Journal Club 2014

노트 제목 2014-04-04

J. Cardy : Therm. & Revivals after Quantum Quench in CFT
arXiv 1403.3040 (II) + PRL 2006 (I) w/ Calabrese

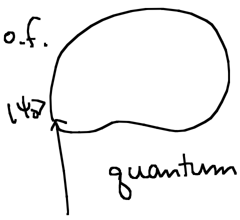
Quantum quench : change Hamiltonian suddenly

$t=0$ $\boxed{|\psi_0\rangle}$ of H_0 (ex) $|\psi_0\rangle$ ground state of H_0)

$t>0$ time evolution by H ← some parameter
"quench" : variation

$\boxed{|\psi(t)\rangle} = e^{-iHt}|\psi_0\rangle$

① finite d.o.f. after some time $|\psi(t_0)\rangle \to |\psi_0\rangle$



quantum recurrence

② "Thermodynamic limit : infi d.o.f" : need not come back.
Then $|\psi(t)\rangle \sim$ stationary is possible as $t \gg 1$?
or thermal ?

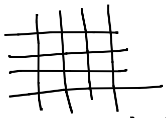
Motivation

* equilibration of quantum system
* quantum coherence is now maintained in long time experimentally
such as cold atom systems

Feshbach resonance tunes coupling to any value
Coherent nonequilibrium has been measured (T. Kinoshita et al)

Result
* if H is near quantum critical point ; universality in large r, t exists
* correlation functions are not power law but exponential
* $|\psi_0\rangle$ provides BC.

Formalism: Consider $\langle \mathcal{O}(t) \rangle$ rather than $|\psi(t)\rangle$



$|\psi_0\rangle \leftarrow$ not eigenstate of H

d-dim lattice + time

Hamiltonian H

$$\langle \mathcal{O}(t, \{\vec{r}_i\}) \rangle \equiv \langle \psi_0 | e^{iHt} \mathcal{O}(\{\vec{r}_i\}) e^{-iHt} | \psi_0 \rangle$$

$$\langle \mathcal{O}(t, \{\vec{r}_i\}) \rangle_\epsilon \equiv \frac{\langle \psi_0 | e^{iHt - \epsilon H} \mathcal{O}(\{\vec{r}_i\}) e^{-iHt - \epsilon H} | \psi_0 \rangle}{\langle \psi_0 | e^{-2\epsilon H} | \psi_0 \rangle}$$ will send $\epsilon \to 0$

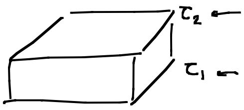
$\boxed{\tau_{1,2} = \mp\epsilon + \hat{\tau}}$

Path integral
$$= \underbrace{\int [\mathcal{D}\phi(\tau, \vec{r})] \langle \psi_0 | \phi(\tau_2, \vec{r}) \rangle \langle \phi(\tau_1, \vec{r}) | \psi_0 \rangle \, \mathcal{O}(\{\vec{r}_i\}) \, e^{-\int_{\tau_1}^{\tau_2} L[\phi] d\tau}}_{(d+1)\text{ dim}}$$

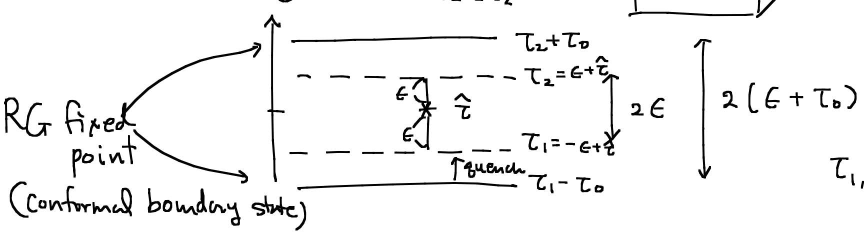
$\hookrightarrow \tau_1 \le \ \le \tau_2$

$\underline{\text{fixed}}$ $t = i\hat{\tau}$

with $\underline{\underline{\tau_1, \tau_2 = \mp\epsilon - it}}$



B.C. by $|\psi_0\rangle$



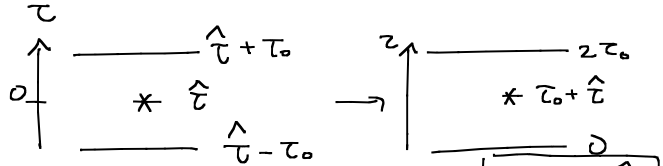
$\tau_{1,2} = \hat{\tau} \pm \epsilon$

$$H_0 = \underset{\substack{\uparrow \\ \text{CFT} \\ \text{(Critical)}}}{H} + \lambda \Phi_{\text{relevant}}$$

