# Journal Club

노트 제목

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## AGT

2d CFT
LFT

$\uparrow$ $\epsilon_1 = \hbar b, \ \epsilon_2 = \frac{\hbar}{b}$

$F_{LFT}$ conf. block

$\shortparallel$

$Z_{inst} = e^{\frac{F}{\epsilon_1 \epsilon_2}}$

$\downarrow\uparrow$

1d q-atum

4d $N=2$ SYM with $\epsilon_1, \epsilon_2$

N M5 branes

$\downarrow$ low

$d=6$ $N=(2,0)$ theory $\xrightarrow{\text{compactify}}$ 4d $N=2$ SYM with $\epsilon_1, \epsilon_2$

$\epsilon_2 = 0$, $\epsilon_1 \neq 0$ $\rightarrow$ 1d Q.M.

$\epsilon_1 = \epsilon_2 = 0$ $\rightarrow$ 4d $N=2$ SYM SW

1d Q.M. $\xrightarrow{\epsilon_1 \to 0}$ 4d $N=2$ SYM SW

$\shortparallel$

classical limit
Bohr Sommerfeld

* can be extended to surface op vs. deg. field insertion

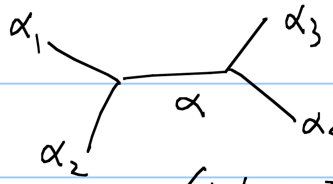
# AGT relation

Conformal Block $\longleftrightarrow$ Nekrasov instanton Partition fct
of Liouville

\# of $u(2)$ gauge group $= 3g-3+n$ "quiver"

$N=2$ $u(2)$ SYM $\longrightarrow \begin{cases} n=4 & \text{on sphere } (g=0) \ast \\ n=1 & \text{on torus } (g=1) \end{cases}$

(ex) $n=4$ on $g=0$




$\Delta_1 = \alpha_1(Q-\alpha_1)$ etc

$Q = b + \frac{1}{b}$ LFT coupling

$\langle V_{\alpha_1}(\infty) V_{\alpha_2}(1) V_{\alpha_3}(q) V_{\alpha_4}(0) \rangle$

$= F^{\alpha_1 \quad \alpha_3}_{\alpha_2 \;\alpha\; \alpha_4}(q)$

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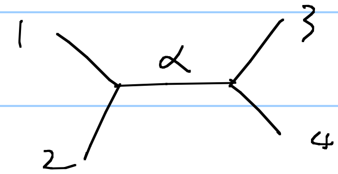
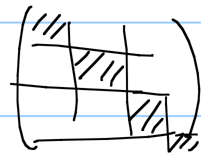
$$F^{\alpha_1 \quad \alpha_3}_{\alpha_2 \;\alpha\; \alpha_4}(q) = \sum_{|Y|=|Y'|} q^{|Y|} \gamma_{\alpha\alpha_1\alpha_2}(Y)\, Q_\Delta^{-1}(Y,Y')\, \gamma_{\alpha\alpha_3\alpha_4}(Y')$$

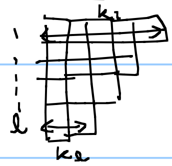



kernel

$Q_\Delta(Y,Y') \equiv \langle \Delta | \mathcal{L}_{Y'} \mathcal{L}_{-Y} | \Delta \rangle \rightarrow$



$\mathcal{L}_{-Y} \equiv L_{-k_l} \cdots L_{-k_2} L_{-k_1}$ $\quad Y = (k_1 \geq k_2 \geq \cdots \geq k_l > 0)$




3-pt

$$\gamma_{\alpha\alpha_1\alpha_2}(Y) = \prod_{i=1}^{l(Y)} \left( \Delta + k_i \Delta_1 - \Delta_2 + \sum_{j<i} k_j \right)$$

# Nekrasov with twists $\epsilon_1, \epsilon_2$

$$\boxed{\epsilon_1 = \hbar b, \quad \epsilon_2 = \frac{\hbar}{b}} \qquad \epsilon = \epsilon_1 + \epsilon_2$$

$\mathcal{N}=2$ SUSY quiver $U(2)$
with $N_f = 4$

$$\mu_1 = m_0 - \tilde{m}_0 = \hbar(\alpha_1 - \alpha_2) + \frac{\epsilon}{2} \qquad \alpha_1 = \frac{m_0}{\hbar}, \quad \alpha_2 = \frac{\tilde{m}_0}{\hbar} + \frac{Q}{2}$$

$$\mu_2 = m_0 + \tilde{m}_0 = \hbar(\alpha_1 + \alpha_2) - \frac{\epsilon}{2}$$

$$\mu_3 = m_1 - \tilde{m}_1 = \hbar(\alpha_3 - \alpha_4) + \frac{\epsilon}{2} \qquad \alpha_3 = \frac{m_1}{\hbar}, \quad \alpha_4 = \frac{\tilde{m}_1}{\hbar} + \frac{Q}{2}$$

