Journal Club

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노트 제목 2009-11-23

Gaiotto Conjecture

$\boxed{d=4\ \ \mathcal{N}=2\ \ U(N)\ \text{gauge theory}}$

$\updownarrow$

compactify $\underbrace{d=6\ \ \mathcal{N}=(2,0)\ \text{theory}}$ on a $\underline{\text{Riemann surface } C}$

↓ low energy limit of N coincident M-5 branes

↓ have punctures labeled by Young

- Seiberg-Witten curve Σ on C : N-folded cover

$$x^N + \phi_2(z)\, x^{N-2} + \cdots + \phi_N(z) = 0$$

ϕ_k : deg. k differential

AGT conjecture : $\underline{N=2}$

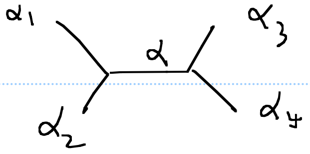
2d conformal block on C vs instanton partition function

(ex) 4-pt Virasoro conf. block of LFT vs $\mathcal{N}=2$ SUSY quiver $U(2)$ with $N_f = 4$

ext.

$$\Delta_k = \alpha_k(Q-\alpha_k)$$

with $\epsilon_1 = b,\ \epsilon_2 = \frac{1}{b},\ Q = b + \frac{1}{b}$



int.

$$\alpha = \frac{Q}{2} + a$$

$$\mu_1 = \alpha_1 - \alpha_2 + \frac{\epsilon}{2} = m_0 - \tilde{m}_0$$

$$\mu_2 = \alpha_1 + \alpha_2 - \frac{\epsilon}{2} = m_0 + \tilde{m}_0$$

$$\mu_3 = \alpha_3 - \alpha_4 + \frac{\epsilon}{2} = m_1 - \tilde{m}_1$$

$$\mu_4 = \alpha_3 + \alpha_4 - \frac{\epsilon}{2} = m_1 + \tilde{m}_1$$

$$\epsilon = \epsilon_1 + \epsilon_2 = Q$$

scalar condensate

$a = a_1 = -a_2$ $\quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$

$$\boxed{(1-q)^{2\alpha_1(Q-\alpha_1)} \mathcal{F}\begin{matrix} \alpha_1 & & \alpha_3 \\ \alpha_2 & \alpha & \alpha_4 \end{matrix}(q) = Z_{\text{inst}}^{U(2),N_f=4}(a,\mu_i,q)} \quad q = e^{2\pi i\tau}$$

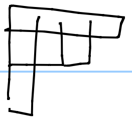
$\alpha_1 = m_0 \qquad \tilde{m}_0 + \frac{Q}{2} = \alpha_2 \to \text{"}\alpha_0\text{"}$

$m_1 = \alpha_3 \qquad \tilde{m}_1 + \frac{Q}{2} = \alpha_4 \to \text{"}\alpha_1\text{"}$

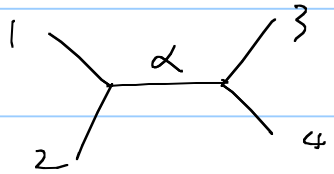
$$Z_{\text{inst}}^{U(2),N_f=4}(a,\mu_i,q) = \sum_{\vec{Y}} q^{|\vec{Y}|} Z_{\text{vector}}(\vec{a},\vec{Y}) Z_{\text{a.f.}}(\vec{a},\vec{Y},\mu_1)\cdots Z_{\text{fun}}(\vec{a},\vec{Y},\mu_4)$$

$$Z_f(\vec{a},\vec{Y},\mu) = \prod_{i=1}^{2}\prod_{s\in Y_i}\left(\phi(a_i,s)-\mu+\epsilon\right), \quad \phi(a,s) \equiv a + \epsilon_1(i-1) + \epsilon_2(j-1)$$

$$Z_{\text{a.f.}}(\vec{a},\vec{Y},\mu) = Z_f(\vec{a},\vec{Y},Q-\mu)$$

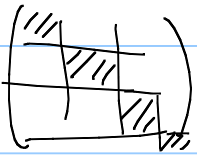


$$\mathcal{F}\begin{matrix} \alpha_1 & & \alpha_3 \\ \alpha_2 & \alpha & \alpha_4 \end{matrix}(q) = \sum_{|Y|=|Y'|} q^{|Y|} \gamma_{\alpha\alpha_1\alpha_2}(Y)\, Q_\alpha^{-1}(Y,Y')\, \gamma_{\alpha\alpha_3\alpha_4}(Y')$$

