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Thermodynamics and form factors of supersymmetric integrable field theories

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Abstract

We study on-shell and off-shell properties of the supersymmetric sinh-Gordon and perturbed SUSY Yang–Lee models using the thermodynamic Bethe ansatz and form factors. Identifying the supersymmetric models with the Eight Vertex Free Fermion Model, we derive an inversion relation for the inhomogeneous transfer matrix and TBA equations and get correct UV results. We obtain two-point form factors of the trace of the energy-momentum tensor using the Watson equations and their SUSY transformations. As an application, we compute the UV central charge using these form factors and the spectral representation of the C -theorem.

1. Introduction

For 2D integrable field theories S -matrices are purely elastic, all incoming momenta are conserved and multi-particle scattering amplitudes are factorized into a product of two-particle S -matrices. These S -matrices, in turn, should satisfy Yang–Baxter equations which often determine the S -matrices completely along with unitarity and crossing symmetry [1]. The S -matrix provides essential tools to understand 2D field theories. First of all, the S -matrix gives information on the UV behaviour of the theory by relating the Casimir energy on the cylinder to the central charge of the corresponding UV conformal field theory (CFT) [2]. This program known as thermodynamic Bethe Ansatz (TBA) [3] has provided consis-

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tency checks for many factorizable scattering theories either with local lagrangians or without them such as perturbed CFTs [4].

S -matrix plays an important role in off-shell physics as well. It can be used to determine off-shell quantities such as correlation functions by computing the matrix elements of an operator on the basis of the on-shell particles. These objects known as form factors (FFs) may be computed exactly using only the S -matrices and particle spectrum (bound states) as input [5,6]. With exact FFs correlation functions are given by an infinite sum over intermediate on-shell states. This form factor approach has the advantage for the computation of correlation functions of massive integrable models that the infinite sum over all intermediate states converges very fast. For many cases, upto two-point FFs give quite accurate results on off-shell quantities [7–10]. Furthermore, the two-point FFs can be related to some exact non-perturbative informations of the underlying theories, such as the wave function renormalization [5,12] and the UV central charges through the spectral representation of the C -theorem [8,13,9]. In this sense, without complete solutions of the FFs one can still extract non-perturbative off-shell informations from the FFs.

While the TBA analysis or the FF computation can be relatively simple for diagonal scattering theory, which has no mass degeneracy, non-diagonal scattering theories entail much more complicacy. By non-diagonal we mean theories with different types of particles of the same mass for which the scattering of two particles can occur in more than one channel. Most of the interesting 2D integrable field theories such as the soliton scattering theories, theories with internal gauge symmetries, and supersymmetric theories belong to this class.

For the non-diagonal theories, the equations for the TBA and FFs are expressed in terms of monodromy and transfer matrices. To solve the equations, one needs to diagonalize these matrices. It is remarkable that with some technical differences the same problem is often met in the study of lattice models [14]. In the lattice model the Yang–Baxter equations are to be satisfied to construct infinite number of conserved charges through the commuting transfer matrices. Partition functions and free energies are expressed in terms of the eigenvalues of the transfer matrices. Due to this common feature, it is often quite useful to connect 2D field theories with lattice models.

There are two types of the models in the lattice and continuum which are connected with each other. The first one is the so-called vertex type; the states are assigned on the lines which form a lattice. For the square lattice, each vertex consists of four lines and an assigned Boltzmann weight depending on the four states of the lines [14]. These lines correspond to the world-lines of incoming and outgoing particles in the scattering theories. While some of these vertex models are associated with field theories with local lagrangians, there remain many vertex-type lattice models still to be related to 2D integrable field theories.

The second type is the interacting-round-face models [15]. The Boltzmann weights are assigned on each vertex on the square lattice, depending on the heights of four faces. As a special case, if the heights are restricted, one obtains restricted solid-on-solid (RSOS) type of models. This wide class of lattice models has been

related to 2D CFTs. Due to the conformal invariance, the corresponding lattice models are at the criticality. Many exact results including correlation functions have been obtained using the CFT techniques. This identification can be continued in the off-critical region. Without the conformal symmetry, the off-critical RSOS models are associated with CFTs perturbed by relevant operators [16–19]. Again, S -matrices of the perturbed CFTs are given by the Boltzmann weights of the RSOS models.

The best known example is the relation between the six-vertex model and the sine-Gordon (SG) model. The SG model has soliton and antisoliton spectrum and the S -matrix can be associated with the R -matrix of the $\widehat{sl}_q(2)$, affine quantum group [18]. The Boltzmann weights of the six vertex model are the same as the S -matrix elements after identifying the up and down arrows assigned on each vertex line with the soliton and antisoliton. In addition, quantum group reduction of the SG model corresponds to the RSOS lattice model obtained from the six vertex model. The TBA analysis of these models have been done by diagonalizing the inhomogeneous transfer matrices of the six-vertex [21–23] and RSOS models [24].

The complete FFs of the SG model have been obtained by Smirnov using quantum inverse scattering methods, providing the only known example with the complete FFs for non-diagonal theories. Based on this information, Smirnov found axioms for the FFs to satisfy [6]. Therefore, the problem to find complete FFs is reduced to solve these axioms for a given theory. However, solving these axiomatic equations completely is very difficult even for diagonal scattering theories except for a few simplest ones such as Ising, Yang–Lee, and sinh-Gordon models [7–9,11]. The problem becomes much more complicated for the non-diagonal cases. As an initial step to the problem, we will concentrate on two-point FFs. Two-point FFs can be determined relatively easily by diagonalizing S -matrix and evaluating the FFs using the Watson equations [5]. For the supersymmetric theories, details can be further simplified due to the SUSY relations between the FFs. As stressed before, the two-point FFs have many useful informations on the underlying theories.

In this paper, we want to apply these frameworks to the $N = 1$ supersymmetric (SUSY) theories. The S -matrices of many SUSY models have been obtained. These S -matrices have the following factorized form [19,26]:

$$S(\theta) = S_S(\theta) \otimes S_0(\theta), \quad (1.1)$$

where the first factor S_S carries the SUSY indices and commutes with the SUSY charges while the second one S_0 is the S -matrices of the models without the SUSY. So far, several SUSY integrable field theories and perturbed super CFTs are solved and their S -matrices are derived. An interesting aspect of the SUSY models is that these S -matrices commuting with SUSY charges are identified with Boltzmann weights of some lattice models.

For example, for the $N = 1$ SUSY CFTs perturbed by the least relevant operator, S_S , which commutes with SUSY charges with central extension due to

the topological charges, is related to the RSOS weights corresponding to the tricritical Ising model [25]. For the $N = 2$ SUSY models, the first factor is identified with the Boltzmann weights of the six-vertex model [20,22]. These relations with lattice models are important not only for the lattice-field theory correspondence but for actual solutions of the models.

The $N = 1$ SUSY sine-Gordon (SSG) model has been solved in a unconventional way. Its soliton S -matrix has been derived from the results on the perturbed super CFTs by the least relevant operator [19]. The SUSY part of the SSG soliton S -matrix is given by the RSOS tricritical Ising model S -matrix while S_0 is the ordinary sine-Gordon S -matrix. The S -matrices of the SSG bound states (breathers) have been derived from multi-soliton scattering amplitudes [27]. In particular, since the lightest bound states are forming a supermultiplet of the fundamental fields appearing in the SSG lagrangian, the lightest breather S -matrix of the SSG model can be analytically continued to get the S -matrix of the supersymmetric sinh-Gordon (SShG) model. This S -matrix is identical to the one derived first by Shankar and Witten by explicitly requiring the commutativity with SUSY charges [28]. Besides, the SSG model with only the lightest breather in the spectrum can be understood as perturbed super CFTs, the SUSY Yang–Lee (SYL) model [26,27]; the simplest nonunitary super CFT perturbed by the least relevant operator. This model includes only one supermultiplet of on-shell states and the S -matrix is identical with that of the SShG model. This S -matrix is our starting point.

These models with $N = 1$ SUSY without a central extension will be identified with the general eight-vertex models with an external field. If the Boltzmann weights of the general eight-vertex model satisfy a “free fermion” condition, the model is exactly solvable and the free energy was derived first from dimer method [29] and later diagonalizing the transfer matrix [30]. Also, this model has been identified with the general XY -spin chain model with a magnetic field [31]. This relation with the lattice model will be very useful in our derivation of TBA equations for the SShG model. It turns out that the SShG model is at the critical point of the XY -spin chain model.

We organize this paper in the following way. In the next section, we write down the lagrangian of the SShG and SSG models and derive the energy–momentum tensor supermultiplet and their relations under the SUSY transformation. Also we present the S -matrices of the models. In sect. 3, we use TBA analysis for the supersymmetric models to derive the UV central charges. In sect. 4, we compute the FFs of the SShG model using the Watson equations and SUSY relations of the energy–momentum tensor. With exact two-point FFs, we derive the UV central charge of the model using the spectral representation of the C -theorem.

2. $N = 1$ SUSY integrable model and factorizable S -matrix

We present the energy–momentum tensor supermultiplet of the $N = 1$ SSG and SShG model and the S -matrix of the theories.