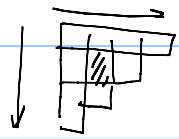
$$\mu_4 = m_1 + \tilde{m}_1 = \hbar(\alpha_3 + \alpha_4) - \frac{\epsilon}{2}$$

$$\epsilon = \epsilon_1 + \epsilon_2 = \hbar Q$$

$$\alpha = \frac{a}{\hbar} + \frac{Q}{2}, \quad \Delta = \alpha(Q - \alpha) = \frac{Q^2}{4} - \frac{a^2}{\hbar^2}$$

$$Z^{U(2)}_{inst}(a, \mu_i, q) = \sum_{\vec{Y}} q^{|\vec{Y}|} Z_{vector}(\vec{a}, \vec{Y}) Z_{a.f.}(\vec{a}, \vec{Y}, \mu_1) \cdots Z_{fu}(\vec{a}, \vec{Y}, \mu_4)$$

$$Z_f(\vec{a}, \vec{Y}, \mu) = \prod_{i=1}^{2} \prod_{s \in Y_i} \left(\phi(a_i, s) - \mu + \epsilon\right), \quad \phi(a, s) \equiv a + \epsilon_1(i-1) + \epsilon_2(j-1)$$



$$Z_{a.f.}(\vec{a}, \vec{Y}, \mu) = Z_f(\vec{a}, \vec{Y}, Q - \mu)$$

## AGT conjecture

scalar condensate
$a = a_1 = -a_2$ $\quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$

$$\boxed{(1-q)^{2\alpha_1(Q-\alpha_1)} \mathcal{F}\begin{matrix} \alpha_1 & & \alpha_3 \\ \alpha_2 & \alpha & \alpha_4 \end{matrix}(q) = Z^{U(2)}_{inst}(a, \mu_i, q)} \qquad q = e^{2\pi i \tau}$$

?? $\epsilon_1$-deformation of Prepotential

$$F(\epsilon_1) \equiv \lim_{\epsilon_2 \to 0} (-\epsilon_1 \epsilon_2) Z_{inst}.$$

& Schrödinger eq.

$$\mathcal{H}\Psi^{(0)}(z) = E\Psi^{(0)}(z), \quad \mathcal{H} = -\epsilon_1^2 \partial_z^2 + V(z, \epsilon_1)$$

$$\Psi^{(0)}(z) \equiv \exp\left(-\frac{1}{\epsilon_1}\int^z P(z'; \epsilon_1) dz'\right)$$

let
$$2\pi i\, a(E, \epsilon_1) = \oint_A P(z, \epsilon_1) dz$$
$$\frac{1}{2}\frac{\partial \hat{F}}{\partial a}(E, \epsilon_1) = \oint_B P(z, \epsilon_1) dz$$

$$\rightarrow \Psi^{(0)}(z + A^i) = \exp\left(-\frac{2\pi i\, a}{\epsilon_1}\right)\Psi^{(0)}(z)$$
$$\Psi^{(0)}(z + B^i) = \exp\left(-\frac{1}{2\epsilon_1}\frac{\partial F}{\partial a}\right)\Psi^{(0)}(z)$$

S.E. $\rightarrow$ $-P^2 + \epsilon_1 P' + V(z, \epsilon_1) = E$, $\quad P = \sum_{k=0}^{\infty} \epsilon_1^k P_k$

$$\oint P = \hat{\mathcal{O}} \oint P_0$$

$$\underbrace{\hat{a}(E, \epsilon_1) = \hat{\mathcal{O}}[a(E)], \quad \frac{\partial \hat{F}}{\partial \hat{a}} = \hat{\mathcal{O}}\left(\frac{\partial F}{\partial a}(E)\right)}_{\Rightarrow \hat{F}(\hat{a}; \epsilon_1) = F}$$

## Mironov - Morozov

0910.5670

Consider <u>pure $U(2)$</u> by taking $\mu_i \to \infty$ [my lecture 2009]
(no hypermultiplets)

$\Delta_i \gg \Delta_\alpha$

<u>3-pt</u>

$$\gamma_{\alpha\alpha_1\alpha_2}(Y) = \prod_{i=1}^{l(Y)} \left( \Delta_\alpha + k_i \Delta_1 - \Delta_2 + \sum_{j<i} k_j \right)$$

$$\approx \prod_{i=1}^{l(Y)} (k_i \Delta_1 - \Delta_2)$$

$Y = $ (single column of length $l$, $|Y|$) dominates $(k_i \equiv 1)$

$$q^{\frac{|Y|}{2}} \gamma_{\alpha\alpha_1\alpha_2}(Y) \sim \left( \sqrt{q}(\Delta_1 - \Delta_2) \right)^{|Y|} \delta(Y - \text{column})$$

$$\left.\begin{array}{l} \mu_1 = \hbar(\alpha_1 - \alpha_2) + \frac{\epsilon}{2} \\ \mu_2 = \hbar(\alpha_1 + \alpha_2) - \frac{\epsilon}{2} \end{array}\right\} \quad \Delta_1 - \Delta_2 = \alpha_1(Q - \alpha_1) - \alpha_2(Q - \alpha_2)$$