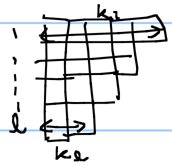


kernel

$$Q_\alpha(Y,Y') \equiv \langle \Delta_\alpha | \mathcal{L}_{Y'} \mathcal{L}_{-Y} | \Delta_\alpha \rangle \to$$



$$\mathcal{L}_{-Y} \equiv L_{-k_\ell} \cdots L_{-k_2} L_{-k_1} \qquad Y = (k_1 \geq k_2 \geq \cdots \geq k_\ell > 0)$$



3-pt

$$\gamma_{\alpha\alpha_1\alpha_2}(Y) = \prod_{i=1}^{\ell(Y)}\left(\Delta_\alpha + k_i\Delta_1 - \Delta_2 + \sum_{j<i} k_j\right)$$

* conformal ward id

E-M tensor $T(z) = \sum_n L_n z^{-n-2}$

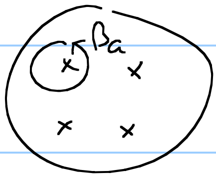
$$\langle T(z) \prod_i \mathcal{O}_i(z_i) \rangle = \sum_j \left[\frac{h_j}{(z-z_j)^2} + \frac{\partial_j}{z-z_j} \right] \langle \prod_i \mathcal{O}_i(z_i) \rangle$$

Define quadratic differential $x^2 = \phi_2(z)$

$$\phi_2(z) = \frac{\langle T(z) \prod_i \mathcal{O}_i(z_i) \rangle}{\langle \prod_i \mathcal{O}_i(z_i) \rangle}$$

as $\epsilon_{1,2} \ll a_i, \mu_i$

$$\phi_2 \longrightarrow \phi_2^{SW}$$

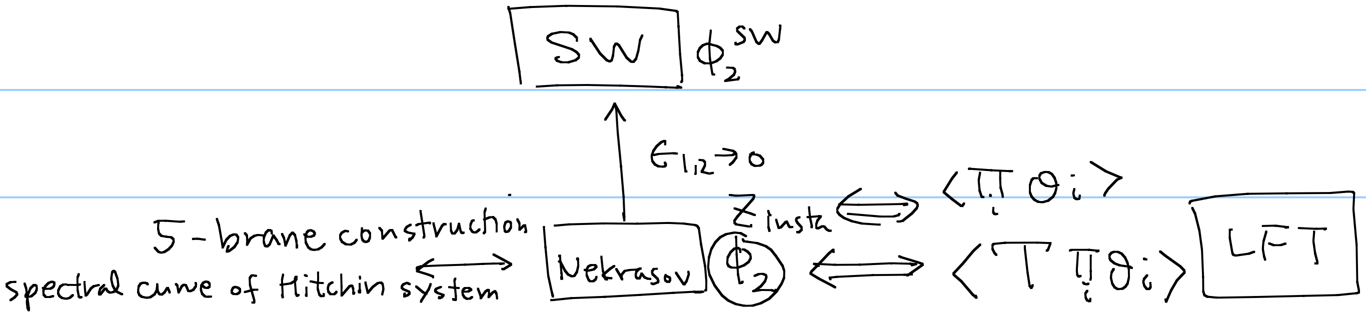


where $m_a = \oint_{\beta_a} \sqrt{\phi_2^{SW}}\, dz$

$a_i = \oint_{\gamma_i} \sqrt{\phi_2^{SW}}\, dz$ moduli

$\sqrt{\phi_2} \sim \frac{\sqrt{h_a}}{z-z_a}$ $\therefore \oint_{\beta_a} \sqrt{\phi_2}\, dz \sim \sqrt{h_a} \to m_a$

$\because h_a = m_a(Q-m_a)$
$a=1,3$



* Matter decoupling limit :

some $\mu_k \to \infty$

non conformal (Asymp. free) $\beta < 0$

→ free pure gauge $\mathcal{N}=2$ theory

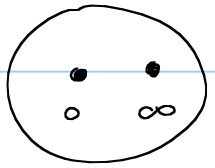
$$\underbrace{Z(Y)}_{} \sim \lim_{\mu_k \to \infty} Z(Y)$$

* What is a corresponding LFT correlator ?

