

COMPLETE S -MATRICES OF SUPERSYMMETRIC SINE-GORDON THEORY AND PERTURBED SUPERCONFORMAL MINIMAL MODEL

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Received 22 October 1990

We derive the first complete S -matrices of the supersymmetric sine-Gordon theory for a general value of coupling constant. The spectrum includes not only solitons and antisolitons but also their bound states. The S -matrices are computed based on the soliton S -matrix which was obtained from the S -matrix of perturbed superconformal unitary model. After constructing a superconformal non-unitary model from the coset CFT with admissible representations, we derive the S -matrices of the perturbed superconformal non-unitary model by restricting the SSG S -matrices. We generalize these results to the theories with the fractional supersymmetries.

1. Introduction

Various two-dimensional field theories have the remarkable property of possessing an infinite number of conserved charges, which implies the theories have no particle production and scattering is elastic. This property ensures the integrability of the theories. The multi-particle scattering amplitudes are decomposed into the products of two-particle scattering amplitudes which can be solved exactly using the factorization equation (or Yang–Baxter equation) and the general principles of analyticity, crossing symmetry and unitarity.

The exact S -matrices were derived for the sine-Gordon (SG) theory, $O(N)$ -symmetric non-linear σ -model, the Gross–Neveu model [1,2], and other integrable quantum field theories [3] by solving the Yang–Baxter equation and imposing on-shell symmetries with which the S -matrices were supposed to commute. For most cases, the on-shell symmetry determines the S -matrix unambiguously*. While this on-shell symmetry is immediately determined for the elementary particles which are created by quantum fields in the lagrangian, the symmetry for the solitons is not easy to find and usually is conjectured based on the spectrum. For the SG theory, on-shell symmetry $O(2)$ is assigned because there are two asymp-

* Overall CDD factors can be always multiplied to the S -matrices. These factors introduce additional poles, hence new particles to the spectra. We will assume there are no CDD factors (minimal solution) throughout this paper.

otic particle states, a soliton and an antisoliton in the spectrum. As far as the same symmetries are involved, the same S -matrices are obtained. For example, the S -matrix for the SG solitons is the same as that of the massive Thirring fermions [4].

The factorizable S -matrix theory can be applied to supersymmetric theories. The fundamental difference of supersymmetry (SUSY) from the other kind of on-shell symmetries is that the ordinary SUSY algebra for the elementary particles is extended by a central charge for the solitons [5]. The authors in ref. [6] first derived the exact S -matrices for the elementary particles of the supersymmetric sine-Gordon (SSG) theory and SUSY $O(N)$ non-linear σ -model. The SUSY algebra, therefore, did not include any central charge. Although this derivation did not include the soliton sector, thus far from the complete solution of the SSG theory, they demonstrated one remarkable property that the supersymmetric S -matrices can be given as tensor products of two S -matrices, one of which is the S -matrix of the non-supersymmetric model and the other is the S -matrix commuting with on-shell SUSY. For example, the S -matrix of the SUSY $O(N)$ model was shown to be the direct product of the S -matrices of ordinary $O(N)$ model and SSG elementary particles. The second factor is commuting with SUSY. The longstanding problem of finding the SSG soliton S -matrix has been recently solved by studying perturbed conformal field theories (CFT) [7].

Initiated by Zamolodchikov [8] the perturbed CFT's by the least relevant operator have been related to integrable field theories. The case studied in the greatest detail is a special perturbation of the minimal model, represented by the coset CFT $SU(2)_1 \otimes SU(2)_L / SU(2)_{1+L}$, by the operator of dimension $(L+1)/(L+3)$. This model is an integrable restriction of the sine-Gordon theory [9–14]. The S -matrix of the SG theory [1] was shown to commute with the quantum group symmetry $\mathcal{U}_q[\mathfrak{sl}(2)]$ for which the SG soliton and antisoliton form a fundamental (spin- $\frac{1}{2}$) representation*. The multi-soliton Hilbert space can be decomposed into irreducible representations with higher spins [11, 14]. The deformation parameter q is uniquely related to the coupling constant of the SG theory. For q a root of unity, the decomposition to the higher spin is restricted up to a maximum spin. In other words, the states corresponding to higher spin than this are truncated from the Hilbert space. This restricted sine-Gordon theory (RSG) is identified with the perturbed minimal model. The S -matrix of this theory, S_{RSG} , is the restricted solid-on-solid (RSOS) form of integrable lattice models. One remarkable property of this S -matrix is that it is commuting with fractional on-shell SUSY due to the quantum group structure [14]. For the case of $L=2$, which corresponds to the tri-critical Ising model in the massless limit, this fractional SUSY becomes SUSY with a central charge. The generalization of these results to the other unitary

* This quantum group symmetry has been recently derived from the lagrangian of the SG theory using the perturbed CFT formalism [15].

models of coset CFT's $G_K \otimes G_L / G_{K+L}$ has been worked out [7]. The integrability of these perturbed CFT's are established by relating them to certain restricted integrable field theories using underlying quantum group structures.

Consider the most interesting case of $G = \text{SU}(2)$. For $K = 1$, this theory corresponds to the integrable perturbation of the conformal unitary model and has on-shell fractional supersymmetric charge $Q^{(L)}$ as mentioned above. Since the perturbed coset CFT is invariant under duality transformation $K \leftrightarrow L$, the S -matrix of the perturbed coset CFT $\text{SU}(2)_K \otimes \text{SU}(2)_L / \text{SU}(2)_{K+L}$ should commute with two independent fractional SUSY charges $Q^{(K)}$ and $Q^{(L)}$. This determines the S -matrix uniquely as a following factorized form:

$$S^{(K,L)} = S_{\text{RSG}}^{(K)} \otimes S_{\text{RSG}}^{(L)}. \quad (1.1)$$

For $K = 2$, the theory is the superconformal unitary model perturbed by a dimension $(L+2)/(L+4)$ operator and can be identified with the restricted supersymmetric sine-Gordon theory (RSSG). The S -matrix for the RSSG, eq. (1.1), is conjectured to be the restriction of the SSG S -matrix. Then the S -matrix for the SSG solitons is derived by reversing the logic of the SG theory, that is, by undoing the restriction of the RSSG theory. Since the restriction depends only on L and since $S_{\text{RSG}}^{(L)}$ is the restricted SG soliton S -matrix, the S -matrix for the SSG solitons should be

$$S_{\text{SSG}} = S_{\text{RSG}}^{(2)} \otimes S_{\text{SG}}^{(L)}. \quad (1.2)$$

The level L is related to the coupling constant of the SSG theory. The spectrum of the SSG theory for the integer value of L contains only the SSG solitons. The dynamics of the SUSY sector is totally included in the first S -matrix in eq. (1.2), reminding the general feature of factorization.

The SSG S -matrix (1.2) is not complete, however. First of all, we do not know what are the on-shell SSG solitons. To compute the SSG soliton S -matrix explicitly, we should know the SUSY charges of these solitons. Secondly, the SSG theory should include not only solitons but also bound states of solitons, referred to the SSG breathers. The breather-soliton and breather-breather S -matrices are necessary to be complete. In this paper, we will solve these problems and apply the results to other integrable models including perturbed superconformal non-unitary model.

This paper is organized as follows. In sect. 2 we briefly repeat the derivation of the SSG soliton S -matrix (1.2). After defining on-shell SUSY soliton states, the explicit S -matrix for the on-shell SSG solitons will be obtained. In sect. 3 we extend L to a general real number. For this value of L , the SSG breathers are shown to exist and their scattering amplitudes with the SSG solitons and SSG breathers are computed by considering multi-soliton scattering processes. We will show as a special case that the S -matrix for the SSG breathers of the lowest mass

is exactly the same as that of the elementary particles of the SSG theory [6] as mentioned earlier. This identification confirms not only our results but also the validity of the SSG soliton S -matrix (1.2) which our derivations are based on. We apply our results on the SSG theory to other interesting integrable theories in sect. 4. The perturbed coset CFT with a rational level L is related to superconformal non-unitary CFT for which we provide the complete S -matrix for the first time. The primary fields of the non-unitary model are constructed with “admissible” representations of $SU(2)$'s with rational levels. Although we work with the non-unitary model, the massive field theory is perfectly well-defined as far as the S -matrices are unitary because the physical states of the massive theory are not the primary and descendent fields but the solitonic states. The method we use to get these results seems equally true for $K \geq 3$. The SSG theory generalizes to the fractional supersymmetric sine-Gordon (FSSG) theories with general couplings. The FSSG breathers can be obtained as bound states of the FSSG solitons. The restrictions of the FSSG theories generate perturbed general non-unitary CFT's with fractional SUSY. The construction of the non-unitary CFT's is done using the coset CFT's with admissible representations. In sect. 5 we conclude with a few comments.

2. The S -matrix of the SSG solitons

In this section we first review the known results on the SG, RSG and SSG theories. The perturbed superconformal unitary model [7] is identified with the RSSG theory and the SSG S -matrix is conjectured from this result. After the SSG soliton spectrum and SUSY transformation are worked out in detail, the SSG soliton S -matrix will be written down.

2.1. THE PERTURBED SUPERCONFORMAL UNITARY MODEL

The action of the supersymmetric generalization of the SG theory (SSG) is given by [16]

$$S = \frac{1}{\beta^2} \int dx dt \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} i \bar{\psi} \not{\partial} \psi + \frac{m^2}{4} \cos^2 \phi - \frac{m}{2} (\cos \phi) \bar{\psi} \psi \right], \quad (2.1)$$

where ϕ is a real scalar field and ψ is a Majorana fermion. β is a coupling constant of the SG theory and m is the mass parameter denoting the deviation from the massless theory.