$$= Q(\alpha_1 - \alpha_2) - (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)$$

$$= \frac{1}{\hbar^2}\left(\mu_1 - \frac{\epsilon}{2}\right)\left(\frac{\epsilon}{2} - \mu_2\right) \approx -\frac{\mu_1 \mu_2}{\hbar^2}$$

let $\sqrt{q}\, M_1 M_2 = \Lambda^2 = \sqrt{q}\, M_3 M_4 = \text{finite}$

$$F_{LFT} = \sum_{|Y| = |Y'|} \left( \frac{\Lambda^4}{\hbar^4} \right)^{|Y|} Q_\Delta^{-1}(Y, Y')\, \delta(Y - \text{column})\, \delta(Y' - \text{column})$$

$$\cong \sum_n \frac{\Lambda^{4n}}{(\epsilon_1 \epsilon_2)^{2n}} Q_\Delta^{-1}(\text{column}, \text{column})$$

$\epsilon_1 \epsilon_2 = \hbar^2$

$\epsilon_1 = \hbar b, \ \epsilon_2 = \frac{\hbar}{b}$

Take limit $\hbar\to 0, b\to\infty$ & $\hbar b =$ finite ; $\epsilon_2\to 0$, $\epsilon_1 =$ finite

$$\Delta = \frac{Q^2}{4} - \frac{a^2}{\hbar^2} = \frac{1}{\hbar^2}\left(\frac{\epsilon^2}{4} - a^2\right) = \frac{1}{\epsilon_1\epsilon_2}\left(\frac{(\epsilon_1+\epsilon_2)^2}{4} - a^2\right)$$

$$c = 1 + 6\,Q^2 = 1 + 6\,\frac{\epsilon^2}{\hbar^2} = 1 + 6\,\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$$

$$\tilde{\Delta} = \epsilon_1\epsilon_2\Delta \to \frac{\epsilon_1^2}{4} - a^2, \quad \tilde{c} = \epsilon_1\epsilon_2 \longrightarrow 6\epsilon_1^2$$

$$Q_\Delta(\square,\square) = \langle\Delta| L_1 L_{-1} |\Delta\rangle = 2\Delta \to Q_\Delta^{-1}(1,1) = \frac{1}{2\Delta}$$

$$Q_\Delta = \begin{array}{c|c} & \begin{array}{cc}\text{⊟} & \square\square\end{array} \\ \hline \begin{array}{c}\text{⊟} \\ \square\square\end{array} & \begin{pmatrix} & \\ & \end{pmatrix}\end{array} \to Q_\Delta^{-1} = \begin{pmatrix} \boxed{\ } & \\ & \end{pmatrix} = \frac{8\Delta + c}{4\Delta(16\Delta^2 + 2c\Delta - 10\Delta + c)}$$

$$Z_{inst} = F_{LFT} = 1 + \frac{\Lambda^4}{\epsilon_1\epsilon_2 2\tilde{\Delta}} + \frac{\Lambda^8}{(\epsilon_1\epsilon_2)^2}\frac{8\tilde{\Delta}+\tilde{c}}{4\tilde{\Delta}^2(16\tilde{\Delta}+2\tilde{c})} + \cdots$$

$$+ \frac{\Lambda^{12}}{(\epsilon_1\epsilon_2)^3}\left(\cdots\cdots\right)$$

$$\equiv \exp\left(\frac{F_{inst}}{\epsilon_1\epsilon_2}\right)$$

$$F_{inst} = -\left(\frac{\Lambda^4}{2a^2} + \frac{5\Lambda^8}{64a^6} + \cdots\right) - \epsilon_1^2\left(\frac{\Lambda^4}{8a^4} + \frac{21\Lambda^8}{128a^4} + \cdots\right) + O(\epsilon_1^4)$$

$$F = F_{inst} + F_{pert}$$

$$= -4a^2\log\frac{a}{\Lambda} - \frac{\epsilon_1^2}{6}\log a + \quad '' \quad + \cdots$$

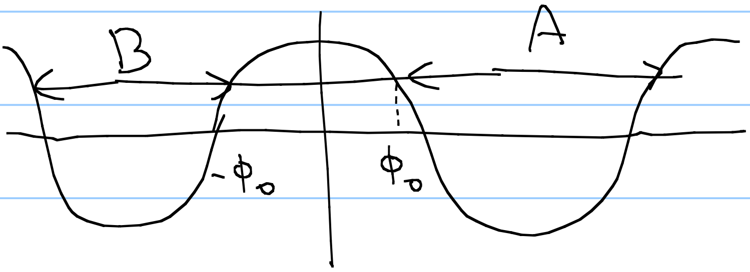
Now, relate to 1d system

Start with 1d SGM

$$S=\int\left(\frac{1}{2}\dot{\phi}^2-\overbrace{\underbrace{\Lambda^2}_{\gamma}\cos\phi}^{V}\right)dt \to \left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\phi^2}+\Lambda^2\cos\phi\right)\Psi=E\Psi$$