ANS $\boxed{\text{"matrix element" of "coherent" states}}$

Λ : scale of asymp free theory

① $N_f = 0$



$$\longmapsto \quad \phi_2 = \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}$$

poles of order 3 at $z=0, \infty$

u: "Coulomb branch moduli"

associate with 2-pt function $\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle$

$$T(z) = \sum L_n z^{-n-2} \sim L_1 z^{-3} + \text{higher order}$$

we need to find $|\Delta, \Lambda^2\rangle \ni;$ $\begin{cases} L_{n\geq 2} |\Delta, \Lambda^2\rangle = 0 \to \underline{\underline{L_2 |\Delta, \Lambda^2\rangle = 0}} \\ L_1 |\Delta, \Lambda^2\rangle = \Lambda^2 |\Delta, \Lambda^2\rangle \end{cases}$

$$[L_2, L_1] = L_3 \to L_3 |\Delta, \Lambda^2\rangle = 0 \quad \text{etc.}$$

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let $|\Delta, \Lambda^2\rangle = \underset{|\Delta\rangle}{\underset{\shortparallel}{|v_0\rangle}} + \Lambda^2 |v_1\rangle + \Lambda^4 |v_2\rangle + \cdots$

$|v_{i>0}\rangle \Rightarrow$ descendent

$$\Lambda^{2n} L_1 |v_n\rangle = \Lambda^2 \Lambda^{2(n-1)} |v_{n-1}\rangle \to \boxed{\begin{array}{l} L_1 |v_n\rangle = |v_{n-1}\rangle \\ L_2 |v_n\rangle = 0 \end{array}}$$

$|v_0\rangle = |\Delta\rangle$

$L_1|v_1\rangle = |\Delta\rangle \rightarrow |v_1\rangle = \frac{1}{2\Delta} L_{-1}|\Delta\rangle$ $\qquad L_1 L_{-1} = 2L_0$

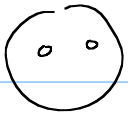
$|v_2\rangle = \frac{1}{\cdots\cdots}\left((c+8\Delta)L_{-1}^2 - 12\Delta L_{-2}\right)|\Delta\rangle$ etc

$$\boxed{\langle\Delta,\Lambda^2|\Delta,\Lambda^2\rangle = \sum_n \Lambda^{4n}\langle v_n|v_n\rangle}$$

2-point

$= Z_{inst}$ (pure gauge)

② $N_f = 1$;



poles of order 3 at $z=0$
order 4 at $z=\infty$

$$\phi_2 = \frac{\Lambda^2}{2z^3} + \frac{2u}{z^2} - \frac{2\Lambda m}{z} - \Lambda^2$$

$$\shortparallel$$

$$\underset{(z=\infty)}{\langle\Delta,\Lambda,m|}\, T\, \underset{(z=0)}{|\Delta,\frac{\Lambda^2}{2}\rangle}$$

$$T = L_{-2} + \frac{L_{-1}}{z} + \frac{L_0}{z^2} + \frac{L_1}{z^3}$$

$$\langle\Delta,\Lambda,m|L_{-2} = -\Lambda^2\langle\Delta,\Lambda,m|$$

$$\langle\Delta,\Lambda,m|L_{-1} = -2\Lambda m\langle\Delta,\Lambda,m|$$

$$\langle\Delta,\Lambda,m|L_{-n>2} = 0$$

$$L_1|\Delta,\frac{\Lambda^2}{2}\rangle = \frac{\Lambda^2}{2}|\Delta,\frac{\Lambda^2}{2}\rangle$$

let $|\Delta,\Lambda,m\rangle = |w_0\rangle + \Lambda|w_1\rangle + \Lambda^2|w_2\rangle + \cdots$

$L_1|w_n\rangle = -2m|w_{n-1}\rangle$ $\qquad L_1|w_0\rangle = L_2|w_0\rangle = 0$