$\tau_0 \sim \frac{1}{m_0}$

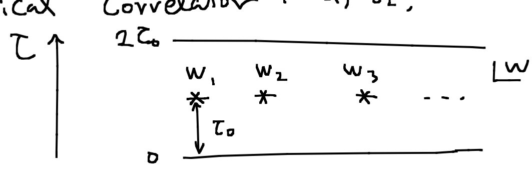
$m_0 \sim \lambda^{2\Delta_{\text{rel.}}}$

Now take $\epsilon \to 0$



$\boxed{\tau = \tau_0 + \hat{\tau}}$

Dynamical correlator: $t_1, t_2, \cdots$



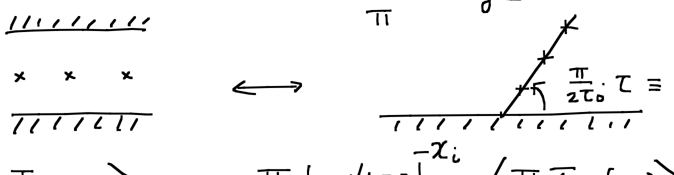
$(\text{Im}\, w = \tau)$

$w = r + i\tau$

$0 \leq \tau \leq 2\tau_0$

Conformal map

$$w = \frac{2\tau_0}{\pi} \log z \quad \text{or} \quad z = e^{\frac{\pi}{2\tau_0} w}$$



$w' = \frac{2\tau_0}{\pi z}$

$$\langle \prod_i \Phi_i(w_i) \rangle_{\text{strip}} = \prod_i |w'(z_i)|^{-x_i} \underbrace{\langle \prod_i \Phi_i(z_i) \rangle_{\text{UHP}}}_{\text{a lot easier}}$$

$$\boxed{w' = \frac{2\tau_0}{\pi} e^{-\frac{\pi}{2\tau_0} w}}$$

① one-point function

$$\langle \Phi(z)\rangle_{UHP} = \frac{A_b^{\Phi}}{|z-\bar z|^{x}}$$

$$\rightarrow \quad \langle \Phi(w)\rangle_{st} = A_b^{\Phi}\frac{|w'(z)|^{-x}}{\left|e^{\frac{\pi}{2\tau_0}w}-e^{\frac{\pi}{2\tau_0}\bar w}\right|^{x}} = A_b^{\Phi}\frac{\left(\frac{2\tau_0}{\pi}\right)^{-x}\left|\cancel{e^{\frac{\pi x}{2\tau_0}(r+i\tau)}}\right|}{\cancel{e^{\frac{\pi r}{2\tau_0}x}}\left|e^{\frac{i\pi}{2\tau_0}\tau}-e^{-\frac{i\pi}{2\tau_0}\tau}\right|^{x}}$$

$$= A_b^{\Phi}\left[\frac{\pi}{4\tau_0}\frac{1}{\sin\left(\frac{\pi\tau}{2\tau_0}\right)}\right]^{x}$$

$$\sin\frac{\pi\tau}{2\tau_0} = \sin\frac{\pi}{2}\left(1-\frac{it}{\tau_0}\right) \quad\Leftarrow\quad \tau=\tau_0+\hat\tau=\tau_0-it$$

$$= \cosh\frac{\pi t}{2\tau_0}$$

$$\therefore \quad = A_b^{\Phi}\left(\frac{\pi}{4\tau_0}\frac{1}{\mathrm{ch}\frac{\pi t}{2\tau_0}}\right)^{x} \underset{t\gg\tau_0}{\sim} A_b^{\Phi}\left(\frac{\pi}{2\tau_0}\right)^{x}e^{-\frac{x\pi t}{2\tau_0}}$$