$$\Pi^{(0)}(A)=\oint_A\sqrt{2(E-\gamma\cos\phi)}\,d\phi$$

Bohr Somerfeld quantization




$$\Pi^{(0)}(A)=2\pi\hbar\left(n+\frac{1}{2}\right)$$

$$V(x)\equiv\gamma\cos x$$

quantum: let $\underline{\psi(\phi)=\exp\left(\frac{i}{\hbar}\int^{\phi}P(\phi)\,d\phi\right)}$ $\phi\to x$

put into Schrödinger $i\hbar P'=P^2-p^2$ $p^2\equiv 2(E-V)$

$$P(x)=\sum_{k=0}^{\infty}\hbar^k P_k(x)$$

let
$$\oint_A P(x)\,dx=2\pi\hbar\left(n+\frac{1}{2}\right)$$
"Exact"

$$\Pi=\frac{1}{\sqrt{2}}\oint p\,dx=\underbrace{\oint\sqrt{E-V}\,dx}_{\equiv\Pi^{(0)}(E,\gamma)}-\frac{\hbar^2}{64}\underbrace{\oint\frac{V'^2}{(E-V)^{5/2}}dx}\cdots$$

$$+\frac{\hbar^2}{96}\oint\frac{V''}{(E-V)^{3/2}}dx=-\frac{\hbar^2\gamma}{24}\partial_E\partial_\gamma\Pi^{(0)}$$

Picard-Fucks eq.

$$\left(\gamma(\partial_E^2+\partial_\gamma^2)+2E\,\partial_E\partial_\gamma\right)\sqrt{E-V}\,dx=d\left(\frac{\sin x}{2\sqrt{E-V}}\right)$$

$$\to\left(\gamma(\partial_E^2+\partial_\gamma^2)+2E\,\partial_E\partial_\gamma\right)\Pi^{(0)}=0$$

$$\Pi^{(0)}(E,\gamma) \sim \begin{cases} \sqrt{E} + O(\gamma) \\ \sqrt{E}\log\frac{E}{\gamma} + O(\gamma) \end{cases} \quad E^{\frac{1}{2}+\varepsilon} \simeq \sqrt{E}\,(1+\varepsilon\log E+\dots) \text{ as } \varepsilon\to 0$$

Try
$$\Pi^{(0)}_{\varepsilon} \equiv E^{\frac{1}{2}+\varepsilon}\left(1+\sum_{n>0} s_n\left(\frac{\gamma}{E}\right)^{2n}\right) = \Pi^{(0)} + \varepsilon\,\Pi^{(0)\prime} + \dots$$

$$\oint_A \qquad \oint_B$$

$$s_{n+1} = \frac{\left(n+\frac{1}{4}-\frac{\varepsilon}{2}\right)\left(n+\frac{1}{4}-\frac{\varepsilon}{2}\right)}{(n+1)(n+1-\varepsilon)}\, s_n = \left(\frac{n^2-\frac{1}{16}}{(n+1)^2} - \frac{n+\frac{1}{16}}{(n+1)^3}\,\varepsilon + O(\varepsilon^2)\right) s_n$$

and

$$\Pi^{(0)}_{\varepsilon} = \underbrace{\sqrt{2E}\left(1-\frac{1}{16}\left(\frac{\gamma}{E}\right)^2 - \frac{15}{2^{10}}\left(\frac{\gamma}{E}\right)^4 + \dots\right)} +$$
$$+\varepsilon\left\{\sqrt{2E}\log E\left(1-\frac{1}{16}\left(\frac{\gamma}{E}\right)^2 - \frac{15}{2^{10}}\left(\frac{\gamma}{E}\right)^4 + \dots\right) - \sqrt{2E}\left(\frac{1}{16}\left(\frac{\gamma}{E}\right)^2 + \frac{13}{2^{11}}\left(\frac{\gamma}{E}\right)^4 + \dots\right)\right\} + O(\epsilon^2)$$

* Seiberg-Witten limit ($\hbar \to 0$)

$$\Pi^{(0)} = a, \qquad \Pi^{(0)\prime} = \frac{1}{4}\frac{\partial F_{SW}}{\partial a}$$

Coulomb moduli

$$\therefore \sqrt{2E} = a\left(1+\frac{1}{4}\left(\frac{\gamma}{a^2}\right)^2 + \frac{3}{64}\left(\frac{\gamma}{a^2}\right)^4 + \dots\right), \quad E = u = \frac{a^2}{2} + \dots$$

$$\to \frac{1}{4}\frac{\partial F_{SW}}{\partial a} = \Pi^{(0)\prime} = -2a\log a + \frac{1}{4}a\left(\left(\frac{\gamma}{a^2}\right)^2 + \frac{15}{32}\left(\frac{\gamma}{a^2}\right)^4 + \dots\right)$$

$$\to F_{SW}(a) = -4a^2(\log a + \text{const.}) \underbrace{- \frac{\gamma^2}{2a^2} - \frac{5\gamma^4}{64a^6} + \dots}_{F_{inst}\,!}$$