The SSG theory is integrable because it is equivalent to Toda theory on the twisted super affine Lie algebra $C^{(2)}(2)$ [17–19]. The equation of motion of the SSG theory can be written as a super zero-curvature condition. An infinite number of conserved charges at the classical level were derived [20] and seem to be preserved at the lowest order of quantum level [21].

As is the case of the ordinary SG theory, the SSG theory can be related to a special perturbation of superconformal minimal model. To show this, we express eq. (2.1) in terms of superfield in two-dimensional euclidean space-time,

$$S = \frac{1}{\beta^2} \int d^2z d^2\theta [D\Phi \bar{D}\Phi + m \cos \Phi], \quad (2.2)$$

where covariant derivatives in the superspace coordinates z, θ (and $\bar{z}, \bar{\theta}$) are

$$D = \partial_\theta + \theta \partial_z, \quad \bar{D} = \bar{\partial}_{\bar{\theta}} + \bar{\theta} \partial_{\bar{z}}. \quad (2.3)$$

We can identify the superfield $\Phi = \phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}F$ with the field of the supersymmetric Feigin–Fuchs (FF) construction for the superconformal model by normalizing the fields by

$$\Phi = \frac{\beta}{\sqrt{4\pi}} \Phi_{\text{FF}}. \quad (2.4)$$

The central charge of the superconformal model and background charges are

$$c = \frac{3}{2} \left(1 - \frac{8}{(L+2)(L+4)} \right), \quad \alpha_0 = \frac{1}{\sqrt{(L+2)(L+4)}}. \quad (2.5)$$

The screening operators are given by

$$V_{\pm} = \int dz d\theta \exp(i\alpha_{\pm} \Phi_{\text{FF}}(z, \theta)),$$

$$\alpha_+ = \sqrt{\frac{L+4}{L+2}}, \quad \alpha_- = -\sqrt{\frac{L+2}{L+4}}. \quad (2.6)$$

The primary fields of the superconformal minimal model are

$$\Phi_{m,n} = \exp(i\alpha_{m,n} \Phi_{\text{FF}}),$$

$$\alpha_{m,n} = \frac{1}{2}(1-m)\alpha_+ + \frac{1}{2}(1-n)\alpha_-, \quad (2.7)$$

with $1 \leq m \leq L+1$, $1 \leq n \leq L+3$. The Neveu–Schwarz sector is given by $n-m = \text{even}$; the Ramond sector by $n-m = \text{odd}$. Since we require one of the operators in the potential of eq. (2.2) to be a screening operator, we can express the constant β in terms of a parameter in the CFT. Expanding $\cos \Phi = \frac{1}{2}[\exp(i\Phi) + \exp(-i\Phi)]$,

we take the $\exp(-i\Phi)$ term to be a screening operator. Therefore, we identify

$$-i\frac{\beta}{\sqrt{4\pi}} = i\alpha_- \Rightarrow \frac{\beta^2}{4\pi} = \frac{L+2}{L+4}. \quad (2.8)$$

We can represent the superconformal unitary model with the coset CFT $SU(2)_K \otimes SU(2)_L/SU(2)_{K+L}$, $K=2$. The primary fields (2.7) of the superconformal unitary model are constructed from the integral representations of $SU(2)$'s with integer levels. This will be extended in sect. 4 to general K and rational L for non-unitary CFT's.

Since the screening operator has dimension 1, the part of the action that includes only the free piece and the screening operator can be considered as CFT; it is a super Liouville theory. The extra term in the action $\exp(i\beta\Phi_{FF}/\sqrt{4\pi})$ is treated as a perturbation and is equivalent to the $\Phi_{1,3}$ primary field of the super-conformal minimal model. This field with the dimension $(L+2)/(L+4)$ is the perturbing operator which preserves the SUSY of the CFT. It must be emphasized that the decomposition of the action (2.2) into a conformal piece and a perturbation is partly heuristic. The action for the conformal piece is not sufficient to encode the truncation of the Hilbert space one performs in the super FF construction (projection of null vectors). Thus the spectrum of the perturbed super CFT is not equivalent to the spectrum of the SSG, but must be obtained as a restriction of it. Also, the ordinary (unrestricted) SSG theory does not have a background charge and corresponds to $c=3/2$ in the massless limit.

As pointed out in sect. 1, the SSG theory is different from the SG theory in that we do not know the SSG S -matrix in the beginning and hence within the quantum group structure. Instead of starting with the S -matrix of the SSG theory, we start with the S -matrix of the RSSG theory. To be consistent with the duality and the results on the SG theory, the S -matrix of the RSSG theory is given by the direct tensor product of two RSG S -matrices,

$$S_{\text{RSSG}}(\theta) = S_{\text{RSG}}^{(K-2)}(\theta) \otimes S_{\text{RSG}}^{(L)}(\theta). \quad (2.9)$$

The deformation parameter of the quantum group $\mathcal{U}_q[\mathfrak{sl}(2)]$ associated $S_{\text{RSG}}^{(L)}(\theta)$ is $q = -\exp[-i\pi/(L+2)]$. And the SSG soliton S -matrix is obtained by unrestricting the RSSG theory as is given in eq. (1.2). The consistency check that the RSSG theory is the perturbed superconformal minimal model comes from relating the maximum topological charge which the primary fields in the CFT can have to the maximum allowed spin for the RSSG theory due to the special value of q .

2.2. THE S -MATRIX FOR THE SSG SOLITONS

As explained, we undo the restriction of the RSSG theory to obtain the SSG S -matrix. In this procedure the factor $S_{\text{RSG}}^{(K)}$ is unaffected; however, the factor $S_{\text{RSG}}^{(L)}$

becomes an ordinary SG soliton S -matrix. Since $\beta^2/4\pi = (L + 2)/(L + 4)$, we are led to define a function γ of the coupling β as

$$\gamma(\beta) \equiv 8\pi(L + 2) = \frac{4\beta^2}{1 - \beta^2/4\pi}. \quad (2.10)$$

The S -matrix of the SSG solitons with this parameter γ is

$$S_{SSG}(\theta) = S_{RSG}^{(K-2)}(\theta) \otimes S_{SG}(x = e^{8\pi\theta/\gamma}, q = -e^{-i8\pi^2/\gamma}). \quad (2.11)$$

The RSG spectrum consists of kinks $|K_{ab}\rangle$ with $a, b \in \{0, \frac{1}{2}, 1\}$, $|a - b| = \frac{1}{2}$. By denoting the SG solitons with topological charge ± 1 by $|A^\pm\rangle$, the SSG states can be expressed by $|K_{ab}^\pm\rangle = |K_{ab}\rangle \otimes |A^\pm\rangle$, where the first quantum number carries the SUSY charges and the second the topological charges.

The S_{SG} is the S -matrix of the ordinary SG solitons [1],

$$S_{++,-,-}^{SG}(\theta) = S_{--,-,-}^{SG}(\theta) = \frac{U(\theta)}{i\pi} \operatorname{sh}\left[\frac{8\pi}{\gamma}(i\pi - \theta)\right],$$

$$S^{SG}(\theta) = \frac{U(\theta)}{i\pi} \begin{pmatrix} & A^+A^- & & A^-A^+ \\ A^+A^- & \operatorname{sh}(8\pi\theta/\gamma) & i \sin(8\pi^2/\gamma) & \\ A^-A^+ & i \sin(8\pi^2/\gamma) & \operatorname{sh}(8\pi\theta/\gamma) & \end{pmatrix}, \quad (2.12)$$

where $U(\theta)$, defined by

$$U(\theta) = \Gamma\left(\frac{8\pi}{\gamma}\right) \Gamma\left(1 + i\frac{8\theta}{\gamma}\right) \Gamma\left(1 - \frac{8\pi}{\gamma} - i\frac{8\theta}{\gamma}\right) \prod_{n=1}^{\infty} \frac{R_n(\theta) R_n(i\pi - \theta)}{R_n(0) R_n(i\pi)},$$

$$R_n(\theta) = \frac{\Gamma(2n(8\pi/\gamma) + i(8\theta/\gamma)) \Gamma(1 + 2n(8\pi/\gamma) + i(8\theta/\gamma))}{\Gamma((2n + 1)(8\pi/\gamma) + i(8\theta/\gamma)) \Gamma(1 + (2n - 1)(8\pi/\gamma) + i(8\theta/\gamma))}, \quad (2.13)$$

satisfies $U(\theta) = U(i\pi - \theta)$. We use the S -matrix convention shown in fig. 1.

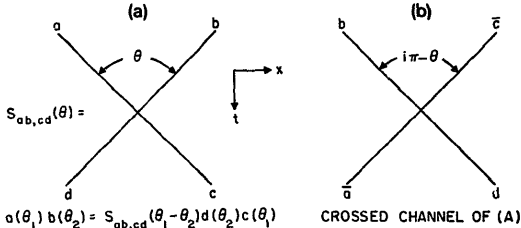


Fig. 1. (a) The convention of the S -matrix. (b) Crossed channel of (a).

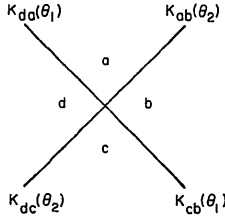


Fig. 2. The RSG scattering amplitude.

