Supersymmetric Integrable Field Theories and Eight-Vertex Free Fermion Models

Changrim AHN

Department of Physics, Ewha Women's University, Seoul

We relate a large class of $N=1$ supersymmetric integrable field theories to the eight-vertex free fermion models. These models correspond to the critical point ($h=-1$) of the vertex model while each has different value of asymmetric parameter Γ. With this relationship, we confirm the proposed S-matrices of the supersymmetric field theories using thermodynamic Bethe ansatz.

§ 1. Introduction

For 2D integrable field theories S-matrices are purely elastic, all incoming momenta are conserved and multi-particle scattering amplitudes are factorized into a product of two-particle S-matrices. These S-matrices, in turn, should satisfy Yang-Baxter equations (YBE) which often determine the S-matrices completely along with unitarity and crossing symmetry.[1] In two dimensions the S-matrix provides essential information on both on- and off-shell physics.

There are two currently active theoretical frameworks to study off-shell physics directly using the S-matrices. The first one is thermodynamic Bethe ansatz (TBA) method to compute exact energy states of the scattering theories.[2] In particular the ground state Casimir energy is related to the central charge of the underlying UV conformal field theory (CFT).[3] The second is to construct correlation functions as infinite sums of form factors.[4] For a limited number of theories one can sum the infinite sums exactly to obtain some rigorous results like differential equations for the correlation functions. While for a wide class of scattering theories this program still remains challenging because of the difficulties to find exact form factors and to sum up the infinite series, this approach can be useful to understand non-perturbative off-shell properties of many massive scattering since form factors upto only two-point are quite sufficient for a required accuracy.

While these frameworks are successfully applied to diagonal scattering theories, it is quite a hard task to implement these on non-diagonal theories for which two incoming particles scatter off in more than one channel. Most of interesting integrable field theories belong to the latter including soliton scattering theories, theories with internal gauge symmmetries, and supersymmetric theories.

One of the most important aspects in the study of 2D integrable models is a close relationship between two quite different branches of physics, namely quantum field theories and 2D lattice models. After Onsager's solution of 2D Ising model, there have been many progresses in 2D integrable lattice models, culminating at Baxter's eight-vertex model.[5] According to modern lattice formulation, the YBE plays a central role providing a sufficient condition for the commutativity of transfer matrices which in turn supplies the infinite number of conserved quantities. The solutions of the YBE are the Boltzmann weights (BWs) which play the role of S-matrices in the field theories.

The lattice-field theory correspondence has achieved many remarkable results in its relation to the CFTs. At the critical points where the scale invariance arises due to infinite correlation length, many interesting lattice models are identified with CFTs and solved with their powerful techniques. One of the most important topics in the study of quantum integrable models is to extend this correspondence to off-critical region and to solve both models exactly.

In this paper we study the breather S-matrices of the supersymmetric sine-Gordon (SSG) model from the above point of view; its correspondence to the lattice model. The lattice model involved here is the so-called eight-vertex free fermion models where an external magnetic field makes the BWs asymmetric under arrow inversion.[6] We show how this correspondence arises and use this result to analyze the UV behaviour of the scattering theories with TBA method. This paper is an extended version of our previous result.[7]

§ 2. Supersymmetric QFTs and eight vertex models

2.1. *SUSY sine-Gordon model*

The SSG model preserves all the interesting properties of the sine-Gordon model like integrability and solitonic solutions besides its new feature; supersymmetry. The S-matrix of the SSG solitons has been derived based on the perturbed superconformal field theories.[8] This S-matrix commutes with the SUSY charge which satisfies an extended SUSY algebra with a topological charge. If the coupling constant of the SSG model*) satisfies $\gamma \equiv 4\beta^2/1-(\beta^2/4\pi) < 8\pi/n$, there can exist n number of the bound states called 'breathers'. The S-matrices of these breathers have been derived by considering scattering amplitudes of two solitons and two anti-solitons which carry rapidities designed so as to give bound state poles.[9] The scattering matrix of the n-th and m-th breather supermultiplet is defined by Zamolodchikov algebra of the bosonic (B_n) and fermionic (F_n) breathers: $(n \geq m)$