* Include $\hbar^2$ $(\partial_E \partial_\gamma \Pi^{(0)})$

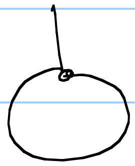
$$F(a,\hbar) = -4a^2(\log a + \dots) - \frac{\hbar^2}{3}\log a - \frac{\gamma^2}{2a^2} - \frac{5\gamma^4}{64a^6} \overbrace{- \frac{\hbar^2\gamma^2}{8a^4} - \frac{21\hbar^2\gamma^4}{128a^8}}^{F_{inst} \sim \epsilon_1^2}$$

(ex2) $\mathcal{N}^*=2$ $U(2)$ theory [Reviewed by Marian]

with an adjoint hypermultiplet with mass $m$ $\longleftrightarrow$ $n=1$, $g=1$ torus one-pt of LFT (conf. block) Fateev Litvinov

$$\langle V_\alpha\rangle_\tau = \mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}V_\alpha(0)\right) = \sum_{\{\Delta\}} C^{\Delta}_{\Delta_\alpha\Delta}\,|q|^{2\Delta-\frac{c}{12}}\left|\mathcal{F}^{(\Delta)}_\alpha(q)\right|^2$$

all decendents



$$\mathcal{F}^{(\Delta)}_\alpha(q) \stackrel{\text{def}}{=} \frac{1}{\langle\Delta|V_\alpha|\Delta\rangle}\left(\langle\Delta|V_\alpha|\Delta\rangle + \frac{\langle\Delta|L_1V_\alpha L_{-1}|\Delta\rangle}{\langle\Delta|L_1L_{-1}|\Delta\rangle}q+\dots\right) =$$

$$= 1+\frac{2\Delta+\Delta^2(\alpha)-\Delta(\alpha)}{2\Delta}q+\dots$$

Can be written in recursive form ..

$$\mathcal{F}^{(\Delta)}_\alpha(q) = \frac{q^{\frac{1}{24}}}{\eta}\mathcal{H}^{(\Delta)}_\alpha$$

$$\mathcal{H}^{(\Delta)}_\alpha(q) = 1+\sum_{m,n} q^{mn}\frac{R_{mn}(\alpha)}{\Delta-\Delta_{mn}}\mathcal{H}^{(\Delta_{m,-n})}_\alpha(q) = 1+\sum_{L=1}^{\infty}H_L(\Delta)q^L$$

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Nekrasov

$$Z_{inst}(\epsilon_1,\epsilon_2,m,\underset{\text{VEV}}{\vec{a}}) = 1+\sum_{k=1}^{\infty}q^k\mathcal{Z}_k\,,\quad \epsilon_1=\hbar b,\ \epsilon_2=\frac{\hbar}{b},\ m=\hbar\alpha,\ \vec{a}=\hbar\vec{P}$$

$$\mathcal{Z}_N = \sum_{\vec{Y}}\prod_{i,j=1}^{2}\prod_{s\in Y_i}\frac{(E_{ij}(s)-\alpha)(Q-E_{ij}(s)-\alpha)}{E_{ij}(s)(Q-E_{ij}(s))},$$

$$E_{ij}(s) = P_i-P_j-bH_{Y_j}(s)+b^{-1}(V_{Y_i}(s)+1).$$

$$Z^{\mathcal{N}=2^*}_{\text{inst}}(\varepsilon_1,\varepsilon_2,m,\vec{a}) = \left(\frac{q^{\frac{1}{24}}}{\eta(\tau)}\right)^{1-2\Delta(\alpha)}\mathcal{F}^{(\Delta)}_\alpha(q)$$

SW limit : $\epsilon_1, \epsilon_2 \to 0$ $(\hbar \to 0)$

$$Z_{inst} \longrightarrow e^{\frac{1}{\epsilon_1 \epsilon_2} F_{inst}}, \quad \Delta_\alpha \to -\frac{m^2}{\hbar^2}, \quad \Delta \to -\frac{a^2}{\hbar^2}$$

$$\mathcal{H}_\alpha^{(\Delta)}(q) \to e^{\frac{1}{\hbar^2}\mathbb{H}(q)}, \quad \mathbb{H}(q) = -\frac{m^4}{2a^2}\mathcal{H}(u|q), \quad u \equiv \left(\frac{m}{a}\right)^2$$

$$\mathcal{H}(u|q) = \sum_{k=1}^{\infty} \underbrace{p_k(u)}_{\text{poly of deg. } 2k-2} q^k$$