2.1. Lagrangian and energy–momentum tensor

We start with a lagrangian of a general $N = 1$ SUSY,

$$\mathcal{L}(\Phi) = \frac{1}{4}\bar{D}\Phi D\Phi + iW(\Phi)|_{\theta_1\theta_2}, \tag{2.1}$$

with a scalar superfield Φ ,

$$\Phi(x, \theta) = \phi + i\bar{\theta}\psi + i\frac{1}{2}\bar{\theta}\theta F, \tag{2.2}$$

and D and \bar{D} , the covariant derivatives,

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu. \tag{2.3}$$

The Grassman variable θ is a Majorana spinor ^{*}. In terms of the component fields, one gets

$$\mathcal{L}(\Phi) = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}i\bar{\psi}[\not{\partial} + W''(\phi)]\psi + \frac{1}{2}[W'(\phi)]^2. \tag{2.4}$$

The SSG model is a particular case of Eq. (2.4) with the superpotential

$$W(\Phi) = \frac{m}{\beta^2}\cos(\beta\Phi). \tag{2.5}$$

The SShG model is the same superpotential with the purely imaginary coupling constant $\beta = i\hat{\beta}$. The $N = 1$ SUSY algebra is generated by the conserved charges Q_1 and Q_2 ,

$$Q_1^2 = P_+, \quad Q_2^2 = P_- \quad \text{and} \quad \{Q_1, Q_2\} = 0, \tag{2.6}$$

with the light-cone momenta defined as $P_\pm = E \pm P$. These charges act on the component fields by

$$\begin{aligned} Q_1\phi &= i\psi_1, & Q_1\psi_1 &= \partial_+\phi, & Q_1\psi_2 &= F, \\ Q_2\phi &= i\psi_2, & Q_2\psi_2 &= -\partial_-\phi, & Q_2\psi_1 &= -F, \end{aligned} \tag{2.7}$$

with $F = -W'(\phi)$.

Integrability of the SSG and SShG models is established because they are equivalent to Toda theory based on the twisted super affine Lie algebra $C^{(2)}(2)$ [32–34]. The equations of motion of the SSG theory can be rewritten as super zero-curvature conditions. An infinite number of conserved charges at the classical level were derived [35] and checked to be preserved at the lowest-order quantum corrections [36].

The energy–momentum tensor supermultiplet can be expressed by [37]

$$J_{\alpha\mu} = [(\not{\partial}\Phi - W'(\Phi))\gamma_\mu D\Phi]_\alpha, \tag{2.8}$$

^{*} Dirac matrices are $\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

or in light-cone coordinates,

$$J_+ = \begin{pmatrix} D_1 \Phi \partial_+ \Phi \\ -W'(\Phi) \Phi D_1 \Phi \end{pmatrix}, \quad J_- = \begin{pmatrix} -W'(\Phi) \Phi D_2 \Phi \\ D_2 \Phi \partial_- \Phi \end{pmatrix}, \quad (2.9)$$

with $x_{\pm} = \frac{1}{2}(x_1 \pm x_0)$ and $\partial_{\pm} = \partial_1 \pm \partial_0$. In terms of the component currents,

$$J_{\pm} = \begin{pmatrix} \Psi_{1\pm} \\ \Psi_{2\pm} \end{pmatrix} + 2i \begin{pmatrix} \theta_2 T_{\pm\pm} \\ \theta_1 T_{-\pm} \end{pmatrix} + i\theta_1 \theta_2 \begin{pmatrix} \chi_{1\pm} \\ \chi_{2\pm} \end{pmatrix}, \quad (2.10)$$

one gets the energy-momentum tensor of the SSG model,

$$T_{++} = \frac{1}{2} [(\partial_+ \phi)^2 + i\psi_1 \partial_+ \psi_1], \quad T_{--} = \frac{1}{2} [(\partial_- \phi)^2 - i\psi_2 \partial_- \psi_2], \\ T_{+-} = T_{-+} = \frac{1}{2} \frac{m^2}{\beta^2} \sin^2 \beta \phi - \frac{im}{4} \bar{\psi} \psi \cos \beta \phi, \quad (2.11)$$

and its superpartner

$$\Psi_{1+} = i\psi_1 \partial_+ \phi, \quad \Psi_{2+} = -i \frac{m}{\beta} \psi_1 \sin \beta \phi, \\ \Psi_{2-} = i\psi_2 \partial_- \phi, \quad \Psi_{1-} = -i \frac{m}{\beta} \psi_2 \sin \beta \phi. \quad (2.12)$$

Including an appropriate normalization factor of 4π , we define the following notation for the SUSY energy-momentum tensor:

$$T = 4\pi T_{++}, \quad \bar{T} = 4\pi T_{--}, \quad \Theta = 4\pi T_{+-}, \quad (2.13)$$

and their SUSY partners,

$$T_F = 4\pi \Psi_{1+}, \quad \bar{T}_F = 4\pi \Psi_{2-}, \quad \Theta_F = 4\pi \Psi_{1-}, \quad \bar{\Theta}_F = 4\pi \Psi_{2+}. \quad (2.14)$$

They are related to each other by the SUSY transformation

$$Q_1 T_F = -2iT, \quad Q_1 T = -\frac{1}{2} \partial_+ T_F, \quad Q_1 \Theta_F = -2i\Theta, \quad Q_1 \Theta = -\frac{1}{2} \partial_+ \Theta_F, \\ Q_2 \bar{T}_F = 2i\bar{T}, \quad Q_2 \bar{T} = -\frac{1}{2} \partial_- \bar{T}_F, \quad Q_2 \bar{\Theta}_F = 2i\Theta, \quad Q_2 \Theta = -\frac{1}{2} \partial_- \bar{\Theta}_F. \quad (2.15)$$

2.2. On-shell particle states and S -matrix

If the coupling constant of the SSG model in Eq. (2.5) becomes purely imaginary, we have a simplest $N = 1$ SUSY field theory, namely the SShG model. Since the potential is not periodic, the soliton spectrum does not exist any more and the spectrum consists of only the fundamental particles appearing in the lagrangian, one scalar and fermion supermultiplet. We will denote on-shell states of these particles by $|b(\theta)\rangle$ and $|f(\theta)\rangle$ with a rapidity θ which is related to the momentum by $E = m \cosh \theta$ and $P = m \sinh \theta$.

The SUSY charges defined in Eq. (2.6) can act on on-shell states as (see subject 4.2)

$$\begin{aligned}
 Q_1 |f(\theta)\rangle &= \sqrt{m} e^{\theta/2} |b(\theta)\rangle, & Q_1 |b(\theta)\rangle &= \sqrt{m} e^{\theta/2} |f(\theta)\rangle, \\
 Q_2 |f(\theta)\rangle &= -i\sqrt{m} e^{-\theta/2} |b(\theta)\rangle, & Q_2 |b(\theta)\rangle &= \sqrt{m} e^{-\theta/2} |f(\theta)\rangle.
 \end{aligned}
 \tag{2.16}$$

It is easy to see that this satisfies $N = 1$ SUSY algebra, Eq. (2.6). The action of SUSY charges on multiparticle on-shell states can be easily worked out using this and the anticommutivity of Q_α and the fermion.

The exact S -matrix of the SShG model was derived using the Yang–Baxter equation, unitarity and crossing symmetry along with the commutativity of the SUSY charges and the S -matrix [28]. In the basis of two-particle on-shell states in the order of $|b_1 b_2\rangle, |f_1 f_2\rangle, |b_1 f_2\rangle, |f_1 b_2\rangle^*$, the S -matrix has been obtained to be ($\theta = \theta_1 - \theta_2$):

$$S(\theta) = Y(\theta) \begin{pmatrix} 1 + \frac{2i \sin \alpha \pi}{\sinh \theta} & \frac{i \sin \alpha \pi}{\cosh \frac{1}{2}\theta} & 0 & 0 \\ \frac{i \sin \alpha \pi}{\cosh \frac{1}{2}\theta} & 1 - \frac{2i \sin \alpha \pi}{\sinh \theta} & 0 & 0 \\ 0 & 0 & 1 & \frac{i \sin \alpha \pi}{\sinh \frac{1}{2}\theta} \\ 0 & 0 & \frac{i \sin \alpha \pi}{\sinh \frac{1}{2}\theta} & 1 \end{pmatrix},
 \tag{2.17}$$

with an arbitrary constant α which will be related to the coupling constant β of the SSG model in a moment. The prefactor $Y(\theta)$ is needed to make the S -matrix unitary and crossing symmetric. The following integral form will be useful later:

$$\begin{aligned}
 Y(\theta) &= \frac{\sinh \frac{1}{2}\theta}{\sinh \frac{1}{2}\theta + i \sin(|\alpha| \pi)} \\
 &\times \exp \left(- \int_0^\infty \frac{dt}{t} \frac{\sinh(|\alpha| t) \sinh((1 - |\alpha|) t)}{\cosh^2(\frac{1}{2}t) \cosh t} \sinh \frac{\theta t}{\pi i} \right).
 \end{aligned}
 \tag{2.18}$$

With $Y(\theta) = Y(i\pi - \theta)$ and a factor of i arising in the crossing relation for the $bb \rightarrow ff$ channel, the S -matrix of Eq. (2.17) is crossing symmetric.