② $\boxed{\text{2-pt correlation}}$

a free boson on UHP

$$\langle\Phi(z_1)\Phi(z_2)\rangle_{UHP} = \left(\frac{z_{1\bar 2}z_{2\bar 1}}{z_{12}z_{\bar 1\bar 2}z_{1\bar 1}z_{2\bar 2}}\right)^{x}\underbrace{F(\eta)}_{\text{universal}}, \quad \eta=\frac{z_{1\bar 1}z_{2\bar 2}}{z_{1\bar 2}z_{2\bar 1}}$$

$\downarrow$

strip

$$\langle\Phi(r,\tau)\Phi(0,\tau)\rangle_{st} = |w'(z_1)|^{-x}|w'(z_2)|^{-x}\langle\Phi(z_1)\Phi(z_2)\rangle_{UHP}$$

$$= \left(\frac{2\tau_0}{\pi}\right)^{-2x}e^{+\frac{\pi}{2\tau_0}(w_1+w_2)x}\left[\quad \wr \quad\right]F(\eta)$$

$$= \left[\left(\frac{\pi}{2\tau_0}\right)^{2}\frac{\mathrm{ch}\left(\frac{\pi r}{2\tau_0}\right)-\cos\left(\frac{\pi\tau}{\tau_0}\right)}{8\,\mathrm{sh}^2\frac{\pi r}{4\tau_0}\,\sin^2\frac{\pi\tau}{2\tau_0}}\right]^{x}F(\eta)$$

take $\tau=\tau_0+it$

$$= \left[\left(\frac{\pi}{2\tau_0}\right)^{2}\frac{\mathrm{ch}\left(\frac{\pi r}{2\tau_0}\right)+\cosh\left(\frac{\pi t}{\tau_0}\right)}{8\,\mathrm{sh}^2\frac{\pi r}{4\tau_0}\,\mathrm{ch}^2\frac{\pi t}{2\tau_0}}\right]^{x}F(\eta)$$

as $r,t\gg\tau_0$

$$\approx\left(\frac{\pi}{2\tau_0}\right)^{2x}\left(\frac{e^{\frac{\pi r}{2\tau_0}}+e^{\frac{\pi t}{\tau_0}}}{e^{\frac{\pi r}{2\tau_0}}\,e^{\frac{\pi t}{\tau_0}}}\right)^{x}F(\eta)$$

$$\eta\sim\frac{e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0}+e^{\pi t/\tau_0}}$$

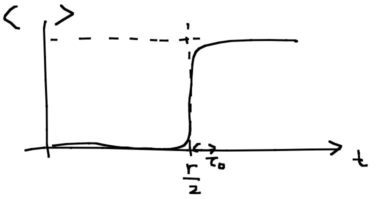
$$F(\eta)\sim\left(A_b^{\Phi}\right)^2\eta^{x_b}$$; x_b: boundary dimension of boundary operator

(1) $\frac{r}{2}>t\gg\tau_0 \rightarrow \frac{r}{2\tau_0}\gg\frac{t}{\tau_0}$; $\eta\sim e^{\frac{\pi}{\tau_0}\left(t-\frac{r}{2}\right)}$

$$\langle\Phi(r,t)\Phi(0,t)\rangle\sim\left(A_b^{\Phi}\right)^2e^{-\frac{\pi x t}{\tau_0}}e^{\frac{\pi x_b}{\tau_0}\left(t-\frac{r}{2}\right)}$$; decays rapidly

(2) $t>\frac{r}{2}$, $\frac{t}{\tau_0}\gg\frac{r}{2\tau_0}$; $\eta\sim 1$

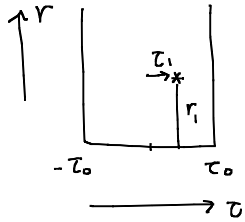
$$\langle\Phi(r,t)\Phi(0,t)\rangle\sim e^{-\frac{\pi x r}{2\tau_0}}$$; becomes stationary



③ $\langle \Phi(r,t)\Phi(0,s)\rangle \sim \begin{cases} e^{-\frac{\pi x}{4\tau_0}(t+s)} & r > t+s \\ e^{-\frac{\pi x}{4\tau_0} r} & t+s > r > t-s \\ e^{-\frac{\pi}{4\tau_0}|t-s|} & r < |t-s| \end{cases}$