$F_{inst}$ : inst. part of prepotential of SW

$$= \frac{m^2}{12}\log q - 2m^2 \log \eta - \frac{m^4}{2a^2}\mathbb{H}(u|q)$$

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$\hbar \to 0$ of conf. block ;

let $\langle \underbrace{V_{-\frac{b}{2}}(z)}_{\text{deg.}} V_\alpha(0) \rangle_\tau = \Theta_1(z)^{\frac{b^2}{2}} \eta^{2\Delta_\alpha - 1 - 2b^2} \Psi(z|\tau)$

deg. satisfies $L_{-1}^2 + L_{-2} = 0$

$$\to \left(\frac{\hbar^2}{b^2}\partial_z^2 + \underbrace{m^2 \wp(z)}_{V}\right)\Psi(z|\tau) = \frac{2i\hbar^2}{\pi}\partial_\tau \Psi(z|\tau) \equiv E(q)\Psi$$

let $\Psi(z|\tau) = \exp\left(\frac{1}{\hbar^2}F(q) + \frac{b}{\hbar}\overset{\text{WKB}}{W}(z|q) + \cdots\right)$

$$\partial_\tau \Psi = \frac{1}{\hbar^2} 2\pi i q F'(q) \Psi \quad \to E(q) = 4 q \partial_q F(q)$$

$$\oint \sqrt{E - V}\frac{dx}{2\pi i} = a \qquad \text{etc.}$$

## $\epsilon_1 = $ finite

$$V(z,\epsilon_1) = m(m-\epsilon_1)P(z)$$

$$\oint p_1 dz = -\frac{1}{2}\frac{\partial}{\partial m}\oint p_0 dz \to \underbrace{\left(1 - \frac{\epsilon_1}{2}\frac{\partial}{\partial m} + \cdots\right)}\oint p_0 dz$$

$$\hat{m}^k = m^k - \frac{\epsilon_1}{2} k m^{k-1} + \cdots$$

$$\therefore \quad \hat{a} = a\Big|_{m^k \to \hat{m}^k}$$

$$\frac{\partial \hat{F}}{\partial \hat{a}} = \frac{\partial F}{\partial a}\Big|_{m^k \to \hat{m}^k}$$

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(ex3) 4 case

$$V = \frac{\tilde{m}_1^2 - \frac{\epsilon_1^2}{4}}{z^2} + \frac{m_0(m_0-\epsilon_1)}{(z-1)^2} + \frac{m_1(m_1-\epsilon_1)}{(z-q)^2} - \frac{m_0(m_0-\epsilon_1)+m_1(m_1-\epsilon_1)+\tilde{m}_1^2-\tilde{m}_0^2}{z(z-1)}$$

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## Origin of 1D quantum system

As already noticed above, "insertion of degenerate" $\equiv$ surface op.

[Related to Kyungkiu]

* algebraic curve

$$\phi_2(z) = \frac{\langle T(z)\prod V_i\rangle}{\langle \prod V_i\rangle} = x^2$$

* Similarly, consider

$$\Psi(a_p, z) = \langle \Phi_{2,1}(z)\prod_{i=1}^{n} V_{\alpha_i}(z_i)\rangle$$

with null condition $\left(b^2 L_{-2} + L_{-1}^2\right) \Phi_{2,1} = 0$

$$\langle \Phi \, \Phi_{21} \Phi \cdots \rangle = 0$$

$$\to \langle \Phi_{21} (\Phi \cdots) \rangle = 0$$

$$L_{-1}^2 \Phi = \partial_z^2 \Phi$$

$$L_{-2} \Phi(z) = \oint \frac{dw}{w-z} T(w) \Phi(z)$$

$$\oint \frac{dw}{w-z} \left\langle T(w) \Phi_{21}(z) \prod_i V_{\alpha_i}(z_i) \right\rangle$$

$$= \oint \frac{dw}{w-z} \left[ \sum_{I=0,i} \frac{\Delta_I}{(w-z_I)^2} + \frac{\partial_{z_I}}{w-z_I} \right] \left\langle \Phi_{21}(z) \prod_i V_{\alpha_i} \right\rangle$$

$$= \oint \frac{dw}{w-z} \sum_i \left( \frac{\Delta_i}{(w-z_i)^2} + \frac{\partial_{z_i}}{w-z_i} \right) \langle \quad \rangle$$

$$= \sum_i \left( \frac{\Delta_i}{(z-z_i)^2} + \frac{\partial_{z_i}}{z-z_i} \right) \Psi(z)$$

$$\therefore \; 0 = \left[ \frac{1}{b^2} \partial_z^2 + \sum_{i=1}^{4} \left( \frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \right] \Psi(z)$$

Since $\left\langle L_{\pm 1, 0} \prod_I^N V_{\alpha_I} \right\rangle = 0$

$$\to \begin{cases} \sum_I \frac{\partial}{\partial z_I} \left\langle \prod_I V_{\alpha_I}(z_I) \right\rangle = 0 \\ \sum_I \left( z_I \partial_{z_I} + \Delta_I \right) \langle \;\text{"}\; \rangle = 0 \\ \sum_I \left( z_I^2 \partial_{z_I} + 2\Delta_I z_I \right) \langle \;\text{"}\; \rangle = 0 \end{cases}$$