The $S_{\text{RSG}}^{(K)}$ is given as the RSOS form [14] (fig. 2)

$$S_{cd}^{ab}(\theta) = \frac{U(\theta)}{2\pi i} \left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]} \right)^{-\theta/2\pi i} \mathcal{R}_{cd}^{ab}(\theta),$$

$$\mathcal{R}_{cd}^{ab}(\theta) = \text{sh} \left(\frac{\theta}{K+2} \right) \delta_{db} \left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]} \right)^{1/2} + \text{sh} \left(\frac{i\pi - \theta}{K+2} \right) \delta_{ac}, \quad (2.14)$$

for the process $|K_{da_1}(\theta_1)\rangle + |K_{a_2b}(\theta_2)\rangle \rightarrow |K_{dc}(\theta_2)\rangle + |K_{c_2b}(\theta_1)\rangle$ with $a_1 = a_2 = a$ and $c_1 = c_2 = c$. If this condition is not met, the scattering amplitude becomes zero. These spins are restricted to $0 \leq j \leq K/2$. The q -number $[n]$ is defined by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad (2.15)$$

where $q = -\exp(-i\pi/K + 2)$. This S -matrix is crossing symmetric $S_{cd}^{ab}(\theta) = S_{ad}^{bc}(i\pi - \theta)$ if we define charge conjugation of kinks by

$$C(|K_{ab}\rangle) = |\bar{K}_{ab}\rangle = |K_{ba}\rangle. \quad (2.16)$$

The kinks $|K_{ab}\rangle$ have on-shell fractional supersymmetry $Q^{(K)}$ which acts on the

kinks by the following rule [14]:

$$\begin{aligned} Q^{(K)}|K_{a_1 a_2}(\theta_1) \dots K_{a_N a_{N+1}}(\theta_N)\rangle &= \sum_{a'} \sum_{j=1}^N e^{(\theta_j/K+2)} \left(\prod_{k=1}^{j-1} \Theta_{a'_k a_{k+1}}^{a'_k a_{k+1}} \right) X_{a'_j}^{a_j a_{j+1}} \\ &\times |K_{a'_1 a'_2}(\theta_1) \dots K_{a'_j a_{j+1}}(\theta_j) \dots K_{a_N a_{N+1}}(\theta_N)\rangle. \end{aligned} \quad (2.17)$$

Similarly, $\bar{Q}^{(K)}$ gives

$$\begin{aligned} \bar{Q}^{(K)}|K_{a_1 a_2}(\theta_1) \dots K_{a_N a_{N+1}}(\theta_N)\rangle &= \sum_{a'} \sum_{j=1}^N e^{(-\theta_j/K+2)} \left(\prod_{k=1}^{j-1} \bar{\Theta}_{a'_k a_{k+1}}^{a'_k a_{k+1}} \right) \bar{X}_{a'_j}^{a_j a_{j+1}} \\ &\times |K_{a'_1 a'_2}(\theta_1) \dots K_{a'_j a_{j+1}}(\theta_j) \dots K_{a_N a_{N+1}}(\theta_N)\rangle. \end{aligned} \quad (2.18)$$

The coefficients X_c^{ab} , \bar{X}_c^{ab} and Θ_{cd}^{ab} , $\bar{\Theta}_{cd}^{ab}$ are expressed in terms of quantum 6- j coefficients [14, 22] and defined up to gauge transformations $X_c^{ab} \rightarrow \epsilon(a; c) X_c^{ab}$, $\Theta_{cd}^{ab} \rightarrow (\epsilon(a; c)/\epsilon(b; d)) \Theta_{cd}^{ab}$, and similarly for \bar{X}_c^{ab} , $\bar{\Theta}_{cd}^{ab}$ with any number $\epsilon(a; c)$.

From now on, we concentrate on the case $K=2$ where $Q^{(K)}$ and $\bar{Q}^{(K)}$ satisfy ordinary SUSY algebra with a central charge due to the topological charges of the SSG solitons. The SUSY acts on the SSG states by

$$\begin{aligned} Q|K_{0\frac{1}{2}}^\pm\rangle &= -i e^{\theta/2} |K_{1\frac{1}{2}}^\pm\rangle, & \bar{Q}|K_{0\frac{1}{2}}^\pm\rangle &= \mp i e^{-\theta/2} |K_{1\frac{1}{2}}^\pm\rangle, \\ Q|K_{1\frac{1}{2}}^\pm\rangle &= i e^{\theta/2} |K_{0\frac{1}{2}}^\pm\rangle, & \bar{Q}|K_{1\frac{1}{2}}^\pm\rangle &= \pm i e^{-\theta/2} |K_{0\frac{1}{2}}^\pm\rangle, \end{aligned} \quad (2.19)$$

and on the charge conjugated states by

$$\begin{aligned} Q|K_{\frac{1}{2}0}^\pm\rangle &= e^{\theta/2} |K_{\frac{1}{2}0}^\pm\rangle, & \bar{Q}|K_{\frac{1}{2}0}^\pm\rangle &= \pm e^{-\theta/2} |K_{\frac{1}{2}0}^\pm\rangle, \\ Q|K_{\frac{1}{2}1}^\pm\rangle &= -e^{\theta/2} |K_{\frac{1}{2}1}^\pm\rangle, & \bar{Q}|K_{\frac{1}{2}1}^\pm\rangle &= \mp e^{-\theta/2} |K_{\frac{1}{2}1}^\pm\rangle. \end{aligned} \quad (2.20)$$

Note that these states need not be on-shell particle states which describe the SSG solitons. To identify the on-shell states, we impose the condition that the on-shell states contain both bosonic and fermionic states which the SUSY transforms into each other. In addition to this, if we require that the charge conjugated states of these on-shell states should also exist in the spectrum, we can identify the

on-shell SSG soliton states as follows:

$$\begin{aligned} |B^\pm\rangle &= \frac{1}{\sqrt{2}} \left\{ |K_{0\frac{1}{2}}^\pm\rangle + |K_{1\frac{1}{2}}^\pm\rangle \right\}, & |F^\pm\rangle &= \frac{1}{\sqrt{2}} \left\{ |K_{0\frac{1}{2}}^\pm\rangle - |K_{1\frac{1}{2}}^\pm\rangle \right\}, \\ |\bar{B}^\pm\rangle &= \frac{1}{\sqrt{2}} \left\{ |K_{\frac{1}{2}0}^\pm\rangle + |K_{\frac{1}{2}1}^\pm\rangle \right\}, & |\bar{F}^\pm\rangle &= \frac{1}{\sqrt{2}} \left\{ |K_{\frac{1}{2}0}^\pm\rangle - |K_{\frac{1}{2}1}^\pm\rangle \right\}. \end{aligned} \quad (2.21)$$

The SUSY transformations for these states are

$$\begin{aligned} Q|B^\pm\rangle &= i e^{\theta/2} |F^\pm\rangle, & \bar{Q}|B^\pm\rangle &= \pm i e^{-\theta/2} |F^\pm\rangle, \\ Q|F^\pm\rangle &= -i e^{\theta/2} |B^\pm\rangle, & \bar{Q}|F^\pm\rangle &= \mp i e^{-\theta/2} |B^\pm\rangle, \\ Q|\bar{B}^\pm\rangle &= e^{\theta/2} |\bar{F}^\pm\rangle, & \bar{Q}|\bar{B}^\pm\rangle &= \pm e^{-\theta/2} |\bar{F}^\pm\rangle, \\ Q|\bar{F}^\pm\rangle &= e^{\theta/2} |\bar{B}^\pm\rangle, & \bar{Q}|\bar{F}^\pm\rangle &= \pm e^{-\theta/2} |\bar{B}^\pm\rangle. \end{aligned} \quad (2.22)$$

Then the SUSY charges satisfy

$$Q^2 = P = e^\theta, \quad \bar{Q}^2 = \bar{P} = e^{-\theta}, \quad Q\bar{Q} + \bar{Q}Q = 2T. \quad (2.23)$$

The central charge T is ± 1 corresponding to the topological charges of the SSG solitons. The charge conjugations of the solitons are

$$C(|B^\pm\rangle) = |\bar{B}^\mp\rangle, \quad C(|F^\pm\rangle) = |\bar{F}^\mp\rangle. \quad (2.24)$$

The S -matrix $S_{\text{RSG}}^{(2)}$ in eq. (2.11) is obtained to be

$$\begin{aligned} S &= \frac{U'(\theta)}{2\pi i} 2^{-\theta/2\pi i - 1} \left(i \text{ch} \frac{\bar{\theta}}{4} + \text{sh} \frac{\bar{\theta}}{4} \right) \\ &\times \left\{ \begin{array}{cc} \begin{array}{cc} B\bar{B} & F\bar{F} \\ B\bar{B} \begin{pmatrix} 1 & \frac{1+i\text{sh}(\bar{\theta}/2)}{\text{ch}(\bar{\theta}/2)} \\ \frac{1+i\text{sh}(\bar{\theta}/2)}{\text{ch}(\bar{\theta}/2)} & 1 \end{pmatrix} & B\bar{F} \begin{pmatrix} 1 & \frac{F\bar{B}}{\text{ch}(\bar{\theta}/2)} \\ \frac{1+i\text{sh}(\bar{\theta}/2)}{\text{ch}(\bar{\theta}/2)} & 1 \end{pmatrix} \end{array} \end{array} \right\}, \\ &= \frac{U'(\theta)}{2\pi i} 2^{-\bar{\theta}/2\pi i - 1} \left(i \text{ch} \frac{\theta}{4} + \text{sh} \frac{\theta}{4} \right) \\ &\times \left\{ \begin{array}{cc} \bar{B}B & \bar{F}F \\ \bar{B}B \begin{pmatrix} 1 & 1 \\ \bar{F}F & 1 \end{pmatrix} & \bar{B}F \begin{pmatrix} 1 & 1 \\ \bar{F}B & 1 \end{pmatrix} \end{array} \right\}, \end{aligned} \quad (2.25)$$

where $U'(\theta)$ is $U(\theta)$ with $\gamma = 32\pi$ in (2.13) and $\bar{\theta} \equiv i\pi - \theta$. Eqs. (2.12) and (2.25) give the S -matrix of the SSG solitons. It is remarkable that all the fermion-number violating amplitudes vanish. This confirms that the on-shell states carry good SUSY charges. The unitarity and crossing symmetry of (2.25) can be checked.