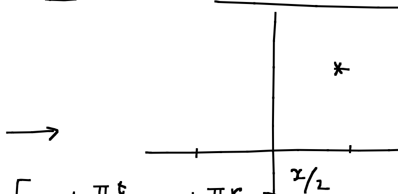
$s \neq t$

④ $\frac{1}{2}$-space



$w = \tau + ir \rightarrow z = \sin\frac{\pi w}{2\tau_0} = \sin\left(\frac{\pi\tau}{2\tau_0} + i\frac{\pi r}{2\tau_0}\right)$

$= \sin\tau\, \mathrm{ch}\, r + i \cos\tau\, \mathrm{sh}\, r$



$$\langle\Phi\rangle_{\frac{1}{2}st} \propto \left[\frac{\mathrm{ch}\frac{\pi t}{\tau_0} + \mathrm{ch}\frac{\pi r}{\tau_0}}{\mathrm{ch}^2\frac{\pi t}{2\tau_0}\ \mathrm{sh}^2\frac{\pi r}{2\tau_0}}\right]^{x/2} \sim \begin{cases} e^{-\frac{\pi x t}{2\tau_0}} & t < r \\ e^{-\frac{\pi x r}{2\tau_0}} & t > r \end{cases}$$

⑤ (nonconformal) Gaussian chain on 1D lattice : Quench $m_0 \to m$

$$H = \frac{1}{2}\sum_r \left[\pi_r^2 + m^2\phi_r^2 + \omega^2(\phi_{r+1}-\phi_r)^2\right]$$

quadratic!

$[\pi_r, \phi_{r'}] = -i\delta_{rr'}$

Fourier $\left(\tilde{f}_k = \frac{1}{\sqrt{N}}\sum_{r=0}^{N-1} e^{\frac{2\pi i k r}{N}} f_r\right)$

$= \sum_k \Omega_k \tilde{A}_k^+ \tilde{A}_k$

$\tilde{A}_k = \frac{1}{\sqrt{2\Omega_k}}(\Omega_k\tilde{\phi}_k + i\tilde{\pi}_k)$

$\tilde{A}_k = \frac{1}{\sqrt{2\Omega_k}}(\Omega_k\tilde{\phi}_{-k} - i\tilde{\pi}_{-k})$

$\Omega_k^2 = m^2 + 4\omega^2\sin^2\frac{\pi k}{N}$

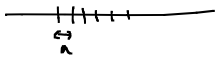
$$\phi_r^H(t) = \sum_{k=0}^{N-1}\sqrt{\frac{2}{N\Omega_k}}\left(e^{i\left(\frac{2\pi k}{N}r - \Omega_k t\right)}\tilde{A}_k + \text{c.c.}\right)$$

$\langle\psi(t)|\phi_r\phi_0|\psi(t)\rangle = \langle\psi_0|\phi_r^H(t)\phi_0^H(t)|\psi_0\rangle$

$\shortparallel$

$$\langle\phi_r^H(t)\phi_0^H(t)\rangle \underset{N\to\infty}{=} \int_{1\text{st BZ } (|k|<\frac{\pi}{a})} \frac{dk}{2\pi} e^{ikr}\frac{\left[(\Omega_{0k}^2+\Omega_k^2)-(\Omega_{0k}^2-\Omega_k^2)\cos(2\Omega_k t)\right]}{\Omega_k^2\,\Omega_{0k}} \quad \leftarrow (*)$$

Continuum limit $(a\to 0)$, $\Omega_k^2 = m^2 + 4\underbrace{\frac{\omega^2\pi^2 k^2}{N^2}}_{\equiv p^2} = m^2 + p^2$, $\Omega_{0k}^2 = m_0^2 + p^2$



$$\langle\phi_r^H(t)\phi_0^H(t)\rangle = \int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ipr}\frac{(m^2-m_0^2)\cos(2\sqrt{m^2+p^2}\,t) + m^2 + m_0^2 + 2p^2}{(m^2+p^2)\sqrt{m_0^2+p^2}}$$

if Quench from massive $m_0 \to m=0$ (conformal) with $\tau_0 = \frac{1}{m_0} \to 0$

$$\underset{\text{as } m_0\to\infty}{=} m_0\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ipr}\frac{1-\cos 2pt}{p^2} = m_0\begin{cases} 0 & t < \frac{r}{2} \\ t - \frac{1}{2}r & t > \frac{r}{2} \end{cases}$$

