Exact results on WLs in $\mathcal{N}=4$ SYM & ABJM

- Motivation: Nonperturbative is beyond imagination and can be understood only by results valid for <u>any coupling</u> in exact, quantitative way. ~~Pert~~, ~~semiclass~~. ~~qualitative~~, ~~BPS~~, ...

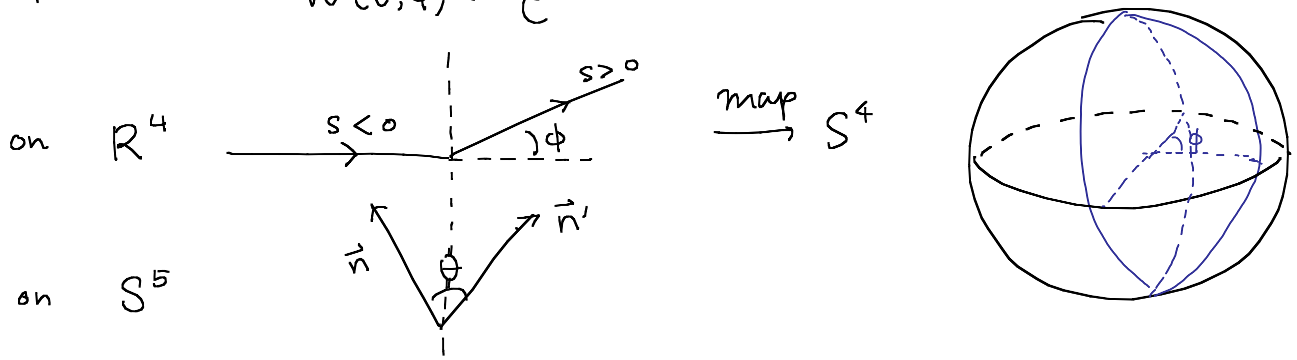
- AdS/CFT provides a promise for exact nonpert. results

- Integrability has been discovered for AdS_{d+1}/CFT_d $d=3,4$ and "spectrum problem" have been exactly solved.

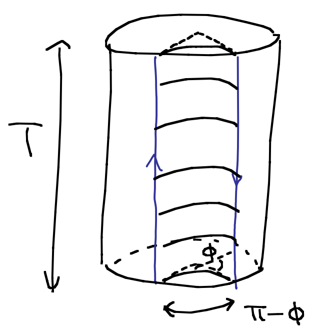
- More exact solutions?
  - higher-point correlation function } some progress but
  - space-time scattering amplitudes } very difficult

★ Exact (any λ) results on ⟨Wilson-Maldacena loops⟩

Cusped WL: $W(\theta,\phi) = e^{\int_C \left(iA_\mu \cdot \dot{x}^\mu + \vec{n}\cdot\vec{\Phi}\,|\dot{x}|\right) ds}$



Map: $R^4 \longrightarrow R\times S^3$



cutoffs

$$\langle W_{cusp}\rangle = e^{-\Gamma_{cusp}(\lambda,\phi,\theta)\log\frac{L}{\varepsilon}}$$

$$\langle W_{Lines}\rangle = e^{-T\,V_{q\bar{q}}(\lambda,\theta,\phi)}$$

Physical observables

① $\phi \to i\varphi$: IR limit scatt. of mass particles , φ: boost

② Regge limit of 4-particle scatt. amplitude of $N=4$ SYM
   ($t \gg s$) $\log A \sim \log t \cdot \Gamma_{cusp}(\varphi)$

③ $\varphi \to \infty$, $\Gamma_{cusp} \sim \varphi \Gamma^{\infty}_{cusp}$: IR div of massless particles

④ quark-$\bar{q}$ potential on S^3

⑤ $\phi \ll 1$, radiation from quark

Non perturbative (any λ) results on WL

⓪ $\theta = \phi = 0$ [Circular WL] $\Gamma_{cusp} = 0$ $\frac{1}{2}$ BPS , $\langle W_0 \rangle (\lambda, N)$

① $\phi = 0$ (No cusp), $\theta \neq 0$

or circle τ with $n_1 + i n_2 = \sin\theta \, e^{i\tau}$

$n_3 = \cos\theta$, $n_{4-6} = 0$

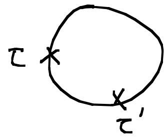
$\boxed{\theta = \frac{\pi}{2} , \frac{1}{4}\text{- BPS}}$.