$L_2|w_n\rangle = -|w_{n-2}\rangle$

$|w_0\rangle = |\Delta\rangle$, $|w_1\rangle = -\frac{m}{\Delta}L_{-1}|\Delta\rangle$,

$|w_2\rangle = \cdots\cdots(\cdots L_{-1}^2 + \cdots L_{-2})|\Delta\rangle$ etc

$$\Rightarrow \quad \langle \Delta, \Lambda, m \,|\, \Delta, \tfrac{\Lambda^2}{2} \rangle = \sum_n \underbrace{\left(\tfrac{\Lambda^2}{2}\right)^n \Lambda^n}_{\to \left(\frac{q}{2}\right)^n} \langle w_n | v_n \rangle$$

(note) $\langle w_m | v_n \rangle = 0$ if $m \neq n$

$\equiv$ instanton partition function for $SU(2)$ $N_f = 1$ w/ $q = \Lambda^3$ with m.

③ $N_f = 2$ pole of order 4 at $z = 0, \infty$

$$\phi_2 = -\frac{\Lambda^2}{z^4} - \frac{2\Lambda m_1}{z^3} + \frac{2u}{z^2} - \frac{2\Lambda m_2}{z} - \Lambda^2$$

$$\Rightarrow \quad \langle \Delta, \Lambda, m_2 | T | \Delta, \Lambda, m_1 \rangle \qquad ✓$$

$$T = L_{-2} + \frac{L_{-1}}{z} + \frac{L_0}{z^2} + \frac{L_1}{z^3} + \frac{L_2}{z^4}$$

$$L_2 |\Delta, \Lambda, m\rangle = -\Lambda^2 |\Delta, \Lambda, m\rangle$$

$$L_1 |\Delta, \Lambda, m\rangle = -2\Lambda m |\Delta, \Lambda, m\rangle$$

$\therefore$ $\langle \Delta, \Lambda, m_2 | \Delta, \Lambda, m_1 \rangle \equiv$ "inst. part. funct" for $SU(2)$ $N_f = 2$ (m_1, m_2) w/ $q = 4\Lambda^2$

$$= \sum_n \underbrace{\Lambda^{2n}}_{\left(\frac{q}{4}\right)^n} \langle w_n(m_2) | w_n(m_1) \rangle$$

upto overall function of Λ^2

(conjecture) $e^{2\Lambda^2}$

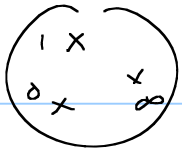
$q \sim \left(\frac{\Lambda}{M}\right)^{4-N_f} \to 0$ as $M \to \infty$

$$(1-q)^{2\tilde{m}_0(Q-\tilde{m}_1)} \to \begin{cases} 1 & \text{if } N_f < 2 \\ \text{const} & \text{if } N_f \geq 2 \end{cases}$$

$(\tilde{m}_0, \tilde{m}_1 \sim M)$

④ 2nd realization : $\left\{ \begin{array}{l} \text{regular punctures at } z=0,1 \\ \quad \text{(poles of order 2)} \\ \text{pole of order 3 at } z=\infty \end{array} \right.$

$$\phi_2 = -\frac{m_+^2}{z^2} - \frac{m_-^2}{(z-1)^2} + \frac{2\tilde{u}}{z(z-1)} + \frac{\Lambda^2}{z}$$



given by 3-point function

$$\phi_2 = \langle \underset{z=\infty}{\Delta,\Lambda^2} |\; \underbrace{T\,\Phi^{(z=1)}_{(\Delta_-,\Lambda^2)}}_{\text{conformal ward id}} \;| \underset{z=0}{\Delta_+,\Lambda^2} \rangle$$

$$\frac{\Delta_-}{(z-1)^2} + \frac{\Delta_+}{z^2} + \cdots$$

$$\Delta_\pm = \left(\frac{Q}{2}\right)^2 - m_\pm^2$$
$$m_\pm = \frac{1}{2}(m_1 \pm m_2)$$

$$T \sim \frac{L_{-1}}{z} + \cancel{\frac{L_0}{z^2} + \cdots}$$
at $z=\infty$

Instanton part. Funct. should be given by the 3-pt fct.