3. The S -matrices of the SSG breathers

In this section we define the bound states of the SSG solitons, the SSG breathers. The S -matrices for the SSG breather and soliton and for the two SSG breathers are computed by studying the multi-soliton scattering amplitudes.

3.1. THE SSG BREATHERS

The SG soliton S -matrix (2.12) has poles in the physical region $0 < \text{Im } \theta < \pi$ for the soliton–antisoliton channel. Since poles of the S -matrix correspond to the bound states arising from the given channel, these poles of the SG theory are matched with the bound states of a soliton and an antisoliton, the breathers. The mass spectrum can be computed from the poles of the S -matrix, which are at

$$m_n = 2m \sin\left(\frac{n\gamma}{16}\right), \quad (3.1)$$

where m is the soliton mass and set to be 1. Since the SSG soliton S -matrix contains the SG S -matrix and the other factor in the SSG S -matrix, S_{SSG} , has no additional pole, the poles of the SSG S -matrix are exactly the same as those of the SG S -matrix. Like the SG theory, the n th bound state reaches the soliton–antisoliton threshold when $\gamma = 8\pi/n$, and when $\gamma \geq 8\pi/n$ it disappears from the spectrum converting into the virtual state. At $\gamma \geq 8\pi$ all bound states including the lowest massive state, which is identified with the elementary particle of the SSG theory, disappear from the spectrum. This condition for no bound states is automatically satisfied with the coupling constant (2.10) for the positive integer L . Therefore, the SSG spectrum consists of only the SSG solitons without any breather for the positive integer L . In terms of the parameter β in (2.1), the SSG threshold arises at $\beta^2/4\pi = 1/3$ compared with the SG threshold $\beta^2/4\pi = 1/2$.

Now, we relax this constraint on L and consider γ as a general real coupling constant. If the SSG theory is well defined at the special coupling, it should be so whatever the coupling is. For this case, there should exist the SSG soliton bound states, the SSG breathers, with the masses of (3.1). These will carry two quantum numbers, one from the two kinks states and the other from the SG breathers, like

$$|K_{ab}K_{bc}\rangle \otimes |A^\pm A^\mp\rangle, \quad a, b, c = 0, \frac{1}{2}, 1. \quad (3.2)$$

There exist six allowed two-kink states for the SSG theory. Any linear combination

of these can be a candidate for the on-shell particles. To identify these on-shell states, we impose the same condition as for the SSG solitons, namely that the SSG breathers form the complete set of bosonic and fermionic states which transform into each other under the SUSY transformation. In addition to this, we require the on-shell states to be invariant under charge conjugation. The reason is the following: Since we identify the SSG breathers of the lowest mass with the elementary particles, which are composed of one real scalar particle and one Majorana fermion, we need the whole spectrum of the SSG breathers to be either real scalars or Majorana fermions. If the lowest massive states are real or Majorana, all the other SSG breathers should be so because the masses of the SSG breathers depend only on the SG sector.

We have found that only the following combination can form the complete set of fermionic and bosonic states which are invariant under the charge conjugation (2.16):

$$\begin{aligned}
|\phi_n^1(\theta)\rangle &= \lim_{\theta_1 - \theta_2 \rightarrow \Delta\theta_n} \frac{1}{\sqrt{2}} \{ |K_{0\frac{1}{2}}(\theta_1) K_{\frac{1}{2}0}(\theta_2)\rangle + |K_{1\frac{1}{2}}(\theta_1) K_{\frac{1}{2}1}(\theta_2)\rangle \}, \\
|\phi_n^2(\theta)\rangle &= \lim_{\theta_1 - \theta_2 \rightarrow \Delta\theta_n} \frac{1}{\sqrt{2}} \{ |K_{\frac{1}{2}0}(\theta_1) K_{0\frac{1}{2}}(\theta_2)\rangle + |K_{\frac{1}{2}1}(\theta_1) K_{1\frac{1}{2}}(\theta_2)\rangle \}, \\
|\psi_n^1(\theta)\rangle &= \lim_{\theta_1 - \theta_2 \rightarrow \Delta\theta_n} \frac{i\alpha_n}{\sqrt{2}} \{ |K_{0\frac{1}{2}}(\theta_1) K_{\frac{1}{2}1}(\theta_2)\rangle - |K_{1\frac{1}{2}}(\theta_1) K_{\frac{1}{2}0}(\theta_2)\rangle \}, \\
|\psi_n^2(\theta)\rangle &= \lim_{\theta_1 - \theta_2 \rightarrow \Delta\theta_n} \frac{\alpha_n}{\sqrt{2}} \{ |K_{\frac{1}{2}0}(\theta_1) K_{0\frac{1}{2}}(\theta_2)\rangle - |K_{\frac{1}{2}1}(\theta_1) K_{1\frac{1}{2}}(\theta_2)\rangle \}. \quad (3.3)
\end{aligned}$$

We defined the rapidity of the bound states to be $\theta = (\theta_1 + \theta_2)/2$ and the phase factor α_n is

$$\alpha_n = \frac{1+i}{\sqrt{2m_n}} (e^{\Delta\theta_n} + i e^{-\Delta\theta_n}), \quad (3.4)$$

where $\Delta\theta_n \equiv i\pi - in\gamma/8$ and m_n is the mass of the n th breather in eq. (3.1). One can check that $|\alpha_n|^2 = 1$. The SSG breathers are direct product of these with the breathers of the SG theory,

$$|\Phi_n^i\rangle = |\phi_n^i\rangle \otimes |B_n\rangle, \quad |\Psi_n^i\rangle = |\psi_n^i\rangle \otimes |B_n\rangle, \quad i = 1, 2, \quad (3.5)$$

where the SG breather $|B_n\rangle$ is given by

$$|B_n(\theta)\rangle = \lim_{\theta_1 - \theta_2 \rightarrow -\theta_n} \frac{1}{\sqrt{2}} \{ |A^+(\theta_1)A^-(\theta_2)\rangle + (-1)^n |A^-(\theta_1)A^+(\theta_2)\rangle \}. \quad (3.6)$$

Using eqs. (2.17) and (2.18), the SUSY transformations for these states are given by

$$\begin{aligned} Q|\Phi_n^i(\theta)\rangle &= \sqrt{m_n} \epsilon e^{\theta/2} |\Psi_n^i(\theta)\rangle, & \bar{Q}|\Phi_n^i(\theta)\rangle &= \sqrt{m_n} \epsilon^* e^{-\theta/2} |\Psi_n^i(\theta)\rangle, \\ Q|\Psi_n^i(\theta)\rangle &= \sqrt{m_n} \epsilon^* e^{\theta/2} |\Phi_n^i(\theta)\rangle, & \bar{Q}|\Psi_n^i(\theta)\rangle &= \sqrt{m_n} \epsilon e^{-\theta/2} |\Phi_n^i(\theta)\rangle, \end{aligned} \quad (3.7)$$

for $i = 1, 2$ and $\epsilon = \exp i\pi/4$. The SUSY algebra satisfies

$$Q^2 = m_n e^\theta = P, \quad \bar{Q}^2 = m_n e^{-\theta} = \bar{P}, \quad Q\bar{Q} + \bar{Q}Q = 0. \quad (3.8)$$

The central charge of the SUSY algebra vanishes for the two-soliton bound states. This is true even for states like $|K_{ab}K_{bc}\rangle \otimes |A^+A^+\rangle$ or $|K_{ab}K_{bc}\rangle \otimes |A^-A^-\rangle$ which have non-vanishing topological charges. This reflects that the central charge of the SUSY algebra is not additive* and explains the failure to find factorizable supersymmetric S -matrices for the case of additive central charges [23].

If we apply the charge conjugation (2.16) on these states, we get

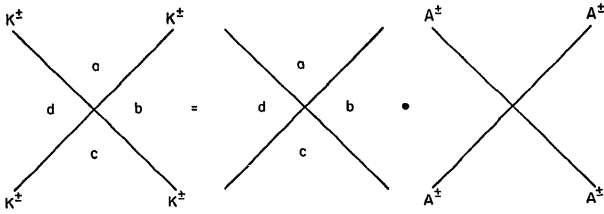
$$C(|\Phi_n^i\rangle) = |\Phi_n^i\rangle, \quad C(|\Psi_n^i\rangle) = |\Psi_n^i\rangle, \quad \text{for } i = 1, 2. \quad (3.9)$$

This means that $|\Phi_n^i\rangle$ are real scalar particles and $|\Psi_n^i\rangle$ are Majorana fermions. This is consistent with the expectation that the breathers of the lowest mass are identified with the elementary particles associated with the real scalar field ϕ and Majorana fermionic field ψ in (2.1). Since there is only one real scalar field and Majorana fermion field in the action, we should identify $|\Phi_n^1\rangle \equiv |\Phi_n^2\rangle$ and $|\Psi_n^1\rangle \equiv |\Psi_n^2\rangle$. Further justification for this identification comes later when we study the scattering amplitudes for these breather states. From now on, we will drop the superscript i .