(In general, $r, t \gg 1$ (*) stationary point $\int dk\, e^{i[f(k)r + g(k)t]}$

$f'(k)r + g'(k)t = 0$

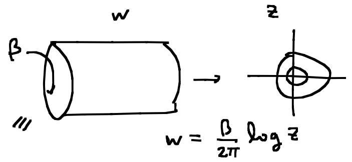
[Sum] universal behaviour

$r \pm 2\Omega_k' t = 0 \rightarrow$ L & R movers

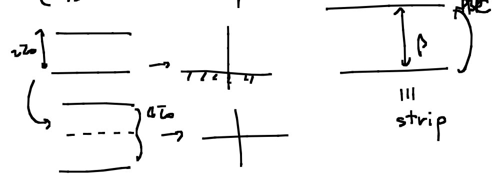
transition when $t = \frac{r}{2} \equiv t^*$

$t > t^* \rightarrow e^{-\frac{x\pi r}{2\tau_0}}$

(cf) finite temperature correlation :



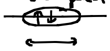
$w = \frac{\beta}{2\pi}\log z$



$\rightarrow \beta = 4\tau_0$

(Picture)

$|\psi_0\rangle \xrightarrow[\text{quench}]{} |\psi(t)\rangle \xrightarrow[t=\frac{r}{2}=t^*]{} |\psi(\text{stationary})\rangle$ & thermal with $\beta = 4\tau_0$

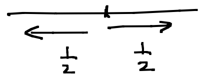
entangled

τ_0

$\longrightarrow$ $|\, L \leftarrow R$ propagation $\rangle$ entangled.

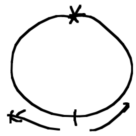
ee = Gibbs entropy with $\beta = 4\tau_0$

II. (1403.3040) deals with finite size effect L

[Motivation] $t \pm \frac{r}{2}$;



$L=\infty$


L finite

quantum recurrence even with ∞ d.o.f.

occurs at $\frac{L}{2} \times n$

$$|\Psi_0\rangle \longrightarrow |\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$$

Fidelity: $F(t) = |\langle\Psi_0| e^{-iHt}|\Psi_0\rangle|$

$\Leftrightarrow$ partition function of annulus

$$H_0 = \underset{\downarrow}{\overset{\text{CFT}}{H}} + \lambda\int\Phi \qquad (m_0(\lambda)^{-1} \ll L)$$

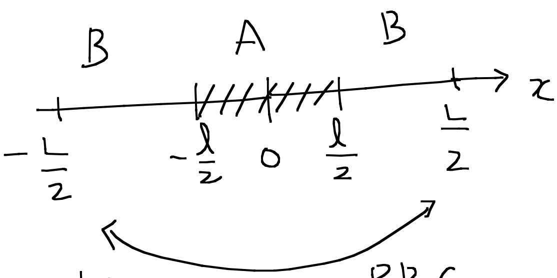
$\downarrow$

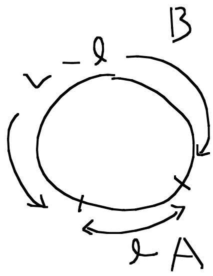
$|B\rangle$ $\qquad (\beta = 4\tau_0)$

Let

$$|\Psi_0\rangle = e^{-\frac{\beta}{4}H}|B\rangle$$

Thermalization of subsystem

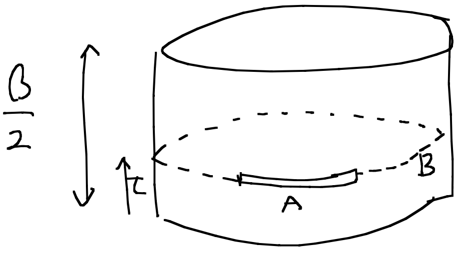


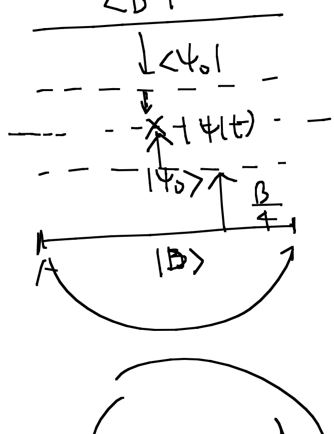


PBC

reduced density

$$\rho_A(\beta,\tau) \propto \mathrm{Tr}_{\mathcal{H}_B}\Big(e^{-\tau H}\underbrace{|\Psi_0\rangle}_{e^{-\frac{\beta}{4}H}|B\rangle}\langle\Psi_0| e^{\tau H}\Big)$$



