Rescaling the variables:

$$x = \hbar^{\frac{2}{2N}} z, \quad u_a = \hbar^{\frac{2N-2a}{2N}} b_a, \quad E = \hbar^{2\frac{2N-2}{2N}} \hat{E}, \quad m = \hat{m},$$

The Schrödinger equation:

$$\Big(-\hbar^2 \frac{d^2}{dx^2} + \big(W'(x)\big)^2 - 2E + \hbar 2m W''(x) \Big) \psi(x) = 0,$$

The basis of the subdominant solutions

$$y(x, u_a, E, m, \hbar) \equiv \hat{y}(z, b_a, \hat{E}, \hat{m}), \quad y_k(x, u_a, E, m, \hbar) \equiv \omega^{\frac{k}{2}} y(x, u_a, E, e^{-i\pi k} m, e^{i\pi k} \hbar).$$

Y-functions

$$Y_{2j}(\hbar, u_a, E, m) = \frac{W_{-j,j} W_{-j-1,j+1}}{W_{-j-1,-j} W_{j,j+1}} (\hbar, u_a, E, m),$$

$$Y_{2j+1}(\hbar, u_a, E, m) = \frac{W_{-j-1,j} W_{-j-2,j+1}}{W_{-j-2,-j-1} W_{j,j+1}} (\hbar, u_a, E, m),$$

Y-system

$$Y_s(e^{\frac{\pi i}{2}} \hbar, e^{-\frac{\pi i}{2}} m) Y_s(e^{-\frac{\pi i}{2}} \hbar, e^{\frac{\pi i}{2}} m) = \big(1 + Y_{s-1}\big)\big(1 + Y_{s+1}\big)(\hbar, m), \quad s = 1, \ldots, 2N-3.$$

$\mathbb{Z}_4$-extended Y-system

$$Y_{a,s}(\hbar) = Y_s(\hbar, e^{\frac{a\pi i}{2}} m), \quad a = 0, 1, 2, 3,$$

$$Y_{0,s}(\theta - \frac{\pi i}{2}) Y_{2,s}(\theta + \frac{\pi i}{2}) = \big(1 + Y_{1,s-1}\big)\big(1 + Y_{1,s+1}\big)(\theta),$$
$$Y_{1,s}(\theta - \frac{\pi i}{2}) Y_{3,s}(\theta + \frac{\pi i}{2}) = \big(1 + Y_{2,s-1}\big)\big(1 + Y_{2,s+1}\big)(\theta),$$
$$Y_{2,s}(\theta - \frac{\pi i}{2}) Y_{0,s}(\theta + \frac{\pi i}{2}) = \big(1 + Y_{3,s-1}\big)\big(1 + Y_{3,s+1}\big)(\theta),$$
$$Y_{3,s}(\theta - \frac{\pi i}{2}) Y_{1,s}(\theta + \frac{\pi i}{2}) = \big(1 + Y_{0,s-1}\big)\big(1 + Y_{0,s+1}\big)(\theta)$$

$s = 1, \ldots, 2N - 3.$

$\mathbb{Z}_4$-extended TBA for SUSY QM

Asymptotics $\theta \to -\infty$

$$\log Y_{a,2k+1} \sim -\frac{1}{i\hbar}\oint_{\gamma_{2k+1}} p_0 dx - i^a \oint_{\gamma_{2k+1}} p_1 dx + \mathcal{O}(\hbar) =: -\frac{m_{2k+1}}{\hbar} + m^{(\frac{1}{2})}_{a,2k+1} + \mathcal{O}(\hbar)$$

$$\log Y_{a,2k} \sim -\frac{1}{\hbar}\oint_{\gamma_{2k}} p_0 dx - i^a \oint_{\gamma_{2k}} p_1 dx + \mathcal{O}(\hbar) =: -\frac{m_{2k}}{\hbar} + m^{(\frac{1}{2})}_{a,2k} + \mathcal{O}(\hbar),$$

TBA equations

$$\log Y_{a,s} = -m_s e^{\theta} + m^{(\frac{1}{2})}_{a,s} + K_+ \star L_{a+1,s-1} + K_+ \star L_{a+1,s+1} + K_+ \star L_{a+3,s-1} + K_+ \star L_{a+3,s+1}, \quad a \equiv a+4,$$

$L_{a,s}(\theta) = \log\big(1 + Y_{a,s}(\theta)\big)$.