$$\underline{\langle W_\theta \rangle (\lambda) = \langle W_0 \rangle (\lambda \cos^2\theta)}$$

if $\theta \ll 1$:
$\lambda' \cong \lambda(1-\theta^2)$

$$\frac{\langle W_\theta \rangle - \langle W_{\theta=0} \rangle}{\langle W_{\theta=0} \rangle} \cong -\theta^2 \lambda \partial_\lambda \log \langle W_0 \rangle$$

$$= \theta^2 \frac{1}{2} \int_0^{2\pi} d\tau \int_0^{2\pi} d\tau' \underbrace{\hat{n}^i(\tau) \hat{n}^j(\tau')}_{\text{from } \hat{n}^1 \,\&\, \hat{n}^2} \langle\langle \Phi^i(\tau) \Phi^j(\tau') \rangle\rangle$$



$\langle\langle \Phi(t) \Phi(0) \rangle\rangle = \frac{\gamma}{t^2}$, $\langle\langle \Phi(\tau) \Phi(0) \rangle\rangle = \frac{\gamma}{2(1-\cos\tau)}$ $\frac{\gamma \delta_{ij}}{2(1-\cos(\tau-\tau'))}$

$$= \theta^2 \left(\frac{2\gamma\pi}{\underset{\text{cut}}{\underbrace{\varepsilon}}} - \pi^2 \gamma \right) = -\pi^2 \gamma \theta^2$$

$$\therefore \quad \underline{\gamma = \frac{1}{\pi^2} \lambda \partial_\lambda \log \langle W_0 \rangle}$$

Localization: $\langle W_0 \rangle = \frac{1}{N} L^1_{N-1}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}}$, $\lambda = g_{YM}^2 N$

$$\lambda \partial_\lambda \langle W_0 \rangle = \frac{1}{2} \frac{\sqrt{\lambda}\, I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} + O\left(\frac{1}{N^2}\right) \equiv 2\pi^2 B$$

② $\theta, \phi \ll 1$

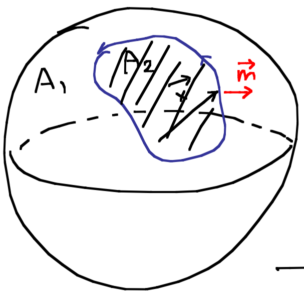
$\theta = \phi \longrightarrow$ BPS WL $\to \Gamma_{cusp} = 0$ if $\theta = \phi$

$\therefore\ \Gamma_{cusp} \simeq (\theta^2 - \phi^2)\, B(\lambda)$

$\Rightarrow\ \theta = 0\ ,\ \phi \ll 1$

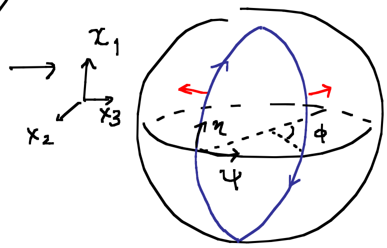
$\Gamma_{cusp} \simeq -\phi^2\, B(\lambda)$

③ $(\theta = \phi)\ \frac{1}{8}$BPS : $\Gamma_{cusp} = 0$



$S^2 \subset S^4$, $\vec{n} = (\vec{m}, 0, 0, 0)$ with $\vec{m} = \vec{x} \times \dot{\vec{x}}$

$\lambda \to \tilde{\lambda} = \lambda \frac{4 A_1 A_2}{(A_1 + A_2)^2}$ $\quad \langle W_\phi \rangle = \langle W_\odot \rangle (\tilde{\lambda})$



$\tilde{\lambda} = \lambda \frac{4(\pi - \phi)(\pi + \phi)}{(2\pi)^2} = \lambda \left(1 - \frac{\phi^2}{\pi^2}\right)$

$\vec{x} = \left(\sin\eta,\ \cos\eta \cos\psi,\ \cos\eta \sin\psi \right)$

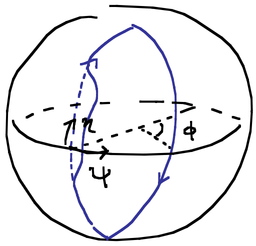
$\sin\eta = \tanh\tau$

$\vec{m} = \dot{\eta}\, (0, \sin\psi, -\cos\psi)$

$\psi = \begin{cases} 0 & \tau < 0 \\ \pi - \phi & \tau > 0 \end{cases} \Rightarrow \begin{array}{l} \dot{\psi} = 0 \\ |\dot{\eta}| = 1 \end{array}$

$= \begin{cases} (0, 0, -1) & \tau < 0 \\ (0, -\sin\phi, -\cos\phi) & \tau > 0 \end{cases}$ $\Big\} \to \underline{\theta = \phi}$ $\quad \vec{m}(\tau>0) \cdot \vec{m}(\tau<0) \equiv \cos\theta$