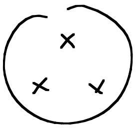
$$\underline{\underline{\langle \Delta, \Lambda^2 | \Phi^{(1)}_{(\Delta_-,\Lambda^2)} | \Delta_+, \Lambda^2 \rangle}}$$ with $q = \Lambda^2$

One can use

$$[L_n, \phi_\Delta(z,\bar{z})] = \Delta(n+1) z^n \phi + z^{n+1} \partial\phi$$

to evaluate 3-pt fct.

⑤ $N_f = 3$ $\begin{cases} \text{regular at } z=0,1 \\ \text{a pole of order 4 at } z=\infty \end{cases}$



$$\phi_2 = -\frac{m_+^2}{z^2} - \frac{m_-^2}{(z-1)^2} + \frac{2\tilde{u}}{z(z-1)} - \frac{2m_3\Lambda}{z} - \Lambda^2$$

$$= \langle \Delta, \Lambda, m_3 | T \, \Phi_{(\Delta_-, \Lambda^2)}(z=1) | \Delta_+, \Lambda^2 \rangle$$

$$T = L_{-2} + \mathcal{O}(\tfrac{1}{z})$$

inst. part. funct: $\langle \Delta, \Lambda, m_3 | \Phi_{(\Delta_-, \Lambda^2)} | \Delta_+, \Lambda^2 \rangle$
with $q = -2\Lambda$

Conclusion

for asymp free gauge theory with $\mathcal{N}=2$
$(\beta < 0)$

$$Z_{\text{inst}} \Longleftrightarrow \langle \Phi \; \Phi \; \Phi \rangle$$

$\nwarrow \; \uparrow \; \nearrow$

special puncture op.
or coherent states

Marshakov Mironov Morozov
0909.2052

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can we get these "coherent states" correlators
as limits $\mu_i \to \infty$ of "primary" correlators?

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① when all $\mu_i \to \infty, i = 1-4$ let $q \to 0$

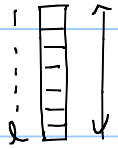
$\exists$: $q \prod_{i=1}^{4} \mu_i = \Lambda^4$ finite

Since $\Delta_i \gg \Delta, 1$

3-pt

$$\gamma_{\alpha\alpha_1\alpha_2}(Y) = \prod_{i=1}^{l(Y)} \left(\Delta_\alpha + k_i \Delta_1 - \Delta_2 + \sum_{j<i} k_j\right)$$

$$\approx \prod_{i=1}^{l(Y)} (k_i \Delta_1 - \Delta_2)$$

$Y =$  dominates $(k_i \equiv 1)$

$$q^{\frac{|Y|}{2}} \gamma_{\alpha\alpha_1\alpha_2}(Y) \sim \left(\sqrt{q}(\Delta_1 - \Delta_2)\right)^{|Y|} \delta(Y -$$  $)$

$$\left.\begin{aligned} \mu_1 &= \alpha_1 - \alpha_2 + \frac{\epsilon}{2} \\ \mu_2 &= \alpha_1 + \alpha_2 - \frac{\epsilon}{2} \end{aligned}\right\}$$

$$\alpha_1(\epsilon - \alpha_1) - \alpha_2(\epsilon - \alpha_2)$$
$$= \epsilon(\alpha_1 - \alpha_2) - (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)$$
$$= \left(\mu_1 - \frac{\epsilon}{2}\right)\left(\frac{\epsilon}{2} - \mu_2\right) \approx -\mu_1\mu_2$$

$$\sqrt{q}\, \mu_1 \mu_2 = \Lambda^2 = \sqrt{q}\, \mu_3 \mu_4$$

$$\Rightarrow \quad F = \sum_{|Y|=|Y'|} \Lambda^{4|Y|} Q_\Delta^{-1}(Y,Y')\,\delta\Big(Y-\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\Big)\,\delta\Big(Y'-\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\Big)$$

$$\cong \sum_n \Lambda^{4n} Q_\Delta^{-1}\Big(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\Big)$$