3.2. THE S -MATRICES FOR THE SSG SOLITONS AND BREATHERS

The scattering amplitudes for the SSG solitons and the breathers can be derived from the residues of the three-soliton scattering amplitudes following the SG case [1, 4]. Since the S -matrix for the SSG solitons (2.11) is the direct product of two S -matrices, three-soliton scattering amplitudes are also factorized into two parts as

*The author thanks A. LeClair for this point.

Fig. 3. The factorized form of the SSG S -matrix.

shown in fig. 3. The SG part is well known [1, 4] to be

$$S_{\text{SG}}^{(n)}(\theta) = \frac{\text{sh } \theta + i \cos(n\gamma/16)}{\text{sh } \theta - i \cos(n\gamma/16)} \prod_{l=1}^{n-1} \frac{\sin^2(\frac{1}{32}(n-2l)\gamma - \frac{1}{4}\pi + \frac{1}{2}i\theta)}{\sin^2(\frac{1}{32}(n-2l)\gamma - \frac{1}{4}\pi - \frac{1}{2}i\theta)}, \quad (3.10)$$

for the process $A^\pm + B_n \rightarrow A^\pm + B_n$.

For the RSG part, we consider two successive scattering processes as shown in fig. 4. Using the definitions of the on-shell particles (2.21) and (3.3) and the S -matrices of the RSOS kinks (2.14), we obtain the following results:

$$\begin{aligned} S_{\text{RSG}}^{(n)} &= X_n(\theta) 2^{-\Delta\theta_n/2\pi i} \\ &\times \left\{ \begin{array}{cc} \Phi_n & \Psi_n F \\ \Phi_n B \left(\text{ch } \frac{1}{4}(2\theta - i\pi) \right) & \sqrt{m_n} \\ \Psi_n F \left(-i\sqrt{m_n} \right) & -\text{sh } \frac{1}{4}(2\theta - i\pi) \end{array} \right\}, \quad \left\{ \begin{array}{cc} \Phi_n F & \Psi_n B \\ \Phi_n F \left(\text{ch } \frac{1}{4}(2\theta - i\pi) \right) & \sqrt{m_n} \\ \Psi_n B \left(-i\sqrt{m_n} \right) & -\text{sh } \frac{1}{4}(2\theta - i\pi) \end{array} \right\} \\ &\times \left\{ \begin{array}{cc} \bar{B}\Phi_n & \bar{F}\Psi_n \\ \bar{B}\Phi_n \left(\text{ch } \frac{1}{4}(2\theta - i\pi) \right) & \sqrt{m_n} \\ \bar{F}\Psi_n \left(i\sqrt{m_n} \right) & \text{sh } \frac{1}{4}(2\theta - i\pi) \end{array} \right\}, \quad \left\{ \begin{array}{cc} \bar{B}\Psi_n & \bar{F}\Phi_n \\ \bar{B}\Psi_n \left(-\text{sh } \frac{1}{4}(2\theta - i\pi) \right) & -i\sqrt{m_n} \\ \bar{F}\Phi_n \left(\sqrt{m_n} \right) & \text{ch } \frac{1}{4}(2\theta - i\pi) \end{array} \right\} \\ &= X_n(\theta) 2^{\Delta\theta_n/2\pi i} \\ &\times \left\{ \begin{array}{cc} \Phi_n \bar{B} & \Psi_n \bar{F} \\ \Phi_n \bar{B} \left(\text{ch } \frac{1}{4}(2\theta - i\pi) \right) & -i\sqrt{m_n} \\ \Psi_n \bar{F} \left(\sqrt{m_n} \right) & -\text{sh } \frac{1}{4}(2\theta - i\pi) \end{array} \right\}, \quad \left\{ \begin{array}{cc} \Phi_n \bar{F} & \Psi_n \bar{B} \\ \Phi_n \bar{F} \left(\text{ch } \frac{1}{4}(2\theta - i\pi) \right) & i\sqrt{m_n} \\ \Psi_n \bar{B} \left(\sqrt{m_n} \right) & \text{sh } \frac{1}{4}(2\theta - i\pi) \end{array} \right\} \\ &\times \left\{ \begin{array}{cc} B\Phi_n & F\Psi_n \\ B\Phi_n \left(\text{ch } \frac{1}{4}(2\theta - i\pi) \right) & i\sqrt{m_n} \\ F\Psi_n \left(\sqrt{m_n} \right) & \text{sh } \frac{1}{4}(2\theta - i\pi) \end{array} \right\}, \quad \left\{ \begin{array}{cc} B\Psi_n & F\Phi_n \\ B\Psi_n \left(-\text{sh } \frac{1}{4}(2\theta - i\pi) \right) & \sqrt{m_n} \\ F\Phi_n \left(-i\sqrt{m_n} \right) & \text{ch } \frac{1}{4}(2\theta - i\pi) \end{array} \right\}. \quad (3.11) \end{aligned}$$

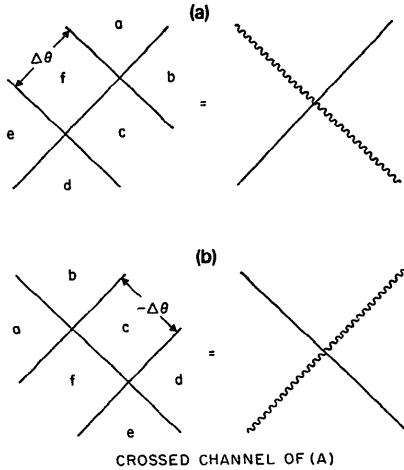


Fig. 4. (a) The breather-soliton scattering amplitude. (b) Crossed channel of (a).

The overall factor $X_n(\theta)$ is given in terms of the prefactor in (2.25),

$$X_n(\theta) = -\frac{U'(\theta + \frac{1}{2}\Delta\theta_n)}{2\pi i} \frac{U'(\theta - \frac{1}{2}\Delta\theta_n)}{2\pi i}. \quad (3.12)$$

Crossing symmetry is easily confirmed with $X_n(\theta) = X_n(i\pi - \theta)$ and the charge conjugation for the breathers changes $\Delta\theta_n \rightarrow -\Delta\theta_n$ (fig. 4b). Unitarity of these scattering amplitudes are also satisfied if the coefficient $X_n(\theta)$ satisfies

$$X_n(\theta)X_n(-\theta) = \frac{1}{\text{sh}^2(\theta/2) + \sin^2(n\gamma/32)}. \quad (3.13)$$

Using the recursive method [6] we can find $X_n(\theta)$ to be

$$X_n(\theta) = x(\theta)Y_n(\theta)Y_n(i\pi - \theta), \quad (3.14)$$

where

$$x(\theta) = \frac{i}{\text{sh}(\theta/2)} \prod_{n=1}^{\infty} \frac{\text{sh} \frac{1}{2}(n\pi i + \theta)}{\text{sh} \frac{1}{2}(n\pi i - \theta)} \quad (3.15)$$

and

$$Y_n(\theta) = \frac{\Gamma(-i\theta/2\pi)}{\Gamma(\frac{1}{2} - i\theta/2\pi)} \prod_{l=1}^{\infty} \left\{ \frac{\Gamma^2(-i\theta/2\pi) + l - \frac{1}{2}}{\Gamma^2(-i\theta/2\pi) + l - 1} \right. \\ \left. \times \frac{\Gamma((n\gamma/32\pi) - (i\theta/2\pi) + l) \Gamma(-(n\gamma/32\pi) - (i\theta/2\pi) + l - 1)}{\Gamma((n\gamma/32\pi) - (i\theta/2\pi) + l + \frac{1}{2}) \Gamma(-(n\gamma/32\pi) - (i\theta/2\pi) + l - \frac{1}{2})} \right\}. \quad (3.16)$$

If we define $X'_n(\theta) \equiv Y_n(\theta)Y_n(i\pi - \theta)$,

$$X'_n(\theta)X'_n(-\theta) = \frac{\text{sh}^2(\theta/2)}{\text{sh}^2(\theta/2) + \sin^2(n\gamma/32)}. \quad (3.17)$$

The scattering amplitudes which are not vanishing are for the processes which preserve fermion numbers. This shows the consistency of our choice of on-shell breathers.

3.3. THE S-MATRICES FOR THE BREATHERS

We follow the same procedure as before to compute the breathers scattering amplitudes. The S -matrices are computed from the four-soliton scattering amplitudes. The SG breather scattering amplitudes were derived also in refs. [1, 4] and the RSG part can be derived from the four-soliton scattering amplitudes of the RSG kinks (fig. 5).

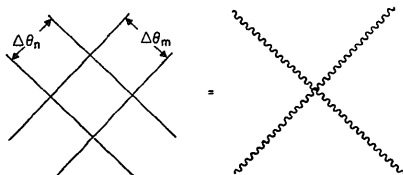


Fig. 5. The breather-breather scattering amplitude.

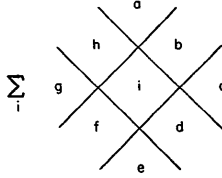


Fig. 6. The four-kink scattering amplitude.