Set $z_0 = z, \; z_1 = \infty, \; z_2 = 1, \; z_3 = \xi, \; z_4 = 0$

$$\left(\sum_i \partial_i + \partial_z\right)\langle\quad\rangle = 0$$

$$\left(\sum_i z_i\partial_i + z\partial_z + \sum\Delta\right)\langle\quad\rangle = 0$$

$$\left(\sum_i z_i^2\partial_i + 2\sum_i \Delta_i z_i + z^2\partial_z + 2\Delta_{21} z\right)\langle\quad\rangle = 0$$

$$\left(\partial_1 + \partial_2 + \partial_q + \partial_4 + \eta\partial_z\right)\langle\quad\rangle = 0$$

$$\left(z_1\partial_1 + \partial_2 + q\partial_q + z\partial_z + \sum\Delta\right)\langle\quad\rangle = 0$$

$$\left(z_1^2\partial_1 + \partial_2 + q^2\partial_q + 2\left(\Delta_1 z_1 + \Delta_2 + \Delta_3 q\right) + z^2\partial_z + 2\Delta_{21}z\right)\langle\quad\rangle = 0$$

$$(z_1^2 - z_1)\partial_1 + (q^2 - q)\partial_q + 2\sum\Delta_i z_i - \sum\Delta_i + \eta(\quad) = 0$$

$$\rightarrow \partial_1 = \frac{1}{z_1^2 - z_1}\left[-(q^2-q)\partial_q + \cdots\right] \sim -\frac{2\Delta_1}{z_1} \rightarrow 0 \text{ as } z_1 \to \infty$$

$$\partial_2 = -\left(z_1\partial_1 + q\partial_q + \sum\Delta_i + \eta\left(z\partial_z + \Delta_{21}\right)\right)$$

$$\rightarrow 2\Delta_1 - \left(q\partial_q + \sum\Delta_i + \eta(z\partial_z + \Delta_{21})\right) \quad \text{as } z_1 \to \infty$$

$$\partial_4 = (q-1)\partial_q + \overset{-2\Delta_1}{\sum\Delta_i} + \eta\left((z-1)\partial_z + \Delta_{21}\right)$$

$$\underset{(\eta=0)}{\hat{\Phi}_2} = \frac{\Delta_4}{z^2} + \frac{\Delta_3}{(z-q)^2} + \frac{\Delta_2}{(z-1)^2} + \underbrace{\frac{1}{z}\partial_4 + \frac{1}{z-1}\partial_2 + \frac{1}{z-q}\partial_q}$$

$$\frac{1}{z}\left((q-1)\partial_q + \underset{-2\Delta_1}{\sum\Delta_i}\right) - \frac{1}{z-1}\left(q\partial_q + \underset{-2\Delta_1}{\sum\Delta_i}\right) + \frac{1}{z-q}\partial_q$$

$$\left(\frac{1}{z} - \frac{1}{z-1}\right)q\partial_q + \left(-\frac{1}{z} + \frac{1}{z-q}\partial_q\right) + \frac{\sum\Delta_i - 2\Delta_1}{z} - \frac{\sum\Delta_i - 2\Delta_1}{z-1}$$

$$-\frac{q\partial_q}{z(z-1)} + \frac{q\partial_q}{z(z-q)} - \frac{\sum\Delta_i - 2\Delta_1}{z(z-1)}$$

$$= \frac{1}{z(z-1)(z-q)}\Big[\underbrace{(z-1)-(z-q)}_{(q-1)}\Big] q\partial_q - \frac{\overbrace{\sum_2^4 \Delta_i - \Delta_1}^{\tilde{\Delta}}}{z(z-1)} \checkmark$$

with $\eta$

$$\frac{(z-1)\partial_z + \Delta_{21}}{z} - \frac{z\partial_z + \Delta_{21}}{z-1} = -\frac{(2z-1)\partial_z + \Delta_{21}}{z(z-1)}$$

Null state eq.

$$\Delta_{21} = -\frac{1}{2} - \frac{3b^2}{4}$$

$\times\hbar^2$

$$\Bigg[b^{-2}\partial_z^2 + \underbrace{\frac{\Delta_4}{z^2} + \frac{\Delta_3}{(z-q)^2} + \frac{\Delta_2}{(z-1)^2}}_{V(z,\epsilon_1)} + \frac{q(q-1)\partial_q}{z(z-1)(z-q)} - \frac{\tilde{\Delta}}{z(z-1)} - \eta\underbrace{\frac{\overbrace{\Delta_{21}}^{0} + (2z-1)\partial_z}{z(z-1)}}_{\hbar^2\to 0}\Bigg]\Psi$$

$\epsilon_1 \equiv \frac{\hbar}{b}$ $\quad \epsilon_2 = \hbar b \quad \hbar, b\to 0$ limit $(\epsilon_2 \to 0)$

$$\Delta \sim \frac{\Delta_f}{\hbar^2}$$

let $\Psi(z) = \exp\left(-\frac{1}{\underset{\epsilon_1\epsilon_2}{\hbar^2}}\left(F(\epsilon_1) + \epsilon_2 W(z,\epsilon_1) + \cdots\right)\right)$

$$\lim_{\epsilon_2\to 0} \frac{F(\epsilon_1)}{\epsilon_1\epsilon_2} = -F_{LFT}\,(\text{conf. block})$$

$$\to \left(-\epsilon_1^2\partial_z^2 + V(z;\epsilon_1) + \frac{(q-1)}{z(z-1)(z-q)}\frac{\partial F(\epsilon_1)}{\partial \log q}\right)\Psi^{(0)}(z) = 0$$