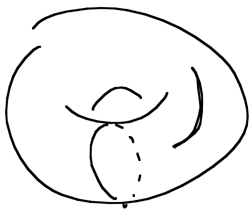


Annulus
$\mathcal{A}$

Thermal density matrix

$$\tilde{\rho}_A(\beta) \propto \mathrm{Tr}_{\mathcal{H}_B} e^{-\beta H}$$

$\uparrow$ indep of t

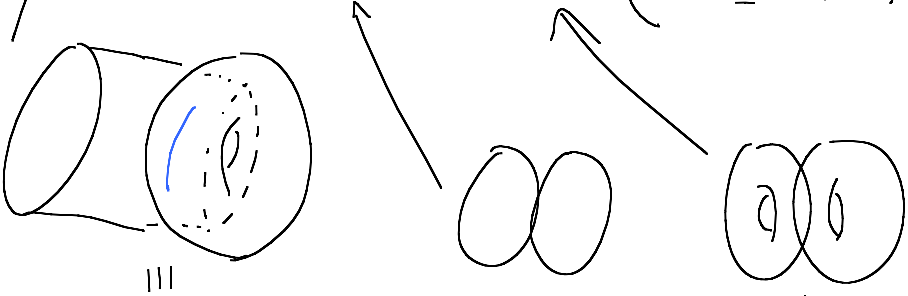


Torus
$\mathcal{T}$

Closeness

$$I(\tau) \equiv \frac{\mathrm{Tr}_{\mathcal{H}_A}(\rho_A \tilde{\rho}_A)}{\sqrt{\mathrm{Tr}_{\mathcal{H}_A}(\rho_A^2)\,\mathrm{Tr}_{\mathcal{H}_A}(\tilde{\rho}_A^2)}} \leq 1$$

$(= 1 \text{ if } \rho_A = \tilde{\rho}_A)$



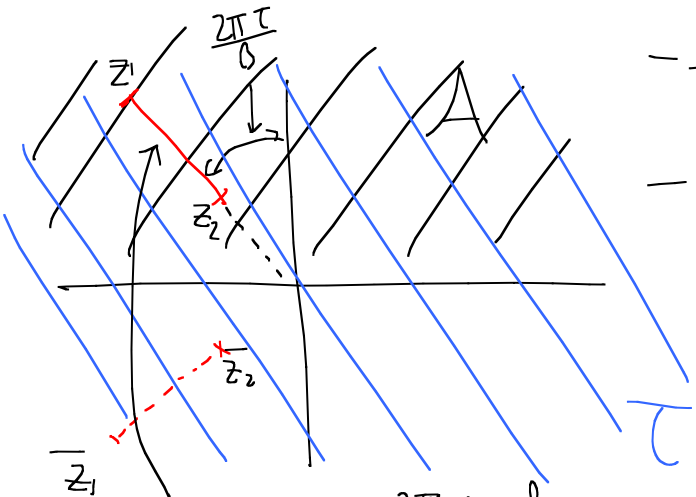
$$\underbrace{Z_{A\oplus T} \qquad Z_{A\oplus A} \qquad Z_{T\oplus T}}_{\text{difficult to compute}}$$

Now, Conformal mapping as $L \to \infty$

$$w = x + i\tau \longrightarrow z = i e^{2\pi w/\beta}$$

$$= i e^{2\pi x/\beta + i\frac{2\pi}{\beta}\tau}$$

$$= e^{2\pi x/\beta}\left(-\sin\frac{2\pi}{\beta}\tau + i\cos\frac{2\pi}{\beta}\tau\right)$$



$-\frac{\beta}{4} \leq \tau \leq \frac{\beta}{4}$ A

$-\frac{\beta}{2} \leq \tau \leq \frac{\beta}{2}$ τ

$$z_{1,2} = i e^{\frac{2\pi}{\beta}\left(\pm\frac{l}{2} + i\underbrace{\tau}_{t}\right)}$$

$z_1 \times$ $\times z_2$

A: $z_1 \times$ $\times z_2$ $\frac{l}{2} > t \gg \beta$ vs. $\beta \ll \frac{l}{2} < t$ $\approx \tau$

