Infinitely many new renormalization group flows between Virasoro minimal models from non-invertible symmetries

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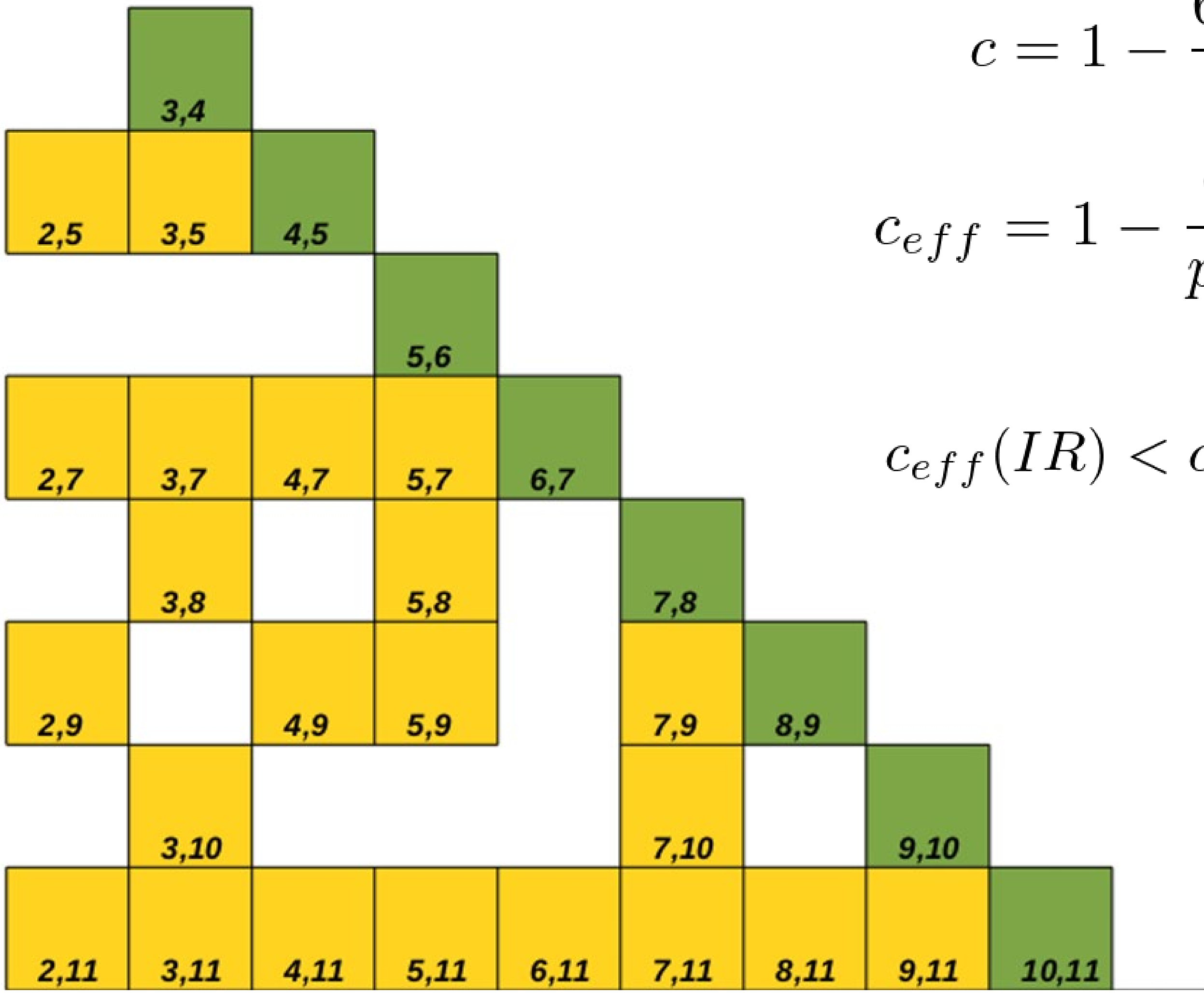
Virasoro minimal models