To determine the constant α we should refer to another derivation of the SSG breather S -matrix. Using the SSG soliton S -matrix, one can compute four soliton scattering amplitudes. By taking bound state poles of the incoming and outgoing

* We use a short notation $|b_1 b_2\rangle = |b(\theta_1) b(\theta_2)\rangle$, etc.

soliton–antisoliton pairs one can derive the S -matrices of the SSG breathers [27]. In particular the S -matrix of the lightest breathers is the S -matrix of the fundamental particles of the SSG and SShG models:

$$S(\theta) = Y(\theta) \cdot \mathcal{R}(\theta) \cdot S_0(\theta),$$

where

$$S_0(\theta) = \frac{\sinh \theta + i \sin(2\alpha\pi)}{\sinh \theta - i \sin(2\alpha\pi)},$$

$$\mathcal{R}(\theta) = \begin{pmatrix} 1 + \frac{2i \sin \alpha\pi}{\sinh \theta} & \frac{\sin \alpha\pi}{\cosh \frac{1}{2}\theta} & 0 & 0 \\ \frac{\sin \alpha\pi}{\cosh \frac{1}{2}\theta} & -1 + \frac{2i \sin \alpha\pi}{\sinh \theta} & 0 & 0 \\ 0 & 0 & 1 & \frac{i \sin \alpha\pi}{\sinh \frac{1}{2}\theta} \\ 0 & 0 & \frac{i \sin \alpha\pi}{\sinh \frac{1}{2}\theta} & 1 \end{pmatrix}. \tag{2.19}$$

The factor S_0 is the lightest breather S -matrix of the SG model. The constant α in Eq. (2.19) is given by the coupling constant of the SSG model [27],

$$\alpha = \frac{\gamma}{16\pi} = \frac{\beta^2/4\pi}{1 - \beta^2/4\pi}. \tag{2.20}$$

For the SShG model with $\beta = i\hat{\beta}$ ($\hat{\beta}$ real), this constant reduces to

$$\alpha = -\frac{\hat{\beta}^2/4\pi}{1 + \hat{\beta}^2/4\pi}, \tag{2.21}$$

and $-\frac{1}{2} < \alpha < 0$.

Two S -matrices, Eqs. (2.17) and (2.19) are equivalent. The sign difference in the $ff \rightarrow ff$ channel is explained because all particles are considered as bosons in Eq. (2.19) by including the exchange factor -1 arising in $ff \rightarrow ff$ in the S -matrix element. In this convention, the crossing relation is satisfied without any extra factor because all particles are bosonic. Besides, for the SShG model with $\alpha < 0$, the S_0 has no pole in the physical strip. Therefore, S_0 is nothing but a CDD factor and can be removed by minimality assumption. For the SSG model, however, with a coupling in $0 < \alpha < \frac{1}{2}$ ($\beta^2 < 4\pi/3$) the S_0 does have a bound state pole corresponding to the second breather.

For a complete description of the SSG model, one should include all the S -matrices of the solitons and breathers as was done in ref. [27]. Depending on the values of the coupling constant of the SSG model, the spectrum of the bound states changes. In particular, if the coupling constant is in the range of $\frac{1}{2} < \gamma/8\pi <$

1, only the lightest bound states can exist along with the soliton and antisoliton in the spectrum. If the solitons are truncated from the theory keeping only the lightest bound states, the scattering theory becomes perturbed CFT by the least relevant operator. The UV CFT is the SUSY extension of the Yang–Lee model [26,27]. The S -matrix is given by Eq. (2.19).

3. Thermodynamics of the $N = 1$ SUSY models

In this section we use the TBA method to derive the central charges of $N = 1$ SUSY models. For the purpose we first show that the S -matrices of $N = 1$ SUSY models are the Boltzmann weights of so-called “free fermion” eight-vertex model. Using this observation, we can derive the TBA equations from inversion relation of diagonalizing the transfer matrix. We apply the TBA equations to both the SShG and SYL models perturbed by the least relevant operator and derive correct UV central charges.

3.1. Free fermion models

After the celebrating solution of the symmetric eight-vertex model by Baxter, Fan and Wu obtained an exact expression of the free energy for the general eight-vertex model with an external field if the Boltzmann weights satisfy some additional constraint, named the free fermion condition [29]. They called this model “Free Fermion” model (FFM) although the name is slightly misleading. The model turned out to be highly non-trivial and interacting.

We start with the Boltzmann weights of the general eight-vertex model:

$$R = \begin{pmatrix} a_+ & 0 & 0 & d \\ 0 & b_+ & c & 0 \\ 0 & c & b_- & 0 \\ d & 0 & 0 & a_- \end{pmatrix}, \tag{3.1}$$

for the following vertex configurations:

$$\tag{3.2}$$

If $R(\theta)$ satisfies the Yang–Baxter equation and the free fermion condition

$$a_+ a_- + b_+ b_- = c^2 + d^2, \tag{3.3}$$

and if the following combinations of the Boltzman weights are independent of the rapidity:

$$\Gamma = \frac{2cd}{a_+b_- + a_-b_+}, \quad h = \frac{a_-^2 + b_+^2 - a_+^2 - b_-^2}{2(a_+b_- + a_-b_+)}, \tag{3.4}$$

the transfer matrix T commutes; $[T(u), T(v)] = 0$. Due to this commutativity, there exist an infinite number of conserved charges including a hamiltonian of the corresponding one-dimensional spin-chain model. This hamiltonian has been identified with that of the XY -model with a magnetic field,

$$\mathcal{H}_{XY} = -J \sum_{j=1}^N [\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \Gamma(\sigma_j^+ \sigma_{j+1}^+ + \sigma_j^- \sigma_{j+1}^-) - h\sigma_j^z], \tag{3.5}$$

where $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ with a conventional Pauli σ^i matrices.

To identify the FFM with $N = 1$ SShG model, we rewrite the S -matrix of the SShG model, Eq. (2.19), by rearranging the two-particle basis. In the order of $|bb\rangle, |bf\rangle, |fb\rangle, |ff\rangle$, the R -matrix of Eq. (2.19) becomes the general form of the FFM with

$$a_\pm = \pm 1 + \frac{2i \sin \alpha\pi}{\sinh \theta}, \quad b_\pm = 1, \quad c = \frac{i \sin \alpha\pi}{\sinh \frac{1}{2}\theta}, \quad d = \frac{\sin \alpha\pi}{\cosh \frac{1}{2}\theta}, \tag{3.6}$$

if we identify \uparrow and \rightarrow with $|b\rangle$ and \downarrow and \leftarrow with $|f\rangle^*$.

It is an easy exercise to check that these weights satisfy the free fermion condition Eq. (3.3). Also, the constants Γ and h become

$$\Gamma = \sin \alpha\pi, \quad h = -1. \tag{3.7}$$

Since $h = -1$ is a critical point of the XY -model, the SShG model corresponds to the critical point of the general eight-vertex model with free fermion condition and with vanishing elliptic modulus.

3.2. Diagonalization of transfer matrix

The periodic boundary condition for the non-diagonal scattering theories becomes

$$e^{im_a L \sinh \theta} \Lambda(\theta | \theta_1, \dots, \theta_N) = 1, \tag{3.8}$$

where Λ is an eigenvalue of the transfer matrix $T(\theta)$ which is defined by

$$\begin{aligned} T(\theta | \theta_1, \dots, \theta_N)_{\{a_i\}}^{\{a'_i\}} \\ = \sum_a \sum_{\{a_i\}} S_{a a_1}^{\alpha_2 a'_1}(\theta - \theta_1) S_{\alpha_2 a_2}^{\alpha_3 a'_2}(\theta - \theta_2) \dots S_{\alpha_N a_N}^{a a'_N}(\theta - \theta_N), \end{aligned} \tag{3.9}$$

acting on $V^{\otimes N}$.

* The Boltzmann weights in Eq. (3.2) become the S -matrix elements if we adopt the convention that time flows from the bottom-left to the upper-right (\nearrow).

We use an inversion relation of the transfer matrix to derive the eigenvalues. As explicitly derived in Appendix A, the inversion relation for $N = 1$ SUSY models looks like:

$$\begin{aligned}
 & T(u | \theta_1, \dots, \theta_N) T(u + i\pi | \theta_1, \dots, \theta_N) \\
 &= (-1)^N \left[\prod_{i=1}^N M_+(u - \theta_i) + \prod_{i=1}^N M_-(u - \theta_i) \right. \\
 &\quad \left. + F \left(\prod_{i=1}^N F_+(u - \theta_i) + \prod_{i=1}^N F_-(u - \theta_i) \right) \right], \tag{3.10}
 \end{aligned}$$

where the fermion index operator F is either $+1$ for the bosonic state or -1 for the fermionic one.

The functions appearing in Eq. (3.10) are expressed in terms of the Boltzmann weights as follows:

$$\begin{aligned}
 M_+ &= a_+ a_- - d^2, & M_- &= a_+ a_- - c^2, \\
 F_+ &= \sinh^2 \phi a_+ b_+ + \cosh^2 \phi a_- b_- - 2 \sinh \phi \cosh \phi cd, \\
 F_- &= -\cosh^2 \phi a_+ b_+ - \sinh^2 \phi a_- b_- + 2 \sinh \phi \cosh \phi cd, \tag{3.11}
 \end{aligned}$$

and

$$\tanh(2\phi) = \frac{2cd}{a_+ b_+ + a_- b_-} = \sin \alpha\pi. \tag{3.12}$$

Using Eq. (3.6) one can find

$$\begin{aligned}
 M_+ &= -\frac{\sinh(\frac{1}{2}\theta + i\alpha\pi)}{\sinh \frac{1}{2}\theta} \frac{\sinh(\frac{1}{2}\theta - i\alpha\pi)}{\sinh \frac{1}{2}\theta}, \\
 M_- &= -\frac{\cosh(\frac{1}{2}\theta + i\alpha\pi)}{\cosh \frac{1}{2}\theta} \frac{\cosh(\frac{1}{2}\theta - i\alpha\pi)}{\cosh \frac{1}{2}\theta}, \\
 F_+ &= -\frac{\cosh(\frac{1}{2}\theta + i\alpha\pi)}{\cosh \frac{1}{2}\theta} \frac{\sinh(\frac{1}{2}\theta - i\alpha\pi)}{\sinh \frac{1}{2}\theta}, \\
 F_- &= -\frac{\sinh(\frac{1}{2}\theta + i\alpha\pi)}{\sinh \frac{1}{2}\theta} \frac{\cosh(\frac{1}{2}\theta - i\alpha\pi)}{\cosh \frac{1}{2}\theta}. \tag{3.13}
 \end{aligned}$$