Kernel function: $K_\pm(\theta) = \frac{1}{4\pi}\left(\frac{1}{\cosh\theta} \pm i\frac{\sinh\theta}{\cosh\theta}\right)$.
constant in source term

cubic superpotential

superpotential $W(x) = \frac{1}{3}x^3 - \frac{1}{4}x$

$$\Big(-\hbar^2 \frac{d^2}{dx^2} + \big(x^2 - \frac{1}{4}\big)^2 - 2E + 4\hbar m x \Big)\psi(x) = 0.$$

turning points: $-a, -b, b, a$, ($a = \sqrt{1/4 + \sqrt{2E}}$, $b = \sqrt{1/4 - \sqrt{2E}}$)

TBA equations decouple into two TBAs:

$$\begin{aligned}
\log Y_{0,1} &= -\, m_1 e^{\theta} + 2\pi i m + K \star L_{1,2}, \\
\log Y_{2,1} &= -\, m_1 e^{\theta} - 2\pi i m + K \star L_{1,2}, \\
\log Y_{1,2} &= -\, m_2 e^{\theta} + K \star \Big[L_{0,1} + L_{2,1} \Big]
\end{aligned}$$

$$\begin{aligned}
\log Y_{1,1} &= -\, m_1 e^{\theta} - 2\pi m + K \star L_{0,2}, \\
\log Y_{3,1} &= -\, m_1 e^{\theta} + 2\pi m + K \star L_{0,2}, \\
\log Y_{0,2} &= -\, m_2 e^{\theta} + K \star \Big[L_{1,1} + L_{3,1} \Big],
\end{aligned}$$

$m_1 = -\frac{2}{a}K(k)$, $m_2 = \frac{4}{a}K(k')$, $k = \sqrt{a^2 - b^2}/a$, $k' = b/a$

D_3-type TBA

$$\hat{Y}_{0,1} = e^{-2\pi i m} Y_{0,1} = e^{2\pi i m} Y_{2,1}, \quad \hat{Y}_{1,1} = e^{2\pi m} Y_{1,1} = e^{-2\pi m} Y_{3,1},$$

Two D_3-type TBA systems:

$$\begin{aligned}\log \hat{Y}_{0,1} &= -m_1 e^{\theta} + K \star \log\left(1 + Y_{1,2}\right),\\ \log Y_{1,2} &= -m_2 e^{\theta} + K \star \log\left(1 + e^{2\pi i m} \hat{Y}_{0,1}\right) + \log\left(1 + e^{-2\pi i m} \hat{Y}_{0,1}\right)\end{aligned}$$

$$\begin{aligned}\log \hat{Y}_{1,1} &= -m_1 e^{\theta} + K \star \log\left(1 + Y_{0,2}\right),\\ \log Y_{0,2} &= -m_2 e^{\theta} + K \star \log\left(1 + e^{-2\pi m} \hat{Y}_{1,1}\right) + \log\left(1 + e^{2\pi m} \hat{Y}_{1,1}\right).\end{aligned}$$

TBA

- $m = 0$ double-well potential, $m = \frac{1}{2}$ SUSY QM
- Asymptotic expansions match with the exact WKB periods
- $\theta \to -\infty$ limit: $\hat{Y}^*_{0,1} = 2\cos\!\left(\frac{2\pi m}{3}\right), \quad Y^*_{1,2} = \frac{\sin(2\pi m)}{\sin\left(\frac{2\pi m}{3}\right)}$
- effective central charge $c_{\text{eff}}(m) = 4(1 \pm 8m^2)$
- PNP relation $\Pi^{(0)}_{\gamma_1}\Pi^{(2)}_{\gamma_2} - \Pi^{(0)}_{\gamma_2}\Pi^{(2)}_{\gamma_1} = -\frac{\pi i}{3}(1 - 8m^2)$
- $|m| > \frac{1}{2}$ analytic continuation of TBA (excited TBA)
  [Dorey-Tateo, BLZ,Fendley, Gavai-Yin, I-Yang]
- $E \to 0$ limit or $m_1 \to 0$ limit of TBA can be taken.
  [Fendley 9706161] SUSY index of $N = 2$ super sine-Gordon model

Asymptotic expansions

Exact WKB period: $\Pi_\gamma = \Pi_\gamma^{(0)} + \Pi_\gamma^{(\frac{1}{2})}\hbar + \sum_{n=1}^{\infty} \hbar^{2n}\Pi_\gamma^{(n)}$

Y-function: $-\log Y_{a,s}(\theta) \sim m_s e^\theta - m_{a,s}^{(\frac{1}{2})} + \sum_{n=1}^{\infty} m_{a,s}^{(n)} e^{(1-2n)\theta}$,

$$m_{0,1} = \frac{1}{i}\Pi_{\gamma_1}^{(0)}, \quad m_{1,2} = \Pi_{\gamma_2}^{(0)}, m_{0,1}^{(\frac{1}{2})} = 2\pi i m, \quad m_{1,2}^{(\frac{1}{2})} = 0$$

$$m_{0,1}^{(n)} = (-1)^n \frac{1}{i}\Pi_{\gamma_1}^{(2n)}(m), \quad m_{1,2}^{(n)} = \Pi_{\gamma_2}^{(2n)}(m).$$

n	$m_{0,1}^{(n)}$	$\Pi_{\gamma_1}^{(n)}/i$	$m_{1,2}^{(n)}$	$\Pi_{\gamma_2}^{(n)}$
0	0.103932897990	0.103932897990	0.146983313914	0.146983313914
1	-0.595015127256	0.595015127355	10.917186649450	10.917186650161
2	$1.135557990286 \cdot 10^2$	$1.135557990512 \cdot 10^2$	$-1.258067561652 \cdot 10^3$	$-1.258067561659 \cdot 10^3$
3	$-7.852584078608 \cdot 10^4$	$7.852584080303 \cdot 10^4$	$1.299686482674 \cdot 10^6$	$1.299686482683 \cdot 10^6$