- We believe we know everything about Virasoro minimal models
- Specified by two coprime integers $\mathcal{M}(p,q)$
- Central charge: $c = 1 - \frac{6(p-q)^2}{pq}$
- When $q = p \pm 1$, unitary $h_{r,s} = h_{q-r,p-s} = \frac{(pr-qs)^2-(p-q)^2}{4pq}.$
- Chiral spectrum from Virasoro rep theory
- Full spectrum: ADE classification
- OPE is known (A-series by Dotsenko-Fateev, D and E, Petkova)

- But RG flows between them are poorly understood

RG flow

Do you know which theory is connected to which by RG flow?



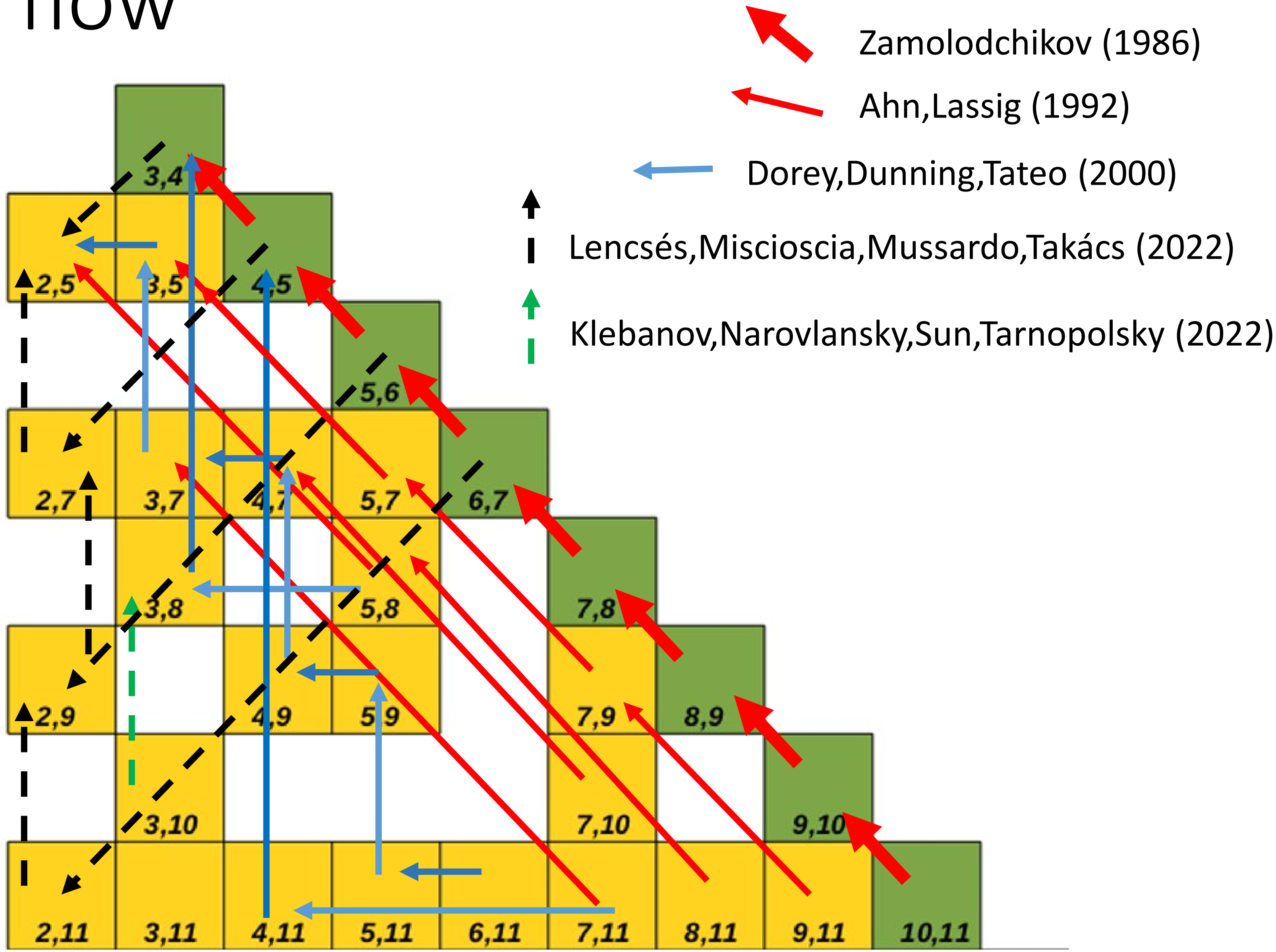
$$c = 1 - \frac{6(p-q)^2}{pq}$$

$$c_{eff} = 1 - \frac{6}{pq}$$

$$c_{eff}(IR) < c_{eff}(UV)$$

(7,11) → (2,5)

RG flow



Zamolodchikov (1986)
Ahn,Lassig (1992)
Dorey,Dunning,Tateo (2000)
Lencsés,Miscioscia,Mussardo,Takács (2022)
Klebanov,Narovlansky,Sun,Tarnopolsky (2022)
3,4
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Infinitely many new RG flows

$$\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q) \quad \text{by} \quad \phi_{1,2k+1}$$

- This is a (infinite) generalization or unification of known (integrable) RG flows
  - Zamolodchikov: $k = I = 1$
  - Ahn, Lassig: $k = 1, I > 1$
  - Dorey,Dunning,Tateo: $k = 2, k = 1/2$
  - Klebanov et al, $k = 3, I = 1, q = 3$ (integrability unknown)
- Conservation of non-invertible symmetry is the key
- All the RG flows preserving $SU(2)_{q-2}$ categorical symmetry are classified by our flows (no less, no more)
- All the known RG flows preserving (invertible) $\mathbb{Z}_2$ are our flows!

Non-invertible symmetries

Invertible symmetries

- Wigner claimed "symmetry" in quantum mechanics must be (projective) unitary (from conservation of probabilities) $U^{-1} = U^\dagger$