From these expressions one can notice that under the change $u \rightarrow u + i\pi$

$$M_{\pm} \rightarrow M_{\mp} \quad \text{and} \quad F_{\pm} \rightarrow F_{\mp},$$

therefore, $T(u)T(u + \pi i) = T(u + \pi i)T(u + 2\pi i)$. This means

$$T(u | \theta_1, \dots, \theta_N) = T(u + 2\pi i | \theta_1, \dots, \theta_N). \tag{3.14}$$

These matrix relations can be easily transformed to equations of the eigenvalues of the transfer matrices; $\Lambda(u|\theta_1, \dots, \theta_N)$ is a $2\pi i$ symmetric function,

$$\Lambda(u|\theta_1, \dots, \theta_N) = \Lambda(u + 2\pi i|\theta_1, \dots, \theta_N), \tag{3.15}$$

and the inversion relation is nicely factorized,

$$\begin{aligned} &\Lambda(u|\theta_1, \dots, \theta_N)\Lambda(u + \pi i|\theta_1, \dots, \theta_N) \\ &= \left[\prod_{i=1}^N \frac{\cosh(\frac{1}{2}(u - \theta_i) + i|\alpha|\pi)}{\cosh(\frac{1}{2}(u - \theta_i))} + F \prod_{i=1}^N \frac{\sinh(\frac{1}{2}(u - \theta_i) + i|\alpha|\pi)}{\sinh(\frac{1}{2}(u - \theta_i))} \right] \\ &\quad \times \left[\prod_{i=1}^N \frac{\cosh(\frac{1}{2}(u - \theta_i) - i|\alpha|\pi)}{\cosh(\frac{1}{2}(u - \theta_i))} + F \sum_{i=1}^N \frac{\sinh(\frac{1}{2}(u - \theta_i) - i|\alpha|\pi)}{\sinh(\frac{1}{2}(u - \theta_i))} \right]. \end{aligned} \tag{3.16}$$

Since $\Lambda(u)$ is a $2\pi i$ -periodic function with poles at $u = \theta_k$ and $u = \theta_k + i\pi$, it can be completely fixed by the location of zeroes on the strip in the complex plain of $-\pi < \text{Im}[u] \leq \pi$ and $-\infty < \text{Re}[u] < \infty$. Also from the fact that $\lambda(u)$ becomes a constant as $u \rightarrow \infty$, we can find that

$$\Lambda(u|\theta_1, \dots, \theta_N) = \text{const.} \times \prod_{k=1}^N \left[\frac{\sinh(\frac{1}{2}(u - z_k^+))}{\sinh(\frac{1}{2}(u - \theta_k))} \frac{\sinh(\frac{1}{2}(u - z_k^-))}{\cosh(\frac{1}{2}(u - \theta_k))} \right], \tag{3.17}$$

where the $2N$ zeroes $\{z_k^+\}$ and $\{z_k^-\}$ located on the strip will be determined as functions of the θ_i .

We defined the zeroes in the way that z_k^+ and z_k^- come from the first and second factors of the r.h.s. of Eq. (3.16), respectively. Therefore, they satisfy

$$\begin{aligned} \prod_{i=1}^N \frac{\tanh(\frac{1}{2}(z_k^+ - \theta_i) + i|\alpha|\pi)}{\tanh(\frac{1}{2}(z_k^+ - \theta_i))} &= -F, \\ \prod_{i=1}^N \frac{\tanh(\frac{1}{2}(z_k^- - \theta_i) - i|\alpha|\pi)}{\tanh(\frac{1}{2}(z_k^- - \theta_i))} &= -F. \end{aligned} \tag{3.18}$$

The solutions of these equations can be written in terms of real variables x_k in the following way:

$$\begin{aligned} z_k^+ &= x_k - i|\alpha|\pi, & x_k - i|\alpha|\pi + i\pi, \\ z_k^- &= x_k + i|\alpha|\pi, & x_k + i|\alpha|\pi - i\pi, \end{aligned} \tag{3.19}$$

where a real number x_k satisfies

$$\prod_{i=1}^N \frac{\tanh(\frac{1}{2}(x_k - \theta_i) - \frac{1}{2}i|\alpha|\pi)}{\tanh(\frac{1}{2}(x_k - \theta_i) + \frac{1}{2}i|\alpha|\pi)} = -F. \tag{3.20}$$

Out of four possible choices of z_k^\pm for $k = 1, \dots, N$ from Eq. (3.19), only two choices are allowed. This can be understood easily if one considers the limit of

$|\alpha| \rightarrow 0$. The Boltzmann weights are either $+1$ or -1 from Eq. (3.6). This means the transfer matrix is just a constant matrix without any dependence on the rapidities. Now from Eqs. (3.17) and (3.20), the only possibility for the eigenvalues of the transfer matrix to be independent of θ_i is when $\{x_k\} = \{\theta_i\}$ and when $z_k^+ - z_k^- = \pm i\pi$ for all k . For example, if one chooses a $(z_k^+, z_k^-) = (\theta_k - i|\alpha|\pi, \theta_k + i|\alpha|\pi)$ pair for some k as $|\alpha| \rightarrow 0$, the eigenvalue will get a term like $\tanh(\frac{1}{2}(u - \theta_k))$. Obviously, this eigenvalue should be excluded for the constant transfer matrix. This leaves only two choices for the zeroes:

$$\begin{aligned} (z_k^+, z_k^-) &= (x_k - i|\alpha|\pi, x_k + i|\alpha|\pi - i\pi) \\ \text{or } (z_k^+, z_k^-) &= (x_k - i|\alpha|\pi + i\pi, x_k + i|\alpha|\pi). \end{aligned} \tag{3.21}$$

From the product form of Eq. (3.16), one notices that if we choose one pair of zeroes in Eq. (3.21) the other pair becomes zeroes of $\Lambda(u + i\pi)$. Since one can choose the zeroes between the two possibilities for each k ($k = 1, \dots, N$), we can construct 2^N different eigenvalues in this way. Also, one can prove that Eq. (3.17) satisfies Eq. (3.16) because if we divide the r.h.s. of Eq. (3.16) with $\Lambda(u)\Lambda(u + i\pi)$ using Eq. (3.17) the final expression has no poles and zeroes while it is bounded. This means the ratio should be a constant.

Using all these results, the eigenvalues are compactly expressed by

$$\lambda(u)_{\epsilon_1, \dots, \epsilon_N} = \text{Const.} \times \left[\frac{\prod_{k=1}^N \lambda_{\epsilon_k}(u - x_k)}{\prod_{i=1}^N \sinh(u - \theta_i)} \right], \quad \epsilon_k = \pm 1, \tag{3.22}$$

with

$$\lambda_{\epsilon}(\theta) = \sinh\left(\frac{\theta}{2} + \epsilon \frac{i|\alpha|\pi}{2}\right) \cosh\left(\frac{\theta}{2} - \epsilon \frac{i|\alpha|\pi}{2}\right). \tag{3.23}$$

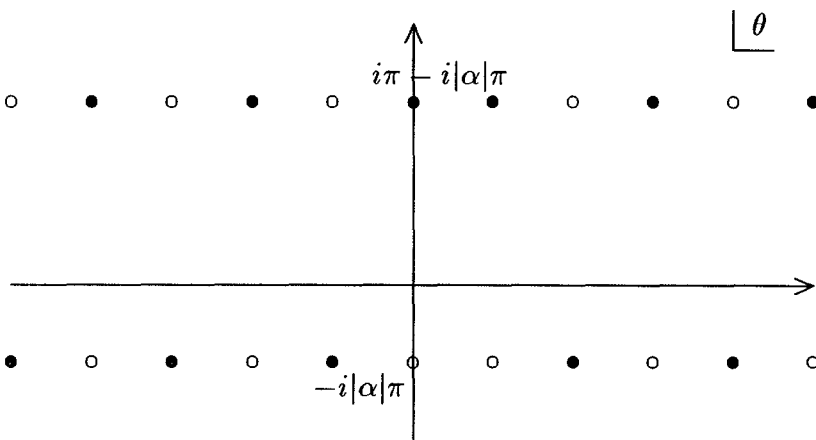


Fig. 1. The zeroes z_k^+ on the complex θ plane.

$\epsilon = +1$ corresponds to the first choice in Eq. (3.21) and $\epsilon = -1$ to the second (see Fig. 1). The real zeroes x_k are determined by Eq. (3.20).