④ $|\theta - \phi| \ll 1$ $\quad (\theta, \phi \sim \mathcal{O}(1))$



slight deformation

displacement operator

$$\delta W = \left[P \cdot \int dt \left(\delta x^j(t)\, \mathbb{D}_j(t) + \delta\vec{n} \cdot \vec{\Theta} \right) \right] \cdot W$$

$$\underset{(\tau<0)}{\vec{m}} = \underset{(0,0,-1)}{\vec{m}_0} + \epsilon \left[(0, \sigma, 0) + \dot{\sigma} \left(\frac{1}{\cosh\tau}, -\tanh\tau, 0 \right) \right]$$

$\delta\phi = \epsilon\sigma, \quad \delta\theta = \epsilon(\sigma - \dot{\sigma}\tanh\tau)$

$\vec{m}_< \cdot \vec{m}_> \equiv \cos\theta = \cos\phi - \epsilon \sin\phi \left[\sigma - \dot{\sigma} \tanh\tau \right] \to \delta\theta = \epsilon [\sigma - \dot{\sigma}\tanh\tau]$

$$\frac{\delta\langle W \rangle}{\langle W \rangle} = \int dt \left[\langle\!\langle \mathbb{D}_i \rangle\!\rangle\, \delta x^i(t) + \langle\!\langle \vec{\Phi} \rangle\!\rangle \cdot \delta\vec{n} \right]$$

$$= \epsilon \int d\tau \Big[\underbrace{\langle\!\langle \mathbb{D} \rangle\!\rangle\, \sigma(\tau) + \langle\!\langle \Phi^2 \rangle\!\rangle \big(\sigma(\tau) - \dot{\sigma} \tanh\tau\big)}_{\tau\text{-indep. (time translation)}} \Big]$$

from $\frac{1}{8}$ BPS condition

$$\frac{\delta \langle W\rangle}{\langle W\rangle} = \epsilon\, \partial_\phi \log\langle W_\phi\rangle = \epsilon \frac{\partial\tilde\lambda}{\partial\phi}\partial_{\tilde\lambda}\log\langle W_0\rangle(\tilde\lambda)$$

$$= -4\epsilon \frac{\phi}{(1-\frac{\phi^2}{\pi^2})} B(\tilde\lambda)$$

if $\dot\sigma = 0$, $\quad \delta\phi = \delta\theta \longrightarrow \delta\langle W\rangle = 0 \quad$ if $\quad \langle\langle D\rangle\rangle + \langle\langle \Phi^2\rangle\rangle = 0$

$$\therefore \frac{\delta\langle W\rangle}{\langle W\rangle} = -\epsilon \langle\langle \Phi^2\rangle\rangle \int_{-\infty}^{\infty} d\tau\, \dot\sigma\, \mathrm{th}\tau = \epsilon \langle\langle\Phi^2\rangle\rangle \underbrace{\int_{-\infty}^{\infty} d\tau \frac{\sigma(\tau)}{\mathrm{ch}^2\tau}}_{\approx \int d\tau \frac{1}{\mathrm{ch}^2\tau} = 2} = 2\epsilon\langle\langle\Phi^2\rangle\rangle$$

$$\Rightarrow \langle\langle D\rangle\rangle = -\langle\langle\Phi^2\rangle\rangle = \frac{2\phi}{1-\frac{\phi^2}{\pi^2}} B(\tilde\lambda)$$

$$\therefore \Gamma_{cusp} = -\langle\langle D\rangle\rangle\delta\phi - \langle\langle\Phi^2\rangle\rangle\delta\theta = (\delta\phi-\delta\theta)\langle\langle\Phi^2\rangle\rangle$$

$$= -(\phi-\theta)\frac{2\phi}{1-\frac{\phi^2}{\pi^2}} B(\tilde\lambda)$$

$$\rightarrow \boxed{\Gamma_{cusp} = -\frac{(\phi^2-\theta^2)}{(1-\frac{\phi^2}{\pi^2})} B(\tilde\lambda)}$$

for any ϕ, θ, $|\phi-\theta| \ll \phi, \theta$

④ Any θ, ϕ [Integrability technique needed]

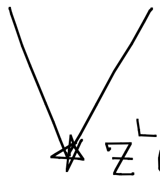
perturb. computation can be done upto λ^2 order

$$\left.\begin{aligned}\Gamma_{cusp}(\lambda,\phi,\theta) &= -\frac{\lambda}{8\pi^2}(\cos\theta-\cos\phi)\frac{\phi}{\sin\phi} \\ &+ \left(\frac{\lambda}{8\pi^2}\right)^2 F(\theta,\phi) + \cdots\end{aligned}\right\}$$

consistent with above for $\lambda \ll 1$

Non pert. calculation for any θ, ϕ & λ

- Review of Integrability [Drukker; Correa, Maldacena, Sever]
- Insert Z^L into WL.