- If the unitary operator commutes with Hamiltonian, it gives a conservation of some charge $[U, H] = 0$

(can be a bit more complicated if the symmetry does not commute with the Hamiltonian but shows conservation like Lorentz boost or dilatation)

- Quantum analogue of Noether's theorem
- Useful to understand RG flows
- As we will see (most) Virasoro minimal models have (only) $\mathbb{Z}_2$ invertible symmetry

Non-Invertible symmetries

- Unitary (→ invertible) $U^{-1} = U^\dagger$
- Commute with Hamiltonian (conservation) $[U, H] = 0$
- We realize abandoning "unitary" may still give something very useful
- Non-invertible q-form symmetry: topological $D - q - 1$

dimensional objects (or topological defects) in QFT_D
(topological → conservation)

- Example: topological lines that have non-trivial fusion rule

$$O_1 \times O_2 = \sum_i O_i$$

- Generically non-invertible (categorical symmetry is a better name…), and cannot be unitary
- But (as long as they are preserved) they can be as useful to understand RG structure as invertible symmetries
  - EX: non-invertible defects in massless QED → non-renormalization of the chiral ABJ anomaly $\pi_0 \to 2\gamma$ (Cordova-Ohmori)

But...

- Unless the original QFT is topological, it is not so easy to find topological defects in general QFTs
- For invertible symmetries, we can use Noether theorem to find topological defects (= conserved currents integrated over a time slice)

$$U = \exp(i \int d^{d-1}x j^0)$$

- No systematic constructions of general topological defects (symTFT approach?)
- For 2D CFTs one can spell out the conditions, but solving them is not so easy
- EXCEPT for A-series Virasoro minimal models where we believe all the topological defect lines are given by Verlinde lines