3.3. Thermodynamic Bethe ansatz

From Eq. (3.8) and Eq. (3.22), the PBC equation becomes

$$\frac{1}{2} e^{im \sinh \theta} \prod_{i=1}^N \left[\frac{Y(\theta - \theta_i)}{\sinh(\theta - \theta_i)} \right] \prod_{k=1}^N \lambda_{\epsilon_k}(\theta | x_1, \dots, x_N) = 1, \tag{3.24}$$

and the constraint equation (3.20) in the limit $N, L \rightarrow \infty$ are expressed by the densities ρ for the allowed states, ρ^1 for the occupied states, P_+ for $\epsilon = +1$ zero-state, and P_- for $\epsilon = -1$. In terms of these densities, one gets

$$\begin{aligned} 2\pi\rho(\theta) &= m \cosh \theta + \int d\theta' [\rho^1(\theta')\Phi_Y(\theta - \theta') + P_+(\theta')\Phi_+(\theta - \theta') \\ &\quad + P_-(\theta')\Phi_-(\theta - \theta')], \\ 2\pi P(\theta) &= \int d\theta' \rho^1(\theta')\Phi_T(\theta - \theta'), \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} \Phi_Y(\theta) &= \frac{\partial}{\partial\theta} \text{Im} \ln \left[\frac{Y(\theta)}{\sinh \theta} \right], \quad \Phi_{\pm}(\theta) = \frac{\partial}{\partial\theta} \text{Im} \ln \lambda_{\pm}(\theta), \\ \Phi_T(\theta) &= \frac{1}{i} \frac{\partial}{\partial\theta} \ln \left[\frac{\tanh(\frac{1}{2}\theta - \frac{1}{2}i|\alpha|\pi)}{\tanh(\frac{1}{2}\theta + \frac{1}{2}i|\alpha|\pi)} \right]. \end{aligned} \tag{3.26}$$

Using Eq. (3.23) and $\lambda_- = (\lambda_+)^*$, one can easily show that the kernels are related by

$$\Phi_T(\theta) = 2\Phi_+(\theta) = -2\Phi_-(\theta) = \frac{1}{i} \frac{\partial}{\partial\theta} \ln \left[\frac{\sinh \theta - i \sin |\alpha|\pi}{\sinh \theta + i \sin |\alpha|\pi} \right], \tag{3.27}$$

which is nothing but the kernel of the sinh-Gordon model. We will denote this kernel by Φ . Also, we can eliminate P_- from the first equation of (3.25) using the second one and $P = P_+ + P_-$ to rewrite it as

$$\begin{aligned} 2\pi\rho(\theta) &= m \cosh \theta \\ &\quad + \int d\theta' [\rho^1(\theta') [\Phi_Y - \frac{1}{2}\Phi * \Phi](\theta - \theta') + P_+(\theta')\Phi(\theta - \theta')], \end{aligned} \tag{3.28}$$

with convolution defined by $[f * g](\theta) = \int_{-\infty}^{\infty} (d\theta'/2\pi) f(\theta - \theta')g(\theta')$.

The TBA equations, therefore, can be expressed by

$$\begin{aligned}
 mR \cosh \theta &= \epsilon(\theta) + \left([\Phi_Y - \frac{1}{2}\Phi * \Phi] * \ln[1 + e^{-\epsilon}] \right)(\theta) \\
 &\quad + (\Phi * \ln[1 + e^{-\mathcal{E}}])(\theta), \\
 0 &= \mathcal{E}(\theta) + (\Phi * \ln[1 + e^{-\epsilon}])(\theta),
 \end{aligned}
 \tag{3.29}$$

in terms of the pseudo-energies ϵ and \mathcal{E} defined by

$$\frac{\rho^1(\theta)}{\rho(\theta)} = \frac{e^{-\epsilon(\theta)}}{1 + e^{-\epsilon(\theta)}}, \quad \frac{P_+(\theta)}{P(\theta)} = \frac{e^{-\mathcal{E}(\theta)}}{1 + e^{-\mathcal{E}(\theta)}}.
 \tag{3.30}$$

3.4. Central charges of the SShG model

The TBA equations in the UV limit ($R \rightarrow 0$) can be easily solved. In particular the ground-state energy can be expressed by

$$E(mR) \sim -\frac{1}{\pi R} \sum_{a=0,1} \left[\mathcal{L}\left(\frac{x_a}{1 \mp x_a}\right) - \mathcal{L}\left(\frac{y_a}{1 \mp y_a}\right) \right],
 \tag{3.31}$$

where $x_0 = \exp[-\epsilon(0)]$, $x_1 = \exp[-\mathcal{E}(0)]$ and $y_0 = \exp[-\epsilon(\infty)] = 0$, $y_1 = \exp[-\mathcal{E}(\infty)] = 1$. The ground-state energy is related to the UV central charge as $R \rightarrow 0$ through the finite-size correction by $E(R) \approx -(\pi/6R)(C - 12(\Delta_0 + \bar{\Delta}_0))$ where Δ_0 and $\bar{\Delta}_0$ are conformal dimensions of vacuum.

Furthermore, the pseudo-energies become independent of the rapidity around $\theta = 0$ upto $\theta \sim -\ln(mR)$ and the TBA equations are reduced to mere algebraic equations,

$$x_0 = (1 + x_0)^a (1 + x_1)^b, \quad x_1 = (1 + x_0)^b,
 \tag{3.32}$$

with

$$a = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} (\Phi_Y - \frac{1}{2}\Phi * \Phi)(\theta), \quad b = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \Phi(\theta).
 \tag{3.33}$$

For the SShG model, $a = 0$ and $b = 1$ from Eqs. (3.23) and (3.26) and $\int_{-\infty}^{\infty} (d\theta/2\pi)\Phi_Y(\theta) = \frac{1}{2}$, $\int_{-\infty}^{\infty} (d\theta/2\pi)\Phi(\theta) = 1$, $\int_{-\infty}^{\infty} (d\theta/2\pi)[\Phi * \Phi](\theta) = [\int_{-\infty}^{\infty} (d\theta/2\pi)\Phi(\theta)]^2 = 1$. Using these values, the solution of Eq. (3.32) can be found easily as $x_0 = x_1 = \infty$. This gives the UV central charge ($\Delta_0 = \bar{\Delta}_0 = 0$)

$$C = \frac{6}{\pi^2} \left[\mathcal{L}(1) + \mathcal{L}(1) - \mathcal{L}\left(\frac{1}{2}\right) \right] = \frac{3}{2}.$$

This is correct UV central charge of the SShG model with a boson and a fermion.

3.5. Central charges of the SUSY Yang–Lee model

As explained in the previous section, one can truncate all solitons from the SSG multi-soliton Hilbert space to have only breathers. In particular, for the coupling constant $\alpha = \frac{1}{3}$, only the lightest breather and its superpartner can exist in the

spectrum with the S -matrix given in Eq. (2.19) [27]. This is the SYL model perturbed by the least relevant operator.

The fundamental difference from the SShG S -matrix is that because of $\alpha > 0$ $S_0(\theta)$ is no more CDD factor. It has a pole which is identified with the particle itself. If we denote the particles as B and F , the bootstrap relations are

$$BB(F F) \rightarrow B \rightarrow BB(F F), \quad BF(F B) \rightarrow F \rightarrow BF(F B).$$

Except this difference, all the TBA analysis of the SShG model is equally applicable to the SYL model.

The SYL conformal theory can be constructed as a coset CFT given by

$$\frac{SU(2)_K \otimes SU(2)_L}{SU(2)_{K+L}}, \quad K = 2 \quad \text{and} \quad L + 2 = \frac{2}{3}.$$

The central charge of the model is $C = -\frac{21}{4}$. Due to the non-unitarity of the model, the lowest conformal dimension is not zero. To determine the lowest conformal dimension of the model, we refer to the general formula of the general coset theories. The primary fields of the coset theory have the conformal dimensions given by the following formula [38]:

$$\Delta_{r,s}^l = \frac{l(l+2)}{4(K+2)} - \frac{l^2}{4K} + \frac{(rp' - sp)^2 - (p' - p)^2}{4Kpp'},$$

$$L + 2 = \frac{p}{q}, \quad p' = p + Kq, \tag{3.34}$$

with the restrictions

$$0 \leq l \leq K, \quad 1 \leq r \leq p - 1, \quad 1 \leq s \leq p' - 1, \quad l = |r - s \bmod 2K|.$$

For the SYL model with the values of $K = 2$, $p = 2$, $p' = 8$, one finds that only $r = 1$, $l = 0, 1, 2$, and $s = 1, 2, 3, 5, 6, 7$ are allowed. The minimal conformal dimension arises when $(8 - 2s)^2$ is minimized, i.e. with $s = 3$. Therefore, the minimal conformal dimension of the SYL model is $\Delta_{\min} = \Delta_{1,3}^2 = -\frac{1}{4}$.

Now we compute the Casimir energy of the model using TBA. Notice that the only change from the SShG TBA is that the kernel Φ_Y in eq. (3.29) gets an extra factor $-(\partial/\partial\theta) \ln S_0(\theta)$ due to S_0 in Eq. (2.19). With $\alpha > 0$ this introduces extra -1 in the exponent a in Eq. (3.32) to make $a = -1$. With this change the algebraic equations now become

$$x_0 = \frac{1 + x_1}{1 + x_0}, \quad x_1 = 1 + x_0, \tag{3.35}$$

and the solutions are $x_0 = \sqrt{2}$ and $x_1 = 1 + \sqrt{2}$. Eq. (3.31) gives $C-24\Delta_{\min}$

$$\frac{6}{\pi^2} \left[\mathcal{L} \left(\frac{\sqrt{2}}{1 + \sqrt{2}} \right) + \mathcal{L} \left(\frac{1 + \sqrt{2}}{2 + \sqrt{2}} \right) - \mathcal{L} \left(\frac{1}{2} \right) \right] = \frac{3}{4},$$

as expected.

4. Form factors of the supersymmetric theories

In this section, we derive FFs of the SShG model from the S -matrix. We compute one-point FFs first to fix overall normalization and two-point FFs using the Watson equations and SUSY relations of the FFs. Using these FFs we derive the UV central charge of the SShG model from the spectral representation of the C -theorem. This provides a consistency check for the FFs and shows the fast convergence of the FFs expansions of correlation functions.