The SG breather–breather scattering amplitudes are

$$\begin{aligned}
 S_{\text{SG}}^{(n,m)}(\theta) &= \frac{\text{sh } \theta + i \sin \frac{1}{16}(n+m)\gamma \text{ sh } \theta + i \sin \frac{1}{16}(n-m)\gamma}{\text{sh } \theta - i \sin \frac{1}{16}(n+m)\gamma \text{ sh } \theta - i \sin \frac{1}{16}(n-m)\gamma} \\
 &\times \prod_{l=1}^{m-1} \frac{\sin^2(\frac{1}{32}(m-n-2l)\gamma + \frac{1}{2}i\theta) \cos^2(\frac{1}{32}(m+n-2l)\gamma + \frac{1}{2}i\theta)}{\sin^2(\frac{1}{32}(m-n-2l)\gamma + \frac{1}{2}i\theta) \cos^2(\frac{1}{32}(m+n-2l)\gamma + \frac{1}{2}i\theta)}, \quad (3.18)
 \end{aligned}$$

for the process $B_n(\theta_1) + B_m(\theta_2) \rightarrow B_n(\theta_1) + B_m(\theta_2)$ with $n \geq m$.

As shown in fig. 6, the RSOS form of four-kink scattering amplitudes can be computed from (2.25). After a long but straightforward computation, we obtain the following results:

$$\begin{aligned}
 S_{\text{RSG}}^{(n,m)} &= Z_{nm}(\theta) \begin{pmatrix} \Phi_n \Phi_m & \Psi_n \Psi_m \\ \Phi_n \Phi_m & 1 + i \frac{\sin(n\gamma/16) + \sin(m\gamma/16)}{2 \text{ch}(\theta/2) \text{sh}(\theta/2)} \frac{2 \sin(n\gamma/16) \sin(m\gamma/16)}{\sqrt{m_n m_m} \text{ch}(\theta/2)} \\ \Psi_n \Psi_m & \frac{2 \sin(n\gamma/16) \sin(m\gamma/16)}{\sqrt{m_n m_m} \text{ch}(\theta/2)} - 1 + i \frac{\sin(n\gamma/16) + \sin(m\gamma/16)}{2 \text{ch}(\theta/2) \text{sh}(\theta/2)} \end{pmatrix} \\
 &= Z_{nm}(\theta) \begin{pmatrix} \Phi_n \Psi_m & \Psi_n \Phi_m \\ \Phi_n \Psi_m & 1 - i \frac{\sin(n\gamma/16) - \sin(m\gamma/16)}{2 \text{ch}(\theta/2) \text{sh}(\theta/2)} \frac{2i \sin(n\gamma/16) \sin(m\gamma/16)}{\sqrt{m_n m_m} \text{sh}(\theta/2)} \\ \Psi_n \Phi_m & \frac{2i \sin(n\gamma/16) \sin(m\gamma/16)}{\sqrt{m_n m_m} \text{sh}(\theta/2)} + i \frac{\sin(n\gamma/16) - \sin(m\gamma/16)}{2 \text{ch}(\theta/2) \text{sh}(\theta/2)} \end{pmatrix} \quad (3.19)
 \end{aligned}$$

The prefactor $Z_{nm}(\theta)$ is defined by

$$Z_{nm}(\theta) = -\frac{i}{4} \operatorname{ch} \frac{1}{2}\theta \operatorname{sh} \frac{1}{2}\theta \frac{U'(\theta + \frac{1}{2}\Delta\theta_n + \frac{1}{2}\Delta\theta_m)}{2\pi i} \frac{U'(\theta + \frac{1}{2}\Delta\theta_n - \frac{1}{2}\Delta\theta_m)}{2\pi i} \\ \times \frac{U'(\theta - \frac{1}{2}\Delta\theta_n + \frac{1}{2}\Delta\theta_m)}{2\pi i} \frac{U'(\theta - \frac{1}{2}\Delta\theta_n - \frac{1}{2}\Delta\theta_m)}{2\pi i} \quad (3.20)$$

and satisfies $Z_{nm}(\theta) = Z_{nm}(i\pi - \theta)$.

Unitarity is satisfied if

$$Z_{nm}(\theta)Z_{nm}(-\theta) = \frac{\operatorname{sh}^2 \frac{1}{2}\theta}{\operatorname{sh}^2 \frac{1}{2}\theta + \sin^2 \frac{1}{32}(n+m)\gamma} \frac{\operatorname{sh}^2 \frac{1}{2}(i\pi - \theta)}{\operatorname{sh}^2 \frac{1}{2}(i\pi - \theta) + \sin^2 \frac{1}{32}(n-m)\gamma}. \quad (3.21)$$

Using eq. (3.21), we can find

$$Z_{nm}(\theta) = Y_{n+m}(\theta)Y_{n+m}(i\pi - \theta)Y_{n-m}(\theta)Y_{n-m}(i\pi - \theta), \quad (3.22)$$

with $Y_n(\theta)$ as given in eq. (3.16). Crossing symmetry is also satisfied. As shown in fig. 7, the indices n and m should be switched and $n\gamma/16 \rightarrow \pi - n\gamma/16$ for the crossed channel. Again, only the processes which preserve fermion numbers have non-vanishing amplitudes.

We want to emphasize that the SSG breathers $|\Phi_n^i\rangle$ and $|\Psi_n^i\rangle$ have given the same S -matrix elements irrespective of the index i . For example, $\langle \Phi_n^1 \Phi_m^1 | S | \Phi_n^1 \Phi_m^1 \rangle$ is the same as $\langle \Phi_n^2 \Phi_m^2 | S | \Phi_n^2 \Phi_m^2 \rangle$. We have checked all other amplitudes and confirmed the results explicitly. This justifies our previous identification of $|\Phi_n^1\rangle = |\Phi_n^2\rangle$ and $|\Psi_n^1\rangle = |\Psi_n^2\rangle$.

Similar to the SG theory, we can identify the lowest SSG breathers with the elementary scalar and Majorana fermion. In the SG theory, this has been justified

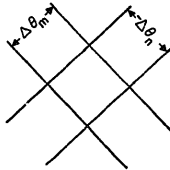


Fig. 7. Crossed channel of fig. 5.

because the breather S -matrix $S_{\text{SG}}^{(1,1)}$ agrees with the result of perturbative computation. We claim that this is also true for the SSG theory because the perturbative S -matrix of the SSG theory will be commuting with SUSY. Therefore, we can identify the elementary fields ϕ, ψ in (2.1) with Φ_1, Ψ_1 . The S -matrix (3.19) for $m = n = 1$ is

$$S(\theta) = Y_2(\theta)Y_2(i\pi - \theta) \times \left\{ \begin{array}{l} \phi\phi \left(1 + \frac{if}{\text{ch}(\theta/2)\text{sh}(\theta/2)} \right) \quad \frac{\psi\psi}{\text{ch}(\theta/2)} \\ \psi\psi \left(\frac{f}{\text{ch}(\theta/2)} \quad -1 + \frac{if}{\text{ch}(\theta/2)\text{sh}(\theta/2)} \right) \end{array} \right\}, \quad (3.23)$$

$$\left\{ \begin{array}{l} \phi\psi \left(\frac{1}{\text{sh}(\theta/2)} \quad \frac{if}{\text{sh}(\theta/2)} \right) \\ \psi\phi \left(\frac{if}{\text{sh}(\theta/2)} \quad 1 \right) \end{array} \right\},$$

where $f = \sin(\gamma/16)$. This result agrees with the known result of the SSG elementary particle S -matrix [6, 24]. This agreement is rather surprising because the two derivations are based on two quite different criteria. The first derivation of the S -matrix for the elementary particles are based on the factorizable S -matrix which commutes with the on-shell SUSY without any central charge as explained in the sect. 1. We derived the result from the SSG soliton S -matrix which was obtained by “unrestricting” the RSSG S -matrix. This is solely based on the perturbed CFT formalism. Therefore, this agreement is a very nice confirmation for the validity of eq. (2.11) and the RSG S -matrix as well as the perturbed CFT formalism. Furthermore, we can identify the arbitrary constant Δ arising from the factorization procedure with the physical parameter in the SSG theory by $\Delta = \gamma/8$.

We close this section with the summary of the complete SSG S -matrices:

$$\begin{aligned} \text{SSG soliton-soliton sector [(2.12), (2.25)]:} & \quad S = S_{\text{RSG}}^{(2)} \otimes S_{\text{SG}}, \\ \text{SSG soliton-breather sector [(3.10), (3.11)]:} & \quad S = S_{\text{RSG}}^{(n)} \otimes S_{\text{SG}}^{(n)}, \\ \text{SSG breather-breather sector [(3.18), (3.19)]:} & \quad S = S_{\text{RSG}}^{(n,m)} \otimes S_{\text{SG}}^{(n,m)}. \end{aligned} \quad (3.24)$$

4. The perturbed superconformal non-unitary model

In this section we restrict the SSG theory to get the perturbed superconformal non-unitary model which is obtained from the coset CFT with a rational level. The generalization to the FSSG theory and corresponding perturbed CFT is considered. Detailed results in this section will be published in ref. [27].