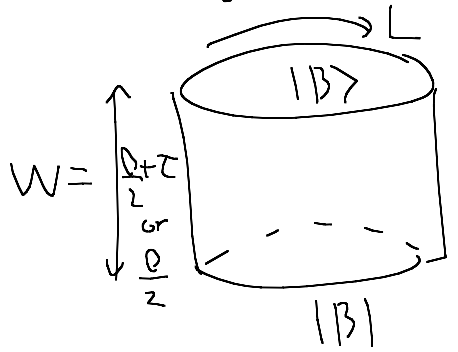
$A \approx \tau \to I \approx 1$

Deviation

$$I - 1 \sim - e^{-4\pi \underbrace{\Delta_{min}}_{\text{min. } \Delta \text{ of } \langle B|\Phi|B\rangle \neq 0} (t - \frac{\ell}{2})/\beta}$$

$\boxed{\text{Return amplitude}}$

$$F = \left| \frac{\langle \Psi_0 | e^{-itH} | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \right| = \left| \frac{\langle B | e^{-\frac{\beta}{4}H} e^{-itH} e^{-\frac{\beta}{4}H} | B \rangle}{\langle B | e^{-\frac{1}{4}\beta H} e^{-\frac{1}{4}\beta H} | B \rangle} \right|$$



no cut

$$= \left| \frac{Z_A(\frac{\beta}{2} + it, L)}{Z_A(\frac{\beta}{2}, L)} \right|$$

Fusion coef.

$$Z_A(W, L) = \sum_{\Delta} |B_\Delta|^2 \chi_\Delta(q) = \sum_{\tilde{\Delta}} n^{\tilde{\Delta}}_{BB} \chi_{\tilde{\Delta}}(\tilde{q})$$

$$B_\Delta = \langle B | \Delta \rangle\rangle$$

$W = \text{time}$ $\qquad$ $L = \text{time}$

$$q \equiv e^{2\pi i \tau} = e^{-4\pi W/L}, \quad \tilde{q} = e^{-2\pi i/\tau} = e^{-\pi L/W}$$

Character

$$\chi_\Delta(q) = q^{-\frac{c}{24} + \Delta} \sum_{N=0}^{\infty} d_N q^N, \quad \text{assume } n^0_{BB} = 1$$

$(B \times \mathbb{1} \sim B)$

$\updownarrow$

$\chi_{\tilde{\Delta}}(\tilde{q})$ by modular transforms

$T: \tau \to \tau + 1$

$S: \tau \to -\frac{1}{\tau}$

$$\tau = 2i\frac{W}{L} = \frac{1}{L}(-2t+i\beta)$$

$$\tilde{\tau} = -\frac{1}{\tau} = \frac{L}{2t-i\beta} = \frac{L(2t+i\beta)}{(2t)^2+\beta^2}$$

$$\tilde{q} = e^{2\pi i\tilde{\tau}} \to |\tilde{q}| = e^{-\frac{2\pi L\beta}{(2t)^2+\beta^2}} \ll 1 \quad \text{if} \quad \boxed{t^2 \ll L\beta}$$

$$\sum_{\tilde{\Delta}} n_{BB}^{\tilde{\Delta}} \chi_{\tilde{\Delta}} \sim \tilde{q}^{-\frac{c}{24}}(1+\cdots) + (\tilde{\Delta}\neq 0)$$

$$\left|Z_A\left(\frac{\beta}{2}+it, L\right)\right| \approx e^{\frac{2\pi L\beta}{(2t)^2+\beta^2}\cdot\frac{c}{24}}$$

$$\left|\frac{Z_A\left(\frac{\beta}{2}+it, L\right)}{Z_A\left(\frac{\beta}{2}, L\right)}\right| \approx e^{\left(\frac{2\pi L\beta}{(2t)^2+\beta^2} - \frac{2\pi L\beta}{\beta^2}\right)\frac{c}{24}}$$

$$-\frac{2\pi L\beta}{\beta^2}\frac{(2t)^2}{(2t)^2+\beta^2}\cdot\frac{c}{24} = -\frac{\pi L}{3\beta}\frac{t^2}{(4t^2+\beta^2)}$$

$$(t \gg \beta) \qquad \approx e^{-\frac{\pi L}{12\beta}} \leftarrow (t \text{ indepedent})$$