4.1. Two-point form factors

FFs are matrix elements of a hermitian operator \mathcal{O} between vacuum and in-coming states and can be expressed by

$$F_{a_1, \dots, a_n}^{\mathcal{O}}(\theta_1, \dots, \theta_n) = \langle 0 | \mathcal{O}(0) | a_1(\theta_1), \dots, a_n(\theta_n) \rangle_{\text{in}}, \tag{4.1}$$

where $|a(\theta)\rangle$ denotes an on-shell particle state of type a with a rapidity θ ($\theta_1 > \theta_2 > \dots > \theta_n$). These FFs satisfy some basic axioms. While the general FFs depend on quite complicated recursion relations and are difficult to solve, the situation becomes quite simple for the two-particle states [6]. The two-point FFs can be written in terms of two factors,

$$F_{a_1 a_2}(\theta_1, \theta_2) = K_{a_1 a_2}(\theta_1, \theta_2) F_{a_1 a_2}^{\text{min}}(\theta_1 - \theta_2), \tag{4.2}$$

where F^{min} satisfies the Watson equation [5] without any pole,

$$F_{a_1 a_2}^{\text{min}}(\theta) = F_{a_1 a_2}^{\text{min}}(-\theta) S_{a_1 a_2}^{a_1' a_2'}(\theta), \quad F_{a_1 a_2}^{\text{min}}(i\pi - \theta) = F_{a_1 a_2}^{\text{min}}(i\pi + \theta), \tag{4.3}$$

and the prefactor $K_{a_1 a_2}(\theta_1, \theta_2)$ has all the required poles and operator dependence. Note we omitted a phase factor from the fermion exchange operator treating all particles as bosonic.

F^{min} can be determined from the following steps. In the basis which diagonalizes the S -matrix, Eq. (4.3) becomes a simple functional relation. Then, using an integral representation for the i th eigenvalue of the S -matrix, one finds

$$\begin{aligned} S_i(\theta) &= \exp \left[\int_0^\infty \frac{dt}{t} f_i(t) \sinh \frac{\theta t}{\pi i} \right] \\ &\rightarrow F_i^{\text{min}} = \exp \left[\int_0^\infty \frac{dt}{t} \frac{f_i(t)}{\sinh \theta} \sin^2 \frac{\hat{\theta} t}{2\pi} \right], \end{aligned} \tag{4.4}$$

where $\hat{\theta} = i\pi - \theta$. Rotating back to the original on-shell two-particle states, one finds the F^{min} . The function $K(\theta_1, \theta_2)$ should be determined by the other axioms. It is a symmetric function of the rapidities if $a_1 = a_2$ and has an $i\pi$ -pole if $\bar{a}_1 = a_2$. The asymptotic behaviour at large rapidities is related to the spin of the operator and the overall normalization of the FFs is fixed by the one-point function.

4.2. One-point form factor of the SShG model

We work out one-point function of the SShG model to fix the normalization of the general FFs. From the Fourier transformation of the elementary fields:

$$\begin{aligned} \phi(x) &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2}k_0} [b_k e^{ik \cdot x} + b_k^\dagger e^{-ik \cdot x}], \\ \psi(x) &= \int \frac{dk}{2\pi} \frac{1}{k_0} [f_k u(k) e^{ik \cdot x} + f_k^\dagger v(k) e^{-ik \cdot x}] \end{aligned} \tag{4.5}$$

with $u^* = v$ and with the commutation relations

$$\{f_k, f_{k'}^\dagger\} = 2\pi k_0 \delta(k - k'), \quad [b_k, b_{k'}^\dagger] = 2\pi k_0 \delta(k - k'). \tag{4.6}$$

On-shell SUSY is determined from the SUSY transformation of the elementary fields, Eq. (2.7) and $|b(\theta)\rangle = b^\dagger(\theta)|0\rangle$ and $|f(\theta)\rangle = f^\dagger(\theta)|0\rangle$:

$$\begin{aligned} Q_1 |f(\theta)\rangle &= -i \frac{P_+}{\sqrt{2}v_1(\theta)} |b(\theta)\rangle, & Q_1 |b(\theta)\rangle &= i\sqrt{2}v_1(\theta) |f(\theta)\rangle, \\ Q_2 |f(\theta)\rangle &= i \frac{P_-}{\sqrt{2}v_2(\theta)} |b(\theta)\rangle, & Q_2 |b(\theta)\rangle &= i\sqrt{2}v_2(\theta) |f(\theta)\rangle. \end{aligned} \tag{4.7}$$

From $0 = \langle 0 | Q_\alpha [\phi(0) | f(\theta)\rangle]$, one gets

$$Q_\alpha |f(\theta)\rangle = -i\sqrt{2}u_\alpha(\theta) |b(\theta)\rangle, \tag{4.8}$$

and comparing this with Eq. (4.7), one can find the spinors

$$v_1(\theta) = -i\sqrt{\frac{m}{2}} e^{\theta/2}, \quad v_2(\theta) = \sqrt{\frac{m}{2}} e^{-\theta/2}.$$

This gives the SUSY transformation of on-shell states given above in Eq. (2.16). Combining this and Eq. (4.5), we fix the one-point function as follows:

$$\langle 0 | \phi(0) | b(\theta)\rangle = \frac{1}{\sqrt{2}}, \quad \langle 0 | \psi_\alpha(0) | f(\theta)\rangle = \sqrt{\frac{m}{2}} \begin{pmatrix} ie^{\theta/2} \\ e^{-\theta/2} \end{pmatrix}. \tag{4.9}$$

4.3. Form factors of the SShG model

We compute two-point FFs of the trace of energy–momentum tensors and their SUSY counterparts, Θ and Θ_F ($\bar{\Theta}_F$), given in Eqs. (2.13) and (2.14). These operators are of particular interest for their role in the C-theorem. First, we derive SUSY relations between the FFs using Eq. (2.15).

From $0 = \langle 0 | Q_\alpha [\mathcal{O} | a_1(\theta_1)a_2(\theta_2)\rangle]$, one finds a relation

$$\langle 0 | Q_\alpha [\mathcal{O}] | a_1(\theta_1)a_2(\theta_2)\rangle = -(-1)^{F(\mathcal{O})} \langle 0 | \mathcal{O} | Q_\alpha [a_1(\theta_1)a_2(\theta_2)]\rangle, \tag{4.10}$$

with F 1 for fermionic and 0 for bosonic. This gives the following relations between the FFs:

$$\begin{aligned}
 F_{bb}^\Theta &= i \frac{\sqrt{m}}{2} \left[\sqrt{x_1} F_{fb}^{\Theta_F} + \sqrt{x_2} F_{bf}^{\Theta_F} \right], & F_{ff}^\Theta &= i \frac{\sqrt{m}}{2} \left[\sqrt{x_1} F_{bf}^{\Theta_F} - \sqrt{x_2} F_{fb}^{\Theta_F} \right], \\
 F_{bb}^{\Theta_F} &= \frac{\sqrt{m}}{2} \left[\frac{1}{\sqrt{x_1}} F_{fb}^{\Theta_F} + \frac{1}{\sqrt{x_2}} F_{bf}^{\Theta_F} \right], & F_{ff}^{\Theta_F} &= \frac{\sqrt{m}}{2} \left[\frac{-1}{\sqrt{x_1}} F_{bf}^{\Theta_F} + \frac{1}{\sqrt{x_2}} F_{fb}^{\Theta_F} \right],
 \end{aligned}
 \tag{4.11}$$

where each FF is a function of θ_1 and θ_2 or of $x_i = e^{\theta_i}$.

A special case of $\alpha \rightarrow 0$

It is useful to consider the case of $\alpha \rightarrow 0$ where the S -matrix is of the diagonalized form $(1, -1, 1, 1)$ from Eq. (2.19). Let us compute F_{bb}^Θ and F_{ff}^Θ . In terms of the solution of the Watson equation ^{*},

$$F_{bb}^{\min} = 1, \quad F_{ff}^{\min} = \sinh \frac{\theta}{2},
 \tag{4.12}$$

the FFs can be written as

$$F_{bb}^\Theta(x_1, x_2) = K_{bb}(x_1, x_2) \quad \text{and} \quad F_{ff}^\Theta(x_1, x_2) = K_{ff}(x_1, x_2) \sinh \frac{\theta}{2}.
 \tag{4.13}$$

Since K should have the $i\pi$ pole (or at $x_1 = -x_2$) and $K(\theta_1 + \Lambda, \theta_2 + \Lambda) = K(\theta_1, \theta_2)$ because the spin of Θ is zero, we can find

$$F_{bb}^\Theta(x_1, x_2) = 2\pi m^2 \quad \text{and} \quad F_{ff}^\Theta(x_1, x_2) = 2\pi m^2 \sinh \frac{\theta}{2}.
 \tag{4.14}$$

Here we fixed the normalization factor as πm^2 by comparing with the perturbative computation using Eq. (2.4).

After finding these, one can derive the other FFs simply using Eq. (4.11) as follows:

$$F_{bf}^{\Theta_F}(x_1, x_2) = 2\pi m^{3/2} \sqrt{x_2}, \quad F_{bf}^{\bar{\Theta}_F}(x_1, x_2) = 2\pi m^{3/2} \frac{-i}{\sqrt{x_2}}.
 \tag{4.15}$$

One can check that the spins of Θ_F and $\bar{\Theta}_F$ can be found correctly as $\pm \frac{1}{2}$ under the rapidity translation. These FFs are consistent with perturbative computation.

^{*} This solution is not unique in the sense that one can multiply any even function of θ satisfying $f(i\pi + \theta) = f(i\pi - \theta)$. If we include these functions in the prefactor K , one can define F^{\min} uniquely.