4.1. COSET CFT WITH A RATIONAL LEVEL

Start with the coset

$$\frac{\mathrm{SU}(2)_K \otimes \mathrm{SU}(2)_L}{\mathrm{SU}(2)_{K+L}}. \quad (4.1)$$

We generalize the FF construction of the coset CFT in sect. 2 to arbitrary K with a rational value of L . Following the standard procedure for integer L [25], we introduce a scalar FF field φ and the Z_K parafermions Ψ_k . The vacuum charge α_0 in (2.5) and the screening operators (2.6) are generalized to

$$\begin{aligned} V_{\pm} &= \oint dz \Psi_{\pm} \exp(i\alpha_{\pm}\varphi), & \alpha_0 &= \sqrt{\frac{K}{2(L+2)(K+L+2)}}, \\ \alpha_- &= -\sqrt{\frac{2(L+2)}{K(K+L+2)}}, & \alpha_+ &= \sqrt{\frac{2(K+L+2)}{K(L+2)}}. \end{aligned} \quad (4.2)$$

The primary fields are

$$\begin{aligned} V_{m,n}^k &= \exp(i\alpha_{m,n}\varphi)\Psi_k, & \alpha_{m,n} &= \frac{1}{2}(1-m)\alpha_+ + \frac{1}{2}(1-n)\alpha_-, \\ 1 \leq m \leq L+1, & & 1 \leq n \leq L+K+1, & & 0 \leq k \leq K. \end{aligned} \quad (4.3)$$

We identify these fields with the primary fields of the coset CFT which are defined from the branching functions of character decomposition by comparing their conformal dimensions. The primary fields of the coset CFT can be expressed in terms of the highest-weight states of each $\mathrm{SU}(2)$, and represented by

$$\Phi_{m,n}^k = \frac{(K;k) \otimes (L;m)}{(K+L;n)}. \quad (4.4)$$

Using the dimension of $(L; m)$, $\Delta(L; m) = \frac{1}{4}(m^2 - 1)/(L + 2)$ with $1 \leq m \leq L + 1$, one can check that the dimensions of $V_{m,n}^k$ and $\Phi_{m,n}^k$ are identical.

To construct general non-unitary CFT's including conformal and superconformal models, we generalize the unitary coset CFT's with integer levels to the coset

CFT's with rational levels [28]. Since any rational number can be represented by two relative prime numbers p, q , we consider one of the level, say L , satisfies

$$L + 2 = \frac{p}{q-p}, \quad q > p. \quad (4.5)$$

Let us consider the SU(2) Kac-Moody algebra with a rational level L . The central charge is the same as that of the integer level, $c = 3L/(L + 2)$. The primary fields are given by "admissible" representation [28] and represented by two integers k, m

$$\phi_{k,m}^{(L)} \quad \text{where } 0 \leq k \leq q-p-1, \quad 1 \leq m \leq p-1. \quad (4.6)$$

The corresponding conformal dimensions are

$$\Delta(\phi_{k,m}^{(L)}) = \frac{[m - k(L + 2)]^2 - 1}{4(L + 2)}. \quad (4.7)$$

The fusion rules, BRST cohomology structure and modular invariant partition functions [29-32] have been studied. The fusion algebra for the admissible representations is given by

$$\phi_{k,n}^{(L)} \times \phi_{k',n'}^{(L)} = \sum_{\substack{m = \lfloor n-n' \rfloor + 1 \\ n+n'-m = 1 \pmod{2}}}^{\min\left\{ \begin{matrix} n+n' \\ 2p-1-(n+n') \end{matrix} \right\}} (-)^{[(k+k')/(q-p)]} \phi_{k+k' \pmod{q-p}, m}^{(L)} \quad (4.8)$$

where $[\cdot/\cdot]$ denotes integer division without remainder. Therefore, the fusion coefficients can be negative integers. For unitary model $q-p=1$, only $k=0$ is allowed and the primary fields are reduced to those of integral representations.

One can see that the $k=0$ sector of the primary fields forms the closed fusion subalgebra of (4.8), which is identical to that of integral representations. Furthermore, the conformal dimensions of the primary fields in this sector are exactly the same form as those of integral representations as one can see easily from (4.7). The dimensions of the representations in this sector are finite. As we introduce the $k \neq 0$ sectors, the number of primary fields which close the fusion algebra increases. These fields in the $k \neq 0$ sectors have some exotic conformal dimensions and correspond to infinite-dimensional representations. The correlation functions for these sectors seem to be problematic [33].

The primary fields of the coset CFT (4.1) with the level (4.5) are defined from the branching functions arising from the character decompositions as usual [28].

Symbolically, we represent the primary fields of non-unitary models by

$$\Phi_{(k_1, m), (k_2, n)}^k = \frac{(K; k) \otimes \phi_{k_1, m}^{(l)}}{\phi_{k_2, n}^{(K+L)}}, \quad (4.9)$$

which are of the same form as (4.4) with two integral representations replaced by admissible ones. The fusion algebra can be immediately derived from (4.8). Again, the $k = 0$ sector of $\Phi_{(0, m), (0, n)}^k$ forms the closed fusion algebra with the smallest number of primary fields and the conformal dimensions of the primary fields in the $k = 0$ sector are exactly of the same form as those of the unitary CFT in (4.4). Recalling that we identified (4.3) and (4.4) because the conformal dimensions are same, the identification can be extended to the rational value of L if we consider only the $k = 0$ sector for the coset CFT's. Only difference from the unitary models is that the allowed ranges of (4.3) are modified according to the allowed ranges of the SU(2) primary fields in (4.6). If we introduce the $k \neq 0$ sectors in the coset CFT's, there will be many additional primary fields with the conformal dimensions which the FF primary fields can not have. In this paper, we are concentrating on the $k = 0$ sector because the SSG theory is identified with the perturbed superconformal minimal model via the FF construction. This does not rule out the possibility of new coset CFT's which contain new sets of primary fields arising from the $k \neq 0$ sectors.

The complete set of primary fields of the non-unitary models which can be related to the FF construction is*

$$\Phi_{(0, m), (0, n)}^k, \quad 1 \leq m \leq p - 1, \quad 1 \leq n \leq p + K(q - p) - 1, \quad 0 \leq k \leq K. \quad (4.10)$$

These modified ranges of the primary fields becomes important later when we relate the restricted SSG theory to the perturbed superconformal non-unitary model.

4.2. THE S-MATRICES OF PERTURBED SUPERMINIMAL CFT'S

Except for a few applications to statistical models such as Yang-Lee edge singularity model, non-unitary models do not have equally good justifications as CFT's. However, if we consider the massive perturbation of these models, the resulting theories can be unitary in the sense that probability can be conserved. The reason is that the Hilbert spaces of the massive quantum field theories are not

* After finishing this paper, the author was informed that the equivalent results were obtained for the non-unitary models independently using the BRST cohomology method [34]. The author thanks H. Tye for the information.

composed of the primary fields and their descendents but of the particle states, either elementary particles or solitons. As far as the S -matrices for these on-shell particles are unitary, the perturbed theories are well defined.

Just like the unitary models [7], consider the perturbing field Φ_{pert} to be $\Phi_{(0,1),(0,3)}^0$ with the dimension

$$\Delta(\Phi_{\text{pert}}) = 1 - \frac{2}{K + L + 2}. \quad (4.11)$$

We can show that there exists a conserved current under the perturbation which is identified with fractional SUSY $Q^{(K)}$. The proof is the same as that for the unitary models because the proof depends on the fusion rules and the conformal dimensions, which are not changed for the rational level L as shown before. The only difference is that the duality $K \leftrightarrow L$ does not exist for the non-unitary models. To relate the perturbed non-unitary CFT's to restricted integrable field theories, consider first the well-known case of the minimal model.

$K = 1$ in (4.1) gives the minimal model. The central charge

$$c = 1 - \frac{6(p-q)^2}{pq} \quad (4.12)$$

is identical with the original model with a finite number of primary fields [26]. The perturbing operator (4.11) corresponds to the $\Phi_{1,3}$ operator. Since the perturbing field and the screening operator give the SG potential, this perturbed theory is the restriction of the SG theory [12, 13]. For this rational value of L , the restriction of the SG S -matrix based on the quantum group symmetry will be modified. The deformation parameter $q = -\exp[-i\pi/(L+2)]$ satisfies now $q^p = \pm 1$. This means that the highest allowed spin of $\mathcal{R}_q[s(2)]$ is $p/2 - 1$. The restricted S -matrix of the SG theory S_{RSG} will be the same form as (2.14) with new bounds on the allowed spin $0 \leq j \leq p/2 - 1$. By comparing the maximum spin with the allowed maximum topological charges of perturbed CFT, we can identify the perturbed minimal model with the RSG theory. Still, the unitarity of this perturbed model need to be checked.

Eguchi and Yang [12] studied the restricted model using the BRST formalism. They found that the coupling constant $\gamma/8\pi$ (γ is related to the coupling by $\gamma = \beta^2/(1 - \beta^2/8\pi)$), different from the case of the SSG coupling (2.10)) should have one of the following values: $p/(q-p)$, $1/(n-1)$, or irrational. The first is the case where the truncation of multi-soliton Hilbert space reduces the SG theory to the perturbed minimal model. The other two cases do not allow truncation and correspond to perturbed $c = 1$ CFT.

Reshetikhin and Smirnov [13] examined the unitarity of the S -matrix of the perturbed non-unitary CFT and further restricted the allowed coupling constants

which can give unitary S -matrix as follows:

$$\frac{\gamma}{8\pi} = L + 2 = \frac{N}{Nl + 1}, \quad \frac{3}{3l + 2} \quad \text{with } N \geq 2, \quad l \geq 0. \quad (4.13)$$

$l = 0$ corresponds to the perturbed unitary CFT which is the RSG theory with no breathers. The S -matrix is just the RSG S -matrix (2.14). For $l \neq 0$, the model corresponds to the perturbed non-unitary minimal model. Since this is related to the restriction of the SG theory for the general coupling constant, the spectrum consists of the RSG kinks and the SG breather and the S -matrices are given by $S_{\text{RSG}}^{(N)}$, $S_{\text{RSG}}^{(n)}$ and $S_{\text{RSG}}^{(n,m)}$, respectively. The maximum allowed spin for the kinks is $N/2 - 1$.