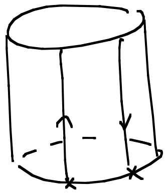


$Z^L(0)$

$$\mathrm{tr}\, P\left[e^{\int_0^\infty ds\,[iA_\mu\dot x^\mu + \vec n\cdot\vec\Phi|\dot x|]}\, Z^L(s=0)\, e^{\int_{-\infty}^0 ds\,[\,\cdot\,]}\right]$$

: non BPS ! except $\theta = \phi$

Symmetry



$\underbrace{R\times S^3}_{SO(4)}$ $\longrightarrow$ time dilation

$SO(4) \longrightarrow SO(3) \approx SU(2)$ rotation around lines

$\vec{n} = (0,0,0,1,0,0)$ $\qquad \vec{n}\cdot\vec{\Phi} = \Phi^4$

$SO(6) \longrightarrow SO(5) \longrightarrow SO(3) \simeq SU(2)$

$Z = \Phi^5 + i\Phi^6$: sym which makes Z^L invariant

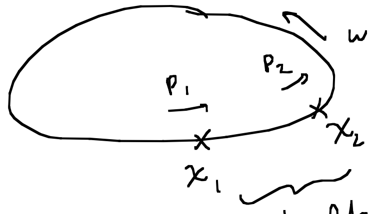
$\Rightarrow$ $SU(2|2)$

$SU(2|2) \otimes SU(2|2) \xrightarrow[\text{breaks}]{} SU(2|2)_D$

Operator space

for single trace operators ; we mapped to (1+1) QFT

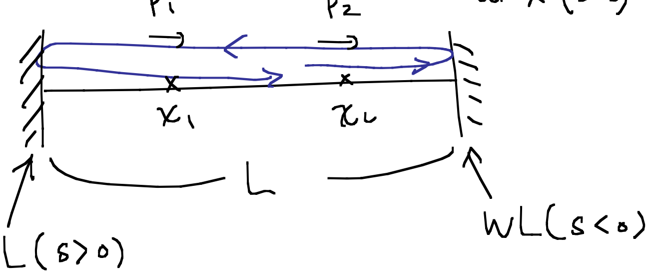
$\mathrm{tr}\left[Z \cdots X_1 \cdots X_2 \cdots Z \right] \rightarrow$



Wilson loop with one-operator insertion

$$\mathrm{tr}\, P\left[e^{\int_0^\infty ds\, [i A_\mu \dot{x}^\mu + \vec{n}\cdot\vec{\Phi}|\dot{x}|]} \underbrace{Z \cdots X_1 \cdots X_2 \cdots Z}_{\text{at } x^\mu(s=0)}\, e^{\int_{-\infty}^0 ds\, [\,\cdot\,]} \right] \;:\; \text{non}$$

$\Rightarrow$

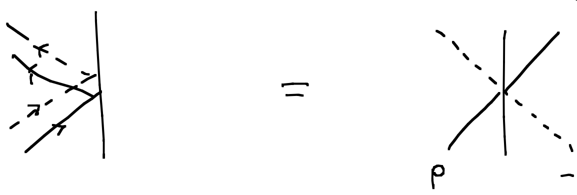


1D space with boundaries

In addition to bulk-scattering, we need "boundary scattering"

$R\ A_i(p) = R_i{}^j\ A_j(-p)$ $\qquad i, j \in \underset{A}{SU(2|2)} \otimes \underset{\dot A}{SU(2|2)}$ $\qquad i = (A\dot{A})$, $j = (B\dot{B})$

$[R, Q_{SU(2|2)_D}] = 0 \quad \Rightarrow \quad R^{(B\dot B)}_{(A\dot A)}(p) \propto S^{B\dot B}_{A\dot A}(p,-p)$

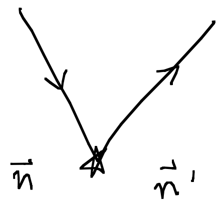


PBC $\longrightarrow$ Boundar BC.

$$e^{2ip_iL}\left[R_L(-p)\,S(p_1,p_i)\cdots R_R(p)\quad S(p_1,-p_i)\cdots S(p_N,-p_i)\right]=1$$

diffent SO(3)