$m = 1/2,\ E = 1/64$

n	$m_{0,1}^{(n)}$	$\Pi_{\gamma_1}^{(n)}/i$	$m_{1,2}^{(n)}$	$\Pi_{\gamma_2}^{(n)}$
0	0	0	$1/3$	$1/3$
1	-1.507964473727	1.507964473723	$-1.904158772073 \cdot 10^{21}$	∞
2	$3.272282907987 \cdot 10$	$3.272282907979 \cdot 10$	$1.620951043969 \cdot 10^{64}$	∞
3	$-3.710585147572 \cdot 10^3$	$3.710585147575 \cdot 10^3$	$-2.573503923004 \cdot 10^{107}$	∞

$m = 1/10,\ E = 0$

Exact Quantization Condition and Voros spectrum

Exact quantization condition: [Zinn-Justin, Alvarez, ...]

$$\frac{1}{\hbar} s_{\mathrm{med}}\Big(\frac{1}{i}\Pi_{\gamma_1}(\hbar)\Big) + \epsilon \arctan\left(e^{-\frac{1}{2\hbar}s(\Pi_{\gamma_2}(\hbar))}\right) = 2\pi(k+\frac{1}{2}), \quad k \in \mathbb{Z}_{\geq 0}$$

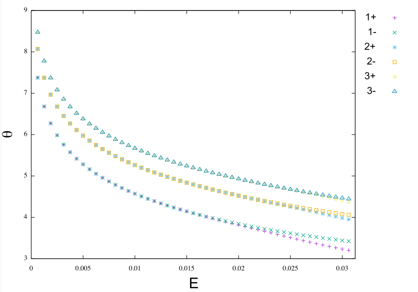
parity parameter $\epsilon = \pm 1$.

For (E, m) and k_ϵ, $\theta_{k_\epsilon} = -\log \hbar_{k_\epsilon}$ is determined (Voros spectrum).

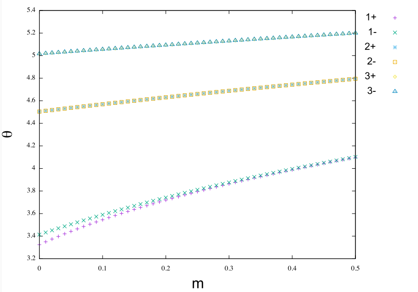
k_ϵ	θ_{k_ϵ}	E_{k_ϵ}
1_+	4.098461939440	0.015625000025
1_-	4.101928506979	0.0156250000105
2_+	4.794626227596	0.015625000012
2_-	4.794647371948	0.015625000011
3_+	5.200323939648	0.015625000010
3_-	5.200323939648	0.015625000010

Diagonalization of the Hamiltonian in terms of eigenfunctions of the harmonic oscillator [Emery,Okun-Burke] ($E = 1/64$, $u_2 = 1/2$)

Voros spectrum from TBA



$m = 1/2$



$E = 1/64$

- $0 < E < \frac{1}{32}$, $0 < m < \frac{1}{2}$
- splitting of energies with different parity (tunneling effects)

Conclusion and Outlook

Conclusion and Outlook

- effective potential: enxtended symmetry in TBA
  - deformed SUSY $\mathbb{Z}_4$-extension of TBA
  - centrifugal potential $A_r \to D_{r+1}$ TBA
- general superpotential $W(x)$
  WKB curve $y^2 = (W'(x))^2 - 2E$: $SU(n)$ SW theory
- $O(\hbar^n)$ deformation of the potential
- higher order ODE
- TBA for degenerate WKB curve
- ODE/IM correspondence for SUSY integrable field theories

Higher order ODE and Duality

- Lax representation affine Toda field equation $\hat{\mathfrak{g}}$
  Linear problem $\mathcal{L}\psi = 0 \Longleftrightarrow$ BAE for $\hat{\mathfrak{g}}^{\vee}$
  Langlands dualty [Dore-Dunning-Masoero-Suzuki-Tateo, Ito-Locke,...]
- WKB expansion p_n of the Linear problem $\Longleftrightarrow$ classical conserved charges in Drinfeld-Sokolov hierarchy [Ito-Zhu 2408.12917]
  $\int p_n(x)dx =$ quantum integrals of motion of $W\mathfrak{g}$-algebra
  ODE/IM =correspondence between classical and quantum IMs
- Duality of AD theories [Cecotti-Neitzke-Vafa]
  $(A_r, A_1) \sim (A_1, A_r)$, $(A_2, A_2) \sim (D_4, A_1) \sim (A_1, D_4)$,
  $(A_2, A_3) \sim (E_6, A_1)$ [Ito-Kondo-Shu, Ito-Yang 2408.01124]

$$(-\frac{d^{r+1}}{dx^{r+1}} + W_G(x))\psi(x) = 0 \quad (A_r, G)$$