Now, consider the superconformal non-unitary model with $K = 2$ in (4.1). With the rational level (4.5), the screening operators (2.6) are dimension 1 and with the perturbing field give the SSG potential. Therefore, the perturbed superconformal minimal model should be related to the restriction of the SSG theory. When L is integer, the S -matrix becomes the RSSG S -matrix (2.9). This is the restriction of the S -matrix of the SSG solitons of (2.14) and (2.25). Recalling the tensor product form of the S -matrices of the perturbed coset theories, the quantum group structure of the theories depends only on the SG sector. Therefore, the unitarity condition on the S -matrix (4.13) applies to this case too. The other sector of the S -matrix $S_{\text{RSG}}^{(2)}$ remains unaffected because the deformation parameter of this sector depends on $K = 2$, not on L . The restriction of the SSG S -matrices (3.24) gives the S -matrices of the perturbed superconformal non-unitary model as follows:

$$\text{RSSG kink-kink sector [(2.14), (2.25)]:} \quad S = S_{\text{RSG}}^{(2)} \otimes S_{\text{RSG}}^{(\rho)},$$

$$\text{RSSG kink-breather sector [(3.10), (3.11)]:} \quad S = S_{\text{RSG}}^{(n)} \otimes S_{\text{SG}}^{(n)},$$

$$\text{RSSG breather-breather sector [(3.18), (3.19)]:} \quad S = S_{\text{RSG}}^{(n,m)} \otimes S_{\text{SG}}^{(n,m)}, \quad (4.14)$$

where the RSOS spins are restricted to be $0 \leq j \leq p/2 - 1$. Since the maximum spin of the RSSG theory can be related to the maximum topological charge which the primary fields (4.10) of the superminimal model can carry, the two models are identified. (Integral topological charges can be carried only by the primary fields $V_{m,1}^k$, $0 \leq m \leq p - 1$, and $t_{\text{max}} = 2j_{\text{max}} + 1 = p - 1$.)

Expression (4.14) needs some additional explanations. Restricting the S -matrices in eq. (3.24), we should consider the quantum group structure of the second factors in eq. (3.24). S_{SG} is restricted to S_{RSG} . Since the SG breathers are singlets of the quantum group, the breather-breather S -matrices will be unchanged. The breather-soliton sector is unchanged modulo the q -Clebsch-Gordan coefficients occurring when we change the basis from the vertex form to the RSOS form. This

is so because the S -matrices of this sector in (3.10) do not depend on the topological charges of the SG solitons. All these S -matrices are unitary and crossing symmetric.

Let us consider the special case of $p = 2$, $q = 2n + 3$. The coupling constant is

$$\frac{\gamma}{8\pi} = \frac{2}{2n+1}. \quad (4.15)$$

Using the above expression, one can see easily that all the SSG solitons are “frozen” and only the SSG breathers are left in the physical spectrum. This is the supersymmetric generalization of ref. [9]. Especially, when $n = 1$ which corresponds to the SUSY Yang–Lee edge singularity model, the spectrum consists of one real scalar and one Majorana fermion with mass $\sqrt{3}$. The S -matrices are just the breather–breather S -matrices [35].

4.3. FRACTIONAL SUPERSYMMETRIC MODELS

All the previous results and analysis for the $K = 2$ theory seem equally valid for the case of $K \geq 3$. The S -matrix of the fractional supersymmetric sine-Gordon theory (FSSG) can be conjectured as an unrestricted form of (1.1), $S = S_{\text{RSG}}^{(K)} \otimes S_{\text{SG}}$ as has been done in ref. [7]. Note that the SG part is exactly the same as that of the SSG theory. Therefore, if we generalize the value of the coupling constant L to the generic one, the theory should contain the fractional supersymmetric breathers in addition to FSSG solitons. The S -matrices of these states can be derived from (2.14), considering three-soliton, and four-soliton scattering processes in the same way as the SSG theory. The SG part remains the same as (3.10) and (3.18).

This can lead to the discovery of the FSSG action if we identify the lowest mass breathers with the elementary fields appearing in the FSSG action. We can guess the action from the perturbative expansion of the exact S -matrices. This is the opposite direction to the identification of the lowest mass breather with the elementary scalar field in the SG theory. Furthermore, as argued in sect. 3, the S -matrices of the FSSG breathers and solitons for the special non-unitary couplings can be related to the perturbed non-unitary CFT's with fractional SUSY [27].

Consider the limit of $K \rightarrow \infty$ with the rational value of L in (4.5). From the structure of the coset CFT (4.1) and its perturbation with dimension (4.11), this theory is easily identified with perturbed $SU(2)$ WZW theory with a rational level perturbed by a $J\bar{J}$ operator where J (\bar{J}) is the current of the theory,

$$S = S_L^{\text{WZW}} + \frac{\lambda}{2\pi i} \int d^2z \sum_a J^a(z) \bar{J}^a(\bar{z}). \quad (4.16)$$

The S -matrix of this model is the limit $K \rightarrow \infty$ of $S_{\text{RSG}}^{(K)} \otimes S_{\text{RSG}}^{(L)}$ where the second

S -matrix includes not only the RSG S -matrix but also the breather S -matrices described above. Since the allowed spin of the RSG theory is unbounded in this limit, the first factor $S_{\text{RSG}}^{(K)}$ becomes unrestricted. The particle spectrum of the theory is then

$$|A^\pm\rangle \otimes |K_{ab}\rangle \quad \text{and} \quad |A^\pm\rangle \otimes |B_n\rangle. \quad (4.17)$$

The spin a, b is restricted up to $p/2 - 1$ and $|B_n\rangle$ denotes the SG breathers with mass m_n .

As argued in ref. [7], the first factor in the S -matrix becomes the S -matrix of the $SU(2)$ Gross-Neveu model denoted by S^{rat} and total S -matrix is given by

$$S_{\text{WZW}}(\theta) = S^{\text{rat}}(\theta) \otimes \{S_{\text{RSG}}^{(L)}(\theta), S_{\text{SG}}^{(n)}(\theta), S_{\text{SG}}^{(n,m)}(\theta)\}, \quad (4.18)$$

where $S^{\text{rat}}(\theta) = \lim_{K \rightarrow \infty} S_{\text{SG}}(x = e^{\theta/K+2}, q = -e^{-i\pi/(K+2)})$ is explicitly computed in ref. [7] and all other factors are given in eqs. (2.14), (3.10), and (3.18).

For the special coupling (4.15), the RSG solitons are completely frozen out and the only spectrum of the perturbed WZW theory is the breathers with the extra topological charge ± 1 , $|A^\pm\rangle \otimes |B_n\rangle$ and the S -matrix is explicitly written as

$$S_{\text{WZW}}(\theta) = S^{\text{rat}}(\theta) \cdot S_{\text{SG}}^{(n,m)}(\theta). \quad (4.19)$$

Although WZW models with rational levels seem non-local because the Wess-Zumino terms are not single-valued, the special cases of $L = -1, -2$ give the local WZW action. For $L = -2$, since the SSG coupling constant γ vanishes, the SG S -matrices become just identity and the WZW S -matrix (4.18) becomes S^{rat} . For $L = -1$, all breathers become unstable and only the SG solitons are left. These negative integer level $SU(2)$ WZW models can be identified with perturbed WZW model based on non-compact group $SU(1, 1)$ with level 1, 2. Note that even though we considered the perturbation of non-unitary CFT, the S -matrices of the massive on-shell particles (4.18) are perfectly unitary and crossing symmetric.

5. Conclusions

In this paper, we derived the complete S -matrices of the SSG theory including the SSG soliton bound states. We have checked the validity of our derivation by comparing one special case of our results with the known result. Since our derivation of the SSG S -matrices is based on the duality of the coset CFT and the result of the SG theory, this check shows the usefulness of the conformal field theoretic approach to the integrable field theories.

We showed that the results of the SSG theory can be applied to the perturbed superconformal minimal model. We defined the non-unitary models by extending the coset construction to $SU(2)$'s with the rational levels and their admissible

representations. While the primary fields of the coset CFT's are richer, we considered only the smallest subset of the primary fields which form the closed fusion algebra because only these fields are appearing in the FF construction. The perturbed superconformal minimal model is related to the restriction of the SSG theory using the FF formalism. Although we study the massive perturbation of non-unitary model, the final S -matrices are unitary if the coupling constants are restricted to the certain values. The very existence of bound states (breathers) reflects the fact that some primary fields have negative conformal dimensions.

The generalization of the superconformal minimal model to the coset CFT's with the higher integer level of K is straightforward. The S -matrices of the FSSG theories with both solitons and breathers can be derived in the same way as the SSG theory. The restriction of the FSSG theories can be also related to the perturbed minimal (including non-unitary) CFT's which are constructed from the coset CFT's as before. Also, we presented some special limiting cases which are interesting. A detailed analysis will be reported in another paper [27].

The quantum group structure of the SSG theory is based on that of the SG theory because the S -matrix of the SSG theory is the tensor product of the S -matrix of the SG theory with the RSG factor. The quantum group symmetry has been defined as an on-shell symmetry acting on the on-shell particles and thus commuting with the S -matrix. The recent derivation of the quantum group symmetry from the SG action [15] which can be applied to the SSG and FSSG theories can justify the existence of these symmetries for the generalized models including the SSG theory. Using these quantum group symmetries, one may construct the on-shell SSG and FSSG solitons explicitly in terms of the elementary fields.

It is a pleasure to thank A. LeClair for valuable comments throughout this work, D. Bernard and again A. LeClair for the previous collaboration. Also, we thank P. Argyres, S. Chung, B. Greene, I.G. Koh, C.K. Lee, K. Shigemoto, F. Smirnov, H. Tye and M. Walton for discussions and H.S. Song for his hospitality at the Center for Theoretical Physics in Seoul National University where part of this work has been done. This work was supported in part by the National Science Foundation.

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