---

correction $\quad (1+|\tilde{q}|^{\alpha}+\cdots)$

$$\alpha = \Delta_{min} \ni;\ n_{BB}^{\tilde{\Delta}} \neq 0$$
$$\Delta_{min} \neq 0$$

<u>Revival</u>

$$t = \frac{L}{2}n \to \tau \simeq -n \quad (\beta \ll L)$$

$$\beta \to 0 \quad \langle B|B\rangle = 1$$

$$\therefore F(t) = |Z_A(it, L)| = \sum_{\Delta} |B_\Delta|^2 \chi_\Delta(q = e^{-2\pi n i})$$

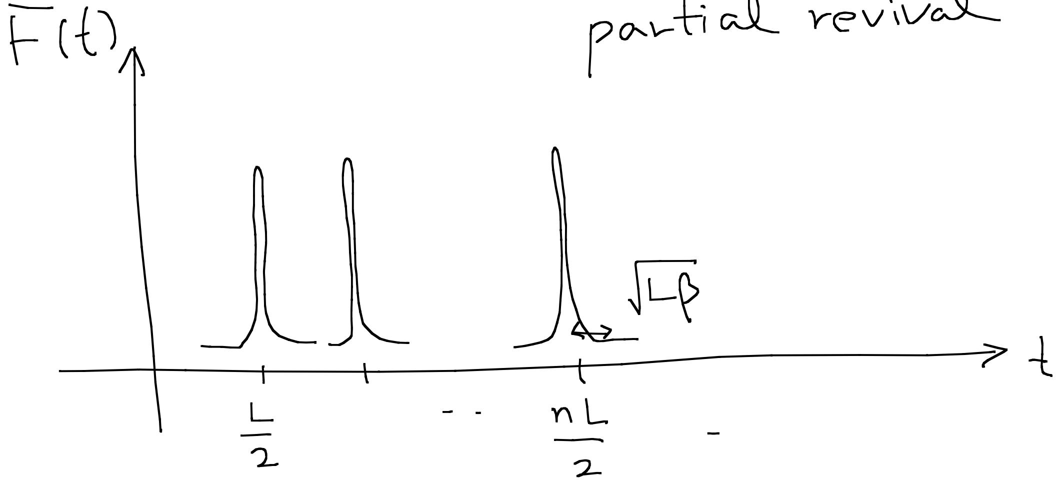
$$e^{2\pi i(1)} \xrightarrow[S;\ \tau \to -\frac{1}{\tau}]{} e^{-2\pi i} \xrightarrow[T^n]{} e^{-2\pi i n}$$

$$= \sum_{\Delta} |B_\Delta|^2 (T^n S)\, \chi_\Delta(\tau=0)$$

$$F\left(\frac{nL}{2}\right) = \sum_{\tilde\Delta} n_{BB}^{\tilde\Delta}\, \underbrace{(T^n S)\chi_{\tilde\Delta}(\tilde\tau)}_{(S T^n S)\,\chi_{\tilde\Delta}(\tau=0)}$$

$= \text{finite}$

partial revival



$$\tilde q = e^{2\pi i \frac{L}{2t}} \qquad \tilde q^{-\frac{c}{24}+\tilde\Delta+N}$$

$$\tilde q^{\tilde\Delta} = e^{-\frac{2\pi i}{n}\tilde\Delta}$$

↑ rational

$F(t)=1$ (complete revival) should happen

if $e^{iE_nT} = 1$ for all E_n

CFT. $E_n = \frac{4\pi}{L}(\Delta_n + N_n) = \frac{4\pi}{ML} \times \text{Integer}$

let $M =$ Common divider of all Δ_n

$\therefore\ t = \frac{ML}{2}$ should give $F=1$

For minimal CFT; $c = 1 - \frac{6(p-q)^2}{pq}$

$$\Delta_{rs} = \frac{(pr-qs)^2-(p-q)^2}{4pq}$$

$$M = 4pq$$

as $c \to 1$ $(pq \to \infty)$ $\longrightarrow$ $(M \to \infty) \Rightarrow$ no revival

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$\tau = 0 \longrightarrow \tau = \frac{n}{m}$ given by $ST^{n_1}ST^{n_2}\cdots$

$n_1, n_2 \cdots$

$$\frac{n}{m} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$$