For general α

For general cases, we should diagonalize the S -matrix first. It turns out that the eigenvalues of the S -matrix of the $F = 0$ sector (bb and ff) are complicated and it is hard to find the integral representations. Instead, we consider $F = -1$ (bf and fb) sector first. The S -matrix is easily diagonalized by the eigenvectors

$$|+\rangle = \frac{1}{\sqrt{2}}(|b_1 f_2\rangle + |f_1 b_2\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|b_1 f_2\rangle - |f_1 b_2\rangle), \quad (4.16)$$

with eigenvalues

$$S_+(\theta) = \exp\left[\int_0^\infty \frac{dt}{t} f_+(t) \sinh \frac{\theta t}{\pi i}\right],$$

$$S_-(\theta) = -\exp\left[\int_0^\infty \frac{dt}{t} f_-(t) \sinh \frac{\theta t}{\pi i}\right], \quad (4.17)$$

with

$$f_\pm(t) = \frac{(1 - \cosh t)(1 + \cosh((1 - 2|\alpha|)t))}{\sinh^2 t} \pm \frac{\cosh((1 - 2|\alpha|)t)}{\cosh t}. \quad (4.18)$$

From these integrals, F^{\min} in the basis of $|+\rangle$ and $|-\rangle$ can be obtained as

$$F_+^{\min} = \exp\left[\int_0^\infty \frac{dt}{t} \frac{f_+(t)}{\sinh t} \sin^2 \frac{\hat{\theta} t}{2\pi}\right],$$

$$F_-^{\min} = \cosh \frac{\hat{\theta}}{2} \exp\left[\int_0^\infty \frac{dt}{t} \frac{f_-(t)}{\sinh t} \sin^2 \frac{\hat{\theta} t}{2\pi}\right], \quad (4.19)$$

where we chose a normalization such that

$$F_\pm \rightarrow 1 \quad \text{as} \quad \alpha \rightarrow 0. \quad (4.20)$$

For numerical computations, we list expressions of F_\pm^{\min} which converge fast in Appendix B.

Now we consider FFs of Θ_F in the following form:

$$F_+^{\Theta_F}(x_1, x_2) = K_+(x_1, x_2) F_+^{\min}, \quad F_-^{\Theta_F}(x_1, x_2) = K_-(x_1, x_2) F_-^{\min}, \quad (4.21)$$

and similarly for $\bar{\Theta}_F$ in terms of \bar{K}_\pm . These K_\pm and \bar{K}_\pm can be determined from the spins of the operators and symmetric properties of the states under the exchange, $|+\rangle \rightarrow |+\rangle$ and $|-\rangle \rightarrow -|-\rangle$ under $x_1 \leftrightarrow x_2$, as follows:

$$K_\pm = A(\sqrt{x_1} \pm \sqrt{x_2}), \quad \bar{K}_\pm = B\left(\frac{1}{\sqrt{x_1}} \pm \frac{1}{\sqrt{x_2}}\right), \quad (4.22)$$

where the constants A, B can be determined by taking the $\alpha \rightarrow 0$ limit and comparing with Eq. (4.15).

Now rotating back to the on-shell states one can find

$$\begin{aligned}
 F_{bf}^{\Theta_F}(x_1, x_2) &= 2\pi m^{3/2} \left[\sqrt{x_1} \frac{(F_+^{\min} - F_-^{\min})}{2} + \sqrt{x_2} \frac{(F_+^{\min} + F_-^{\min})}{2} \right], \\
 F_{bf}^{\bar{\Theta}_F}(x_1, x_2) &= -2\pi im^{3/2} \left[\frac{1}{\sqrt{x_1}} \frac{(F_+^{\min} - F_-^{\min})}{2} + \frac{1}{\sqrt{x_2}} \frac{(F_+^{\min} + F_-^{\min})}{2} \right].
 \end{aligned}
 \tag{4.23}$$

Also from Eq. (4.11) one can obtain other FFs,

$$\begin{aligned}
 F_{bb}^{\Theta}(x_1, x_2) &= 2\pi m^2 \left[\frac{(F_+^{\min} + F_-^{\min})}{2} + \frac{(F_+^{\min} - F_-^{\min})}{2} \cosh \frac{\theta}{2} \right], \\
 F_{ff}^{\Theta}(x_1, x_2) &= 2\pi m^2 \frac{(F_+^{\min} + F_-^{\min})}{2} \sinh \frac{\theta}{2}.
 \end{aligned}
 \tag{4.24}$$

We checked these FFs using the first-order perturbative computations. FFs for other components of the energy–momentum tensor can be written down by just multiplying P_+/P_- to the above FFs.

4.4. Spectral sum rule

The C -theorem, first introduced by A.B. Zamolodchikov, plays an important role in the study of off-critical models [39]. The C -function, describing a degree of freedom of the 2D models, connects smoothly two renormalization group (RG) fixed points as the length scale of the theory increases from UV limit to IR. For some specific models like the perturbed minimal CFTs by the least relevant operator with positive coefficient, the renormalization group (RG) flow connects two RG fixed points corresponding to two adjacent minimal CFTs [40]. This RG flow will end up at the massive point with $C = 0$.

This theorem can be neatly expressed in the following integral of the two-point correlation function of the trace of the energy–momentum tensor following Cardy [41]:

$$\begin{aligned}
 \Delta C &= \frac{3}{4\pi} \int_{|x|>\epsilon} d^2x x^2 \langle \Theta(x) \Theta(0) \rangle = \int_0^\infty d\mu C_1(\mu, \Lambda), \\
 C_1(\mu, \Lambda) &= \frac{6}{\pi^2} \frac{1}{\mu^3} \text{Im} \left[\int d^2x e^{-ip \cdot x} \langle \Theta(x) \Theta(0) \rangle \right]_{p^2 = -\mu^2}.
 \end{aligned}
 \tag{4.25}$$

Expanding the correlation function in terms of intermediate on-shell states, the spectral density function C_1 can be expressed in terms of the FFs by

$$C_1(\mu, \Lambda) = \frac{12}{\mu^3} \sum_{\alpha} |\langle 0 | \Theta(0) | \alpha \rangle|^2 \delta^2(q - p_{\alpha}),
 \tag{4.26}$$

where p_α is the energy–momentum vector of the multi-particle state α and the vector q is defined as $q = (\mu, 0)$.

For the massive theory, the sum rule of ΔC effectively gives the UV central charge since C_{IR} vanishes. Although one needs the infinite number of the FFs to compute it rigorously, there are many evidences that the sum in Eq. (4.26) converges very fast for the massive theories [7–10]. With this observation, one can compute the UV central charge using the two-point FFs of Θ quite accurately. In next stage, we will compute this numerically using the FFs of the SShG model derived in the previous section.

4.5. Sum rule for the SShG model

The two-point contribution to the sum rule becomes

$$\begin{aligned}
 C^{(2)} &= \frac{12}{\mu^3} \int \frac{d\theta_1 d\theta_2}{2(2\pi)^2} \sum_{a_1, a_2} |F_{a_1 a_2}^\Theta(\theta_1, \theta_2)|^2 \\
 &\quad \times \delta(m \cosh \theta_1 + m \cosh \theta_2 - \mu) \delta(m \sinh \theta_1 + m \sinh \theta_2) \\
 &= \frac{3}{8\pi^2 m^4} \int_0^\infty \frac{d\theta}{\cosh^4 \theta} \left[|F_{bb}^\theta(\theta, -\theta)|^2 + |F_{ff}^\theta(\theta, -\theta)|^2 \right], \tag{4.27}
 \end{aligned}$$

with the FFs given in Eq. (4.24).

For the special case of $\alpha = 0$ where the SShG model becomes free with a boson and fermion, one can insert Eq. (4.14) into Eq. (4.27) and using

$$\int_0^\infty \frac{d\theta}{\cosh^4 \theta} = \frac{2}{3}, \quad \int_0^\infty d\theta \frac{\sinh^2 \theta}{\cosh^4 \theta} = \frac{1}{3},$$

one can easily find $C = \frac{3}{2}$.

For the generic value of α we integrate numerically using the regularized expressions for the F_\pm in Appendix B. Using these we list $\Delta C^{(2)}$ for several values of the coupling constant in Table 1. This shows a good agreement with the UV central charge $C = \frac{3}{2}$. The convergence of the SShG model seems slow compared with the sinh-Gordon result [9]. This suggests in the SShG model one arrives at the

Table 1
The first two-particle form factor in the Sum Rule of $\Delta \hat{C} = \frac{2}{3} \Delta C$

$\beta^2/4\pi$	α	$\Delta \hat{C}^{(2)}$
$\frac{1}{999}$	0.001	0.9993
$\frac{1}{199}$	0.005	0.9953
$\frac{1}{99}$	0.01	0.9902
$\frac{1}{49}$	0.02	0.9800
$\frac{3}{97}$	0.03	0.9697
$\frac{1}{19}$	0.05	0.9495
$\frac{1}{9}$	0.1	0.9093

strong coupling region earlier than the sinh-Gordon model as one can see from the fact that the limit of the SShG coupling constant is $\hat{\beta}^2/8\pi = \frac{1}{2}$ while $\hat{\beta}^2/8\pi = 1$ in the sinh-Gordon model.

5. Conclusion

In this paper we obtained two results on the $N = 1$ SUSY integrable models. The first one is the computation of the UV central charges from TBA method. The non-diagonal TBA of the SShG and SYL models has been rigorously derived from the essential observation that the $N = 1$ SUSY models can be identified with the eight-vertex free fermion models. These TBA equations produced correct UV central charges.