"derivation"

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$N^* = 2$ as before

# Away from conformal [irregular]

① pure SYM

$$Z_{inst} = \langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle$$

let $\Psi(z) = \langle \Delta', \Lambda^2 | \Phi_{21}(z) | \Delta, \Lambda^2 \rangle$

$$\Delta = \Delta(\alpha - \tfrac{b}{4}), \quad \Delta' = \Delta(\alpha + \tfrac{b}{4})$$

$$\langle \Delta', \Lambda^2 | T(w)\, \Phi_{21}(z) | \Delta, \Lambda^2 \rangle \qquad T(w) = \sum_n \frac{L_n}{w^{n+2}}$$

$$= \sum_{n \geq -1} \frac{1}{w^{n+2}} \langle \Delta', \Lambda^2 | L_n \Phi_{21}(z) | \Delta, \Lambda^2 \rangle \qquad L_{n \geq 2} | \Delta, \Lambda^2 \rangle \equiv 0$$

$$[L_n, \Phi_{21}] + \Phi_{21} L_n$$

$$= \sum_{n=0}^{\infty} \frac{1}{w^{n+2}} \langle \Delta', \Lambda^2 | [L_n, \Phi_{21}(z)] | \Delta, \Lambda^2 \rangle + \frac{1}{w} \langle \Delta', \Lambda^2 | L_{-1} \Phi_{21} | \Delta, \Lambda^2 \rangle$$

$$+ \frac{1}{w^2} \langle \Delta', \Lambda^2 | L_0 \Phi_{21}(z) | \Delta, \Lambda^2 \rangle + \frac{1}{w^3} \langle \Delta', \Lambda^2 | \Phi_{21} L_1 | \Delta, \Lambda^2 \rangle$$

$$\to \langle \Delta', \Lambda^2 | L_{-2} \Phi_{21}(z) | \Delta, \Lambda^2 \rangle + b^{-2} \langle \Delta', \Lambda^2 | L_{-1}^2 \Phi_{21} | \Delta, \Lambda^2 \rangle$$

$$= \left[ -\frac{1}{z} \partial_z + \Lambda^2 \left( \frac{1}{z} + \frac{1}{z^3} \right) + \frac{1}{2z^2} \left( \frac{\Lambda}{2} \frac{\partial}{\partial \Lambda} + \Delta + \Delta' - \Delta_{21} - z \partial_z \right) \right] \Psi(z)$$

$$+ b^{-2} \partial_z^2 \Psi(z) = 0$$

$$0 = \left[ b^{-2} z^2 \partial_z^2 + \Lambda^2 \left( z + \frac{1}{z} \right) - \frac{3}{2} \cancel{z \partial_z} + \frac{\Lambda}{4} \partial_\Lambda + \frac{\Delta + \Delta' - \Delta_{21}}{2} \right] \Psi(z)$$

rescale $\Delta \to \frac{\Delta}{\hbar^2}$, $\Lambda^2 \to \frac{\Lambda^2}{\hbar^2}$ and let $\hbar \to 0$, $b \to 0$

$$\left[\epsilon_1^2 z^2 \partial_z^2 + \Lambda^2\left(z + \frac{1}{z}\right)\right] \Psi^{(0)} = \underbrace{[\cdots]}_{E} \Psi^{(0)}$$

$$z = e^{i\theta} \rightarrow \text{SGM} \checkmark$$

② 1 flavor

$F_{inst} = \langle \Delta, \Lambda, m | \Delta, \frac{\Lambda^2}{2} \rangle$

$\rightarrow \Psi(z) = \langle \Delta', \Lambda, m | \Phi_{21}(z) | \Delta, \frac{\Lambda^2}{2} \rangle$

Similar manipulation

$$\left[\epsilon_1^2 z^2 \frac{\partial^2}{\partial z^2} + z^2\left(\frac{\Lambda^2}{2z^3} - \frac{2m\Lambda}{z} - \Lambda^2\right)\right] \Psi^{(0)} = E\, \Psi^{(0)}$$

③ 2 flavors

$\Psi(z) \equiv \langle \Delta', \Lambda, m_2 | \Phi_{21}(z) | \Delta, \Lambda, m_1 \rangle$

$$\rightarrow \left[\epsilon_1^2 z^2 \frac{\partial^2}{\partial z^2} + z^2\left(-\frac{\Lambda^2}{z^4} - \frac{2m_1\Lambda}{z^3} - \frac{2m_2\Lambda}{z} - \Lambda^2\right)\right] \Psi = E\, \Psi$$