Non-invertible symmetries in A-series minimal models

Invertible symmetries in Virasoro minimal models $\mathcal{M}(p,q)$

$$c = 1 - \frac{6(p-q)^2}{pq}$$

- Convention: Unlike yellowbook , we always fix the order of p and q

$$\mathcal{M}(5,4) \to \mathcal{M}(3,4) \qquad h_{r,s} = h_{q-r,p-s} = \frac{(pr-qs)^2-(p-q)^2}{4pq}.$$

- Fusion rules:

$$\phi_{(r,s)} \times \phi_{(m,n)} = \sum_{\substack{k=1+|r-m| \\ k+r+m=1 \mod 2}}^{\min(r+m-1,2q-1-r-m)} \sum_{\substack{l=1+|s-n| \\ l+s+n=1 \mod 2}}^{\min(s+n-1,2p-1-s-n)} \phi_{(k,l)}.$$

- Due to the operator identification, the symmetry of (A-series) minimal model is $\mathbb{Z}_2$ (see e.g. Lassig)
  - (Even,Odd): r-1 mod 2
  - (Odd,Even): s-1 mod 2
  - (Odd,Odd): r+s-1 mod2 (anomalous in the 't Hooft sense)
- $\mathbb{Z}_2$ of (Odd,Odd) model cannot be gauged (if it were gauged, D-series should exist, but (Odd,Odd) has only A-series modular invariant partition function)
- No RG flows between (Odd,Even) and (Odd,Odd) unless we break $\mathbb{Z}_2$

Verlinde lines in (A-series) minimal models

- It has one-to-one correspondence with (chiral) Virasoro character

$$h_{r,s} = h_{q-r,p-s} = \frac{(pr-qs)^2-(p-q)^2}{4pq}.$$

- Action of topological lines on states

$$L_{(r,s)}\left|\phi_{(\rho,\sigma)}\right\rangle = \frac{S_{(r,s),(\rho,\sigma)}}{S_{(1,1),(\rho,\sigma)}}\left|\phi_{(\rho,\sigma)}\right\rangle ,$$

- Explicit modular S-matrix can be found in any CFT textbook

$$S_{(r,s),(\rho,\sigma)} = 2\sqrt{\frac{2}{pq}}(-1)^{1+s\rho+r\sigma}\sin\Big(\pi\frac{p}{q}r\rho\Big)\sin\Big(\pi\frac{q}{p}s\sigma\Big),$$

- Fusion rule is same as

(chiral) Virasoro fusion rule

$$\begin{aligned} L_{(r,s)}\times L_{(m,n)} &= \sum N^{(k,l)}_{(r,s)(m,n)}L_{(k,l)} \\ &= \sum_{\substack{k=1+|r-m| \\ k+r+m=1 \mod 2}}^{\min(r+m-1,2q-1-r-m)} \sum_{\substack{l=1+|s-n| \\ l+s+n=1 \mod 2}}^{\min(s+n-1,2p-1-s-n)} L_{(k,l)}. \end{aligned}$$

- Example: Duality topological defect lines → Tambara-Yamagami fusion category

$$\eta\times\eta = 1+N \;,\eta\times N = \eta \;,N\times N = 1$$

- The consistency (e.g. Cardy condition) is guaranteed by the Verlinde formula

$$N_{ab}^{c} = \sum_{d}\frac{S_{ad}S_{bd}S_{dc}}{S_{0d}},$$

RG constraint from non-invertible symmetries

- Assume the deformation preserves a topological defect line

$$L_a \phi_b |\Phi\rangle = \phi_b L_a |\Phi\rangle \quad \text{on any states } |\Phi\rangle$$

(in unitary theories checking on vacuum is sufficient)

- Assume CFT1 becomes CFT2 (in our case we assume it will be another A-series Virasoro minimal model)
- What kind of properties of topological defect lines are preserved?
  - Quantum dimensions of topological defect lines
  - Spin contents of topological defect lines

  (It will turn out that the constraints are the same)

  See e.g. Chang-Lin-Shao-Wang-Yin

Quantum dimensions of topological defect lines

- Defined by the action of topological defect lines on the vacuum states

$$L_{(r,s)}\left|0\right\rangle = d_{(r,s)}\left|0\right\rangle = \frac{S_{(r,s),(1,1)}}{S_{(1,1),(1,1)}}\left|0\right\rangle ,$$