The second result is two-point FFs of the SShG model using the FF axioms. Here the difficulty arising from the non-diagonal scattering theories has been avoided from the SUSY relations of the FFs. The spectral representation of the C -theorem showed that two-point FFs can give good approximations in the infinite sum of the intermediate states even in nondiagonal theories.

Our results suggest some interesting directions to proceed further. Actually, we notice that a wider class of $N = 1$ scattering theories are belonging to the eight-vertex FFMs which will be reported in separate publication [42]. The relationship between these SUSY models and the eight-vertex FFM may have some deep structure because the FFMs seem to have interesting hidden symmetries [43]. In particular, it has been noticed recently that the FFMs have a hidden quantum group symmetry [44]. It would be interesting to see how this quantum group symmetry will be related to the $N = 1$ supersymmetry in the trigonometric limit.

In this paper, we could not say much on the general FFs of the theories. The solution of the FF bootstrap equations are very difficult and are limited to only a few simplest diagonal theories. We can reduce, however, the non-diagonal bootstrap equations to the level of diagonal theories by diagonalizing the inhomogeneous transfer matrix. It will need some more work to solve these reduced bootstrap equations completely.

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Appendix A

Inversion relation for the Free Fermion Model

We follow Felderhof to diagonalize the transfer matrix of the FFM [30]. We want to point out, first, the difference of our derivation from the lattice model

computation. The first difference is that we want to diagonalize the inhomogeneous transfer matrix. This difference often introduces much difficulty for the computation. However, this difficulty can be avoided by the second difference, which is that we are working with the FFM at the critical point. With this advantage, we can derive the inverse matrix of the FFM transfer matrix and, furthermore, express it using the original transfer matrix with a slight change in the rapidity u .

It is convenient to re-express the Boltzmann weights (3.1) in terms of the σ -matrices,

$$R(\theta) = \begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix}, \tag{A.1}$$

$$\begin{aligned} A &= a_+ \sigma^+ \sigma^- + b_+ \sigma^- \sigma^+, & B &= d \sigma^+ + c \sigma^-, \\ C &= c \sigma^+ + d \sigma^-, & D &= b_- \sigma^+ \sigma^- + a_- \sigma^- \sigma^+. \end{aligned} \tag{A.2}$$

Then, the transfer matrix becomes

$$T(u | \theta_1, \dots, \theta_N) = \text{Tr}_2 \left[\prod_{i=1}^N R(u - \theta_i) \right]. \tag{A.3}$$

Now we define a new transfer matrix T_1 corresponding to new Boltzmann weights defined by

$$a_{\pm}^1 = -b_{\pm}, \quad b_{\pm}^1 = a_{\pm}, \quad c^1 = c \quad \text{and} \quad d^1 = -d. \tag{A.4}$$

In the same way as before, one can express T_1 by

$$T_1(u | \theta_1, \dots, \theta_N) = \text{Tr}_2 \left[\prod_{i=1}^N R_1(u - \theta_i) \right], \quad R_1(\theta) = \begin{pmatrix} A_1(\theta) & B_1(\theta) \\ C_1(\theta) & D_1(\theta) \end{pmatrix}, \tag{A.5}$$

where

$$\begin{aligned} A_1 &= -b_+ \sigma^+ \sigma^- + a_+ \sigma^- \sigma^+, & B_1 &= -d \sigma^+ + c \sigma^-, \\ C_1 &= c \sigma^+ - d \sigma^-, & D_1 &= a_- \sigma^+ \sigma^- - b_- \sigma^- \sigma^+. \end{aligned} \tag{A.6}$$

One can check that these new Boltzmann weights again satisfy the free fermion condition (3.3).

The next step is to show that $TT_1 \propto 1$. For this purpose, we multiply the two matrices,

$$T(u)T_1(u) = \text{Tr}_2 \left[\prod_{i=1}^N R_i \right] \text{Tr}_2 \left[\prod_{i=1}^N R_{1,i} \right] = \text{Tr}_{2 \otimes 2} \left[\prod_{i=1}^N R_i \otimes R_{1,i} \right]. \tag{A.7}$$

Defining the 4×4 matrix $R_i \otimes R_{1,i}$ as S_i , one can find a similarity transformation $S'_i = X_i S_i X_i^{-1}$ where S'_i is of triangular form. The X and S' are given by

$$X = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \cosh \phi & 0 & 0 & -\sinh \phi \\ -\sinh \phi & 0 & 0 & \cosh \phi \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$S' = \begin{pmatrix} M_+ & * & * & * \\ 0 & F_- \sigma^z & 0 & * \\ 0 & 0 & F_+ \sigma^z & * \\ 0 & 0 & 0 & M_- \end{pmatrix}, \tag{A.8}$$

where M_{\pm} , F_{\pm} , and ϕ are given in Eqs. (3.11) and (3.12). We did not specify the unnecessary non-vanishing components (*).

The most important observation is that $\tanh \phi$ becomes just a constant for the $N = 1$ supersymmetric theory. This means one can make all the S_i in the trace of triangular form by the same similarity transformation X . Therefore, $TT_1 = \text{Tr}_4 \prod S'_i$ and from Eq. (A.8) one can derive

$$T(u)T_1(u) = \left[\prod_{i=1}^N M_+(u - \theta_i) + \prod_{i=1}^N M_-(u - \theta_i) + F \left(\prod_{i=1}^N F_+(u - \theta_i) + \prod_{i=1}^N F_-(u - \theta_i) \right) \right], \tag{A.9}$$

with $F = \prod \sigma_i^z$ is either 1 (bosonic) or -1 (fermionic).

Now, consider a translation $u \rightarrow u + i\pi$. Under this the Boltzmann weights of the SSHG model change,

$$a_{\pm} \rightarrow -a_{\mp}, \quad b_{\pm} \rightarrow b_{\mp}, \quad c \rightarrow d \quad \text{and} \quad d \rightarrow -c. \tag{A.10}$$

Again this satisfies the free fermion condition. Now, the transfer matrix with translated rapidity can be expressed in terms of $R_2(u - \theta) = R(u + i\pi - \theta)$ by

$$T(u + i\pi) = \text{Tr}_2 \left[\prod_{i=1}^N R_2(u - \theta_i) \right], \quad R_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}, \tag{A.11}$$

where

$$A_2 = -a_+ \sigma^+ \sigma^- + b_+ \sigma^- \sigma^+, \quad B_2 = -c \sigma^+ + d \sigma^-, \tag{A.12}$$

$$C_2 = d \sigma^+ - c \sigma^-, \quad D_2 = b_- \sigma^+ \sigma^- - a_- \sigma^- \sigma^+.$$

From Eq. (A.6), one can notice that

$$A_2 = -D_1, \quad B_2 = -C_1, \quad C_2 = -B_1 \quad \text{and} \quad D_2 = -A_1. \tag{A.13}$$

Considering the R -matrices as 2×2 matrices, the R_1 and R_2 are related by

$$R_2 = -\sigma^x R_1 \sigma^x,$$

where σ^x is the usual Pauli spin matrix. This gives

$$T(u + i\pi | \theta_1, \dots, \theta_N) = (-1)^N T_1(u | \theta_1, \dots, \theta_N), \tag{A.14}$$

and from Eqs. (A.9) and (A.14), the inversion relation Eq. (3.10).

Appendix B

Regularized expression for the form factors

For the numerical computation we can rewrite F_{\pm}^{\min} in Eq. (4.21) as follows:

$$F_{\pm}^{\min}(\theta) = C_{\pm}(\theta) \left[\prod_{k=1}^n G_k(\alpha, \theta) [H_k(\alpha, \theta)]^{\pm 1} \right] \times \exp \left[\int_0^{\infty} \frac{dt}{t} \frac{f_n^{\pm}(\alpha, t)}{\sinh t} \sin^2 \frac{\hat{\theta} t}{2\pi} \right], \tag{B.1}$$

with $C_+ = 1$, $C_-(\theta) = \cosh \frac{1}{2} \hat{\theta}$ and

$$G_k(\alpha, \theta) = \frac{P_k(2|\alpha| + 1, \theta)^2 P_k(0, \theta)^2}{P_k(1, \theta)^2 P_k(2|\alpha|, \theta) P_k(2|\alpha| + 2, \theta)},$$

$$P_k(x, \theta) = \left[\left(1 + \frac{\hat{\theta}/(2\pi)}{(k + (1+x)/2)} \right) \left(1 + \frac{\hat{\theta}/(2\pi)}{(k + (1-x)/2)} \right) \right]^{k(k+1)/4}, \tag{B.2}$$

$$H_k(\alpha, \theta) = \left[1 + \frac{\hat{\theta}/(2\pi)}{(2k + (2|\alpha| + 3)/2)} \right]^{1/2} \left[1 + \frac{\hat{\theta}/(2\pi)}{(2k - (2|\alpha| - 1)/2)} \right]^{1/2},$$

and the exponents are given by

$$f_n^{\pm}(\alpha, t) = \frac{(1 - \cosh t)(1 + \cosh((1 - 2|\alpha|)t)) D_n(t)}{2 \sinh^2 t} \pm \frac{\cosh((1 - 2|\alpha|)t) e^{-4nt}}{\cosh t},$$

with $D_n(t) = [(n + 1)(n + 2) - 2n(n + 2) + n(n + 1) e^{-4t}] e^{-2nt}$.

If one choose $n = 0$, this reduces to Eq. (4.19). For the fast convergence, one can increase n although the final expression is independent of n .

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