Define a coherent state

$$|\Delta,\Lambda^2\rangle \equiv \sum_Y C_Y \mathcal{L}_{-Y}|\Delta\rangle$$

$$\langle\Delta,\Lambda^2|\Delta,\Lambda^2\rangle = \sum_{Y,Y'} C_Y\, Q(Y,Y')\, C_{Y'}$$

if $C_Y = \Lambda^{2|Y|} Q_\Delta^{-1}\Big(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},Y\Big)$

$$\therefore \quad |\Delta,\Lambda^2\rangle = \sum_Y \underbrace{\Lambda^{2|Y|} Q_\Delta^{-1}\Big(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},Y\Big)}_{C_Y} \mathcal{L}_{-Y}|\Delta\rangle$$

& one can check

$$\begin{cases} L_1|\Delta,\Lambda^2\rangle = \Lambda^2|\Delta,\Lambda^2\rangle \\ L_k|\Delta,\Lambda^2\rangle = 0 \qquad k\geq 2 \end{cases}$$

② if some $\mu_i \to \infty$

(ex) $\mu_{2,3,4}\to\infty$; let $\epsilon\,\mu_2\mu_3\mu_4 = \Lambda_1^3$

$\mu_1 = \text{finite}$ $\rightarrow$ $\alpha_1,\alpha_2\to\infty$ $\ni$; $\dfrac{\Delta_1-\Delta_2}{\sqrt{\Delta_1}} = \text{finite}$

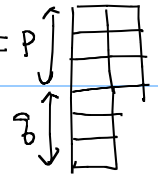
$\mu_2 = \infty$ $\qquad (\Delta_1,\Delta_2\to\infty)$

$$\left(\begin{aligned}\Delta_1-\Delta_2 &= -(\alpha_1-\alpha_2)(\alpha_1+\alpha_2-\epsilon)\\ &\approx (2\mu_1-\epsilon)\sqrt{\Delta_1}\end{aligned}\right)$$

in $\gamma_{\alpha\alpha_3\alpha_4}(Y)$ $Y =$  dominates

but in $\gamma_{\alpha\alpha_1\alpha_2}(Y)$, it does <u>NOT</u>

$\left(\because \quad k_i\Delta_1 - \Delta_2 \sim \sqrt{\Delta_1} \ll \Delta_1, \Delta_2 \text{ if } k_i = 1 \right)$

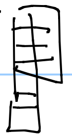
instead $Y = $  , $\ell(Y) = p+q$ dominates

$|Y| = 2p+q$

$$\therefore \quad \gamma_{\alpha\alpha_1\alpha_2}(Y) \sim (2\Delta_1 - \Delta_2)^p (\Delta_1 - \Delta_2)^q$$

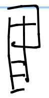
$$\sim (2\mu_1 - \epsilon)^q \Delta_1^{p+\frac{q}{2}} \sim \underbrace{(2\mu_1 - \epsilon)}_{2m}^q \left(\frac{\mu_2}{2}\right)^{|Y|}$$

$F \cong \sum_{|Y|=|Y'|} \sum_p (2m)^{\overbrace{|Y|-2p}^{q}} \Big(\underbrace{\frac{q\,\mu_2\mu_3\mu_4}{2}}_{\frac{\Lambda_1^3}{2}} \Big)^{|Y|} Q_\Delta^{-1}(Y,Y')\, \delta($  $- Y)\, \delta($  $, Y')$

$= \sum_{n,p} (2m)^{n-2p} \left(\frac{\Lambda_1^3}{2}\right)^n Q_\Delta^{-1}($  ,  $)$

$$= \left\langle \Delta, \frac{\Lambda_1}{2}, 2m \,\middle|\, \Delta, \Lambda_1^2 \right\rangle$$

if we define

$|\Delta, \Lambda, m\rangle = \sum_Y \sum_p m^{|Y|-2p} \Lambda^{|Y|} Q_\Delta^{-1}($  $, Y) \mathcal{L}_{-Y}|\Delta\rangle$

and can show

$$L_1 |\Delta, \Lambda, m\rangle = m\Lambda |\Delta, \Lambda, m\rangle$$

$$L_2 |\Delta, \Lambda, m\rangle = \Lambda^2 |\Delta, \Lambda, m\rangle$$

$$L_k |\Delta, \Lambda, m\rangle = 0 \qquad k \geqq 3$$