- Interpret it as the expectation value on the cylinder

$$d_a = \langle 0|L_a|0\rangle = \langle L_a\rangle \qquad a := (r,s)$$

- Satisfies the fusion constraints (→ only discrete solutions)

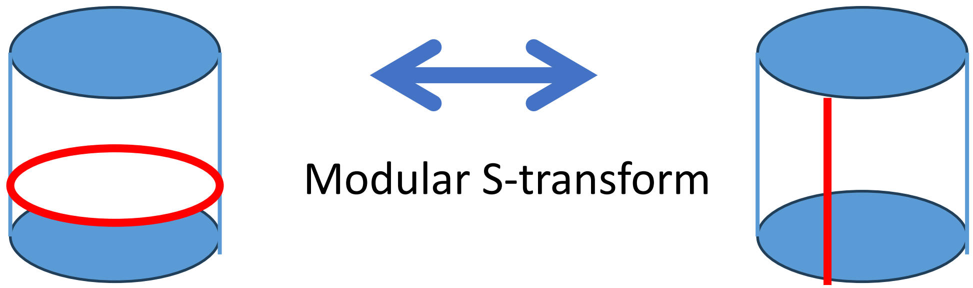
$$\langle L_a\rangle\langle L_b\rangle = \sum_d N_{ab}^d \langle L_d\rangle$$

- E.g. invertible Z2 symmetry $\langle L_{\mathbb{Z}_2}\rangle\langle L_{\mathbb{Z}_2}\rangle = \langle 1\rangle$
  - Plus → non-anomalous
  - Minus → anomalous

$$\langle L_{\mathbb{Z}_2}\rangle = \pm 1$$

- Cannot be changed by continuous deformations → RG invariants (rigidity of modular tensor category)

Spin contents of Verlinde lines



- We focus on spin contents of the defect Hilbert space

$$Z_{L_a}(\tau,\bar{\tau}) = \sum_{b,c,d} \frac{S_{ab}}{S_{0b}} S_{bc} S_{bd} \chi_c(\tilde{\tau}) \bar{\chi}_d(\bar{\tilde{\tau}})$$
$$= \sum_{c,d} N^a_{cd} \chi_c(\tilde{\tau}) \bar{\chi}_d(\bar{\tilde{\tau}})$$

- Spin $h_c - \bar{h}_d$ in defect Hilbert space may not be (half) integer (but OK)
- Since RG flow commute with rotation, spin contents (mod integer) will be conserved! (← novel RG constraints from categorical symmetry!)

New RG flows from non-invertible symmetries

$\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q)$ by $\phi_{1,2k+1}$

- We can show $\phi_{1,2k+1}$ or more generally $\phi_{1,2l+1}$ preserves $L_{(1,1)}, \cdots, L_{(q-1,1)}$ (when k is integer)

$$L_a \phi_b |\Phi\rangle = \phi_b L_a |\Phi\rangle$$

- Proof: direct computation

$$L^{UV}_{(i,1)} = L^{IR}_{(i,1)} \qquad L_{(q-1,1)} = \mathbb{Z}_2$$

- We can show that under these proposed RG flows
  - Quantum dimensions of $L_{(i,1)}$ are preserved
  - Spin contents of $L_{(i,1)}$ are preserved
- We can further show that the proposed flows are sufficiently fine-grained: each connected RG flows have different quantum dimensions/spin contents

Quantum dimensions under our RG flows

- In $\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q)$ by $\phi_{1,2k+1}$
  $L_{(1,1)}, \cdots, L_{(q-1,1)}$ are preserved
- Check of the matching of quantum dimensions of $L_{(r,1)}$

$$\frac{d^{UV}_{(r,1)}}{d^{IR}_{(r,1)}} = \frac{\sin(\pi\frac{kq+I}{q}r)}{\sin(\pi\frac{kq-I}{q}r)}\frac{\sin(\pi\frac{kq-I}{q})}{\sin(\pi\frac{kq+I}{q})} = \frac{\sin(\pi\frac{I}{q}r)}{\sin(-\pi\frac{I}{q}r)}\frac{\sin(-\pi\frac{I}{q})}{\sin(\pi\frac{I}{q})} = 1.$$