With cusp $\quad \vec{n}=(\underbrace{0,0,0}_{SO(3)},1,0,0)\quad,\ \vec{n}'=(\underbrace{0,0,\sin\theta,\cos\theta}_{\text{diffent }SO(3)},0,0)$



$$x^\mu(s)=\begin{cases}(s,\underbrace{0,0,0}) & s\le 0 \leftarrow SO(3)\\ (s\cos\phi,\ s\sin\phi,\ 0,\ 0) & s\ge 0 \leftarrow \text{diffent } SO(3)\end{cases}$$

can be described by rotation around $\left(\begin{array}{l}5\text{-}6 \text{ plane}\\ x^1\end{array}\right.$

Lorentz, IR $\rightarrow$ SU(2|2)

$$m=\left(\begin{array}{cc|cc} e^{i\phi} & & & \\ & e^{-i\phi} & & \\ \hline & & e^{i\theta} & \\ & & & e^{-i\theta}\end{array}\right)$$

$$\underline{R_L=\left(m^{-1}R_R\,m\right)}$$

diagonalize $\rightarrow$ new Asymp. BAE $\longrightarrow$ Boundary TBA, Y, Luscher.

If no bulk excitation ; with only Z^L answer.

$$\boxed{\Delta E\sim-\frac{\cos\phi-\cos\theta}{\sin\phi}\sum_{a=1}^{\infty}(-1)^a\left(\frac{-1+\sqrt{1+16g^2/a^2}}{1+\sqrt{1+16g^2/a^2}}\right)^{1+L}\frac{a\cdot\sin a\phi}{\sqrt{\quad''}}\ F(a,g)}$$

* weak coupling ($g\ll1$) expansion (& $L\to 0$)

$$\Delta E\to\Gamma_{cusp}=-4g^2\ \frac{\cos\phi-\cos\theta}{\sin\phi}\sum_{a=1}^{\infty}(-1)^a\frac{\sin a\phi}{a}=2g^2\ \frac{\cos\phi-\cos\theta}{\sin\phi}\ \phi\quad\checkmark$$

$$\boxed{\text{Now, I am interested in ABJM.}}$$

supersymmetric WLs

1/6 BPS [Drukker-Plefka-Young; Chen-Wu; Rey-Suyama-Yamaguchi]

$\star$ $\boxed{\text{1/2 BPS WL [Drukker Trancanelli]}}$

WL

$$[\hat{A}_\mu] \leftarrow M\times M$$

$$\underset{N\times N}{[A_\mu]} = [\psi] = 1\ , \quad [C^I] = \frac{1}{2} \qquad \underbrace{U(N)_k \times U(M)_{-k}}$$

bifundamental

Try

$$L = A_\mu \dot{x}^\mu + |\dot{x}| M^I_J C_I \bar{C}^J + \cancel{|\dot{x}| \bar{\psi}}$$

$N\times M$

$$\therefore \quad L = \begin{array}{c|cc} & \longleftarrow N \longrightarrow & \longleftarrow M \longrightarrow \\ \hline N & A_\mu \dot{x}^\mu + \frac{2\pi}{k}|\dot{x}| M^I_J C_I \bar{C}^J & \sqrt{\frac{2\pi}{k}}\, |\dot{x}|\, \eta^\alpha_I \bar{\psi}^I_\alpha \\ M & \sqrt{\frac{2\pi}{k}}\, |\dot{x}|\, \psi^\alpha_I \bar{\eta}^I_\alpha & \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k}|\dot{x}| \hat{M}^I_J \bar{C}^J C_I \end{array}$$

$$W_R \equiv \mathrm{Tr}_R\, \mathcal{P} \exp\left(i \oint_C L\, d\tau \right)$$

↑ $U(N|M)$ rep.

$$A_\mu \dot{x}^\mu = A_0 \quad \text{let } \mathcal{A}_0 \equiv A_0 + \frac{2\pi}{k} M^I_J C_I \bar{C}^J$$

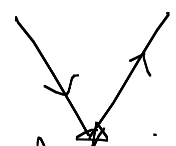
$$\hat{\mathcal{A}}_0 \equiv \hat{A}_0 + \frac{2\pi}{k} \hat{M}^I_J \bar{C}^J C_I$$

if $M^I_J = \hat{M}^I_J = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\eta^+_1, \bar{\eta}^1_+ \neq 0$

Insert Bulk BPS operator.

$$\mathrm{Tr}_R\, \mathcal{P}\left[e^{i\int_0^\infty L\, d\tau} \left[(C_1 \bar{C}^1) \cdots (C_1 \bar{C}^1) \right](0)\, e^{i\int_\infty^0 L\, d\tau} \right]$$

- without cusp ; "$\frac{1}{6}$ BPS" ✓
- with cusp : How much is known by "Localization" ?



- Integrability of ABJM .
- symmetry of boundary s-matrix .