- They have different quantum dimensions. The most severe constraint comes from $L_{(2,1)}$

$$d_{(2,1)} = -2\cos\left(\frac{p}{q}\pi\right)$$

- Takes a distinct values for different p (mod q) with a given q. (There exist $\varphi(q)$ different RG paths)

Examples and physical interpretations

$\mathcal{M}(3k+I,3) \rightarrow \mathcal{M}(3k-I,3)$ and anomaly matching

- Preserved topological line defect is only $\mathbb{Z}_2$ invertible symmetry $L_{(2,1)} = \mathbb{Z}_2$
- Our RG flows predict (Odd, 3) → (Odd,3) and (Even,3) → (Evem,3)
- They are classified by the quantum dimensions of $L_{(2,1)}$

$$d_{(2,1)} = \pm 1$$

- This is nothing but the 't Hooft anomaly matching
  - Recall (Odd,Odd) case is anomalous
- Our RG flow knows 't Hoof anomaly matching

$\mathcal{M}(4k+I,4) \to \mathcal{M}(4k-I,4)$ and duality defects

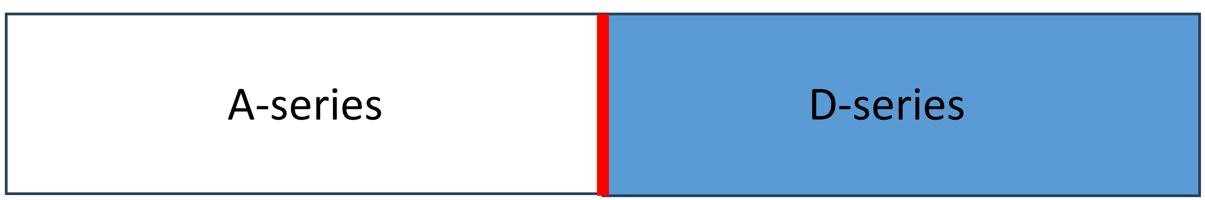
- Preserved topological defect lines are $\mathbb{Z}_2$ invertible symmetry $L_{(3,1)}$ and non-invertible "duality defect" $L_{(2,1)}$
- That has Tambara-Yamagami (=Ising) fusion rule

$$N = L_{(3,1)}$$

$$\eta \times \eta = 1 + N \;, \eta \times N = \eta \;, N \times N = 1$$

$$\eta = L_{(2,1)}$$

- Only (p,4) minimal models have a duality defect
- Suppose we gauge $\mathbb{Z}_2$ in half space-time

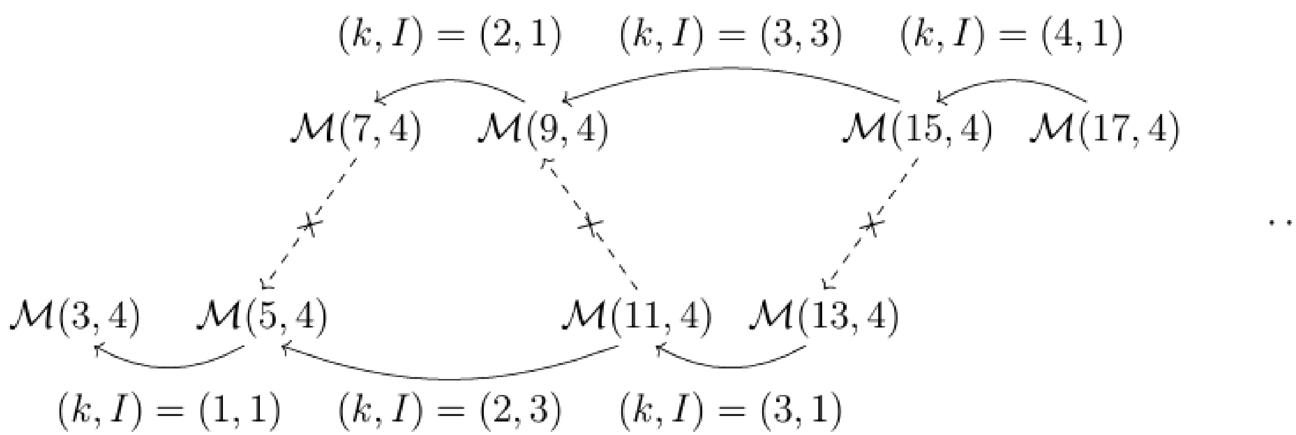


- If $\mathbb{Z}_2$ is non-anomalous we will get D-series minimal model
- But only in (p,4), A-series and D-series are same (self-dual)
- Half gauging gives the non-trivial topological defect line

$$\eta = L_{(2,1)}$$

$\mathcal{M}(4k+I,4) \to \mathcal{M}(4k-I,4)$ and duality defects

- We have two distinct duality defect lines
- Quantum dimensions $d_{(2,1)} = \pm\sqrt{2}$ distinguish $\varphi(4) = 2$ distinct RG flows



- If you study the spectrum, the number of "singlet" relevant deformations decrease one by one along the proposed flow (but not in the forbidden flow)
- Dotted arrows can be realized in half integer k flow which breaks the duality symmetry

Application: fate of non-SUSY Yukawa fixed point (Nakayama-Kikuchi)

- Study Yukawa theory in d=4-epsilon (Fei-Giombi-Klebanov-Tarnopolsky)

$$S = \int d^d x (\partial_\mu \phi \partial^\mu \phi + i\bar{\psi}\gamma^\mu \partial_\mu \psi + g_1 \phi \bar{\psi}\psi + g_2 \phi^4)$$

- One (stable) fixed point is supersymmetric → fermionic $\mathcal{M}(5,4)$ in d=2
- The other (unstable) fixed point without SUSY → fermionic $\mathcal{M}(?,4)$ in d=2?
- Chiral symmetry = non-invertible duality defect
- Must flow to $\mathcal{M}(5,4)$
- Cannot be $\mathcal{M}(7,4)$ or $\mathcal{M}(9,4)$ but $\mathcal{M}(11,4)$!

Discussions and Conclusions

Comments on integrability

- $\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q)$ by $\phi_{1,2k+1}$
- I = 1 case seems integrable in the TBA sense
- I > 1 case: no TBA is known, but may be integrable in the sense of "integral equations" (e.g. Dorey-Dunning-Tateo)
- I need your help

Massive flow and TQFTs

- $\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q)$ by $\phi_{1,2k+1}$
- One sign gives massless flow (our flow)
- The other sign should give massive flow
- We may study massive excitations of S-matrix (e.g. Coppeti-Cordova-Komatsu), vacuum structure and IR TQFTs
- (Due to the non-invertible symmetry, "ground states" must be degenerate → Non-trivial TQFTs)
- I need your help

Puzzle in (2,q) and (p,2)

- $\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q)$ by $\phi_{1,2k+1}$
- We can formally put q=2: (multi-critical Lee-Yang flow)
  $$\mathcal{M}(2k+I,2) \to \mathcal{M}(2k-I,2)$$
- q = 2 case has no $\mathbb{Z}_2$
- But $kq-I$ can be 2
- $\mathbb{Z}_2$ preserving deformations of $\mathbb{Z}_2$ symmetric theory gives no $\mathbb{Z}_2$? SSB or decoupling?

Summary

- Non-invertible symmetries give very powerful constraint on RG flows

- Infinitely many constraints and classifications than just invertible symmetries

- Other 2D CFTs?
- Even in unitary minimal models E-series flows are barely understood
- There should be very powerful constraints in higher dimensions from non-invertible symmetries (if we can find them systematically)