$$B_n(\theta_1)B_m(\theta_2) = a_+(\theta_1-\theta_2)B_m(\theta_2)B_n(\theta_1) + d(\theta_1-\theta_2)F_m(\theta_2)F_n(\theta_1)\,,$$

$$F_n(\theta_1)F_m(\theta_2) = a_-(\theta_1-\theta_2)F_m(\theta_2)F_n(\theta_1) + d(\theta_1-\theta_2)B_m(\theta_2)B_n(\theta_1)\,,$$

$$B_n(\theta_1)F_m(\theta_2) = b_+(\theta_1-\theta_2)F_m(\theta_2)B_n(\theta_1) + c(\theta_1-\theta_2)B_m(\theta_2)F_n(\theta_1)\,,$$

$$F_n(\theta_1)B_m(\theta_2) = b_-(\theta_1-\theta_2)B_m(\theta_2)F_n(\theta_1) + c(\theta_1-\theta_2)F_m(\theta_2)F_n(\theta_1)\,, \tag{1}$$

where the amplitudes are given by

*) The lagrangian of the SSG model is $\mathcal{L} = \frac{1}{\beta^2}[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}i\bar{\psi}\partial\!\!\!/\psi + \frac{m^2}{4}\cos^2\phi - \frac{m}{2}(\cos\phi)\bar{\psi}\psi]$.

$$a_{\pm}=Y_{nm}\left[\pm\sinh\theta+\frac{i}{2}(M_n+M_m)\right],\quad b_{\pm}=Y_{nm}\left[\sinh\theta\mp\frac{i}{2}(M_n-M_m)\right],$$
$$c=Y_{nm}\left[i\sqrt{M_nM_m}\cosh\frac{\theta}{2}\right],\qquad d=Y_{nm}\left[\sqrt{M_nM_m}\sinh\frac{\theta}{2}\right],\tag{2}$$

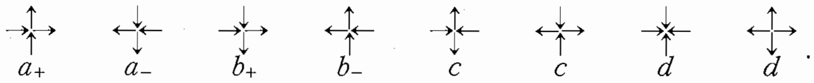
and the prefactor $Y_{nm}(\theta)$ can be determined by unitarity and crossing symmetry[9] and M_n is the mass of n-breather, $2\sin(n\gamma/16)$. Equation (1) may be written in terms of 4×4 matrix

$$S_{n,m}(\theta)=\begin{pmatrix} a_+ & 0 & 0 & d \\ 0 & b_+ & c & 0 \\ 0 & c & b_- & 0 \\ d & 0 & 0 & a_- \end{pmatrix}.\tag{3}$$

These S-matrices commute with the $N=1$ SUSY charges and describe the scattering amplitudes of the SSG breathers and bound states of the perturbed superconformal (non-unitary) field theories by the least relevant operators.[9] In particular, if there is only one kind of breather ($n=m=1$), Eq. (3) becomes the S-matrix of the supersymmetric sinh-Gordon model (SShG)[10] and the perturbed super Yang-Lee model.

2.2. *Eight-vertex free fermion models*

The general eight-vertex model is defined by BWs



(4)

Comparing with Eq. (3), one can relate the BWs with S-matrix of the SSG breathers if one identifies ↑ and → with $|B_n\rangle$ and ↓ and ← with $|F_n\rangle$.*)

There are two classes of exactly solvable models of the type (4) which satisfy the YBE: the first one is the Baxter model where $a_+=a_-$ and $b_+=b_-$. This model is equivalent to XYZ spin chain model and is reduced to the six-vertex model in the limit of the vanishing elliptic modulus. Also the six-vertex model is related to the sine-Gordon field theory by the relationship of the S-matrix and the BWs. At a special coupling it is known to have $N=2$ supersymmetry.[11] The second class is the free fermion model[6] if the weights satisfy the 'free fermion condition'

$$a_+a_-+b_+b_-=c^2+d^2,\tag{5}$$

and if two combinations of the BWs

$$\Gamma=\frac{2cd}{a_+b_-+a_-b_+},\quad h=\frac{a_-^2+b_+^2-a_+^2-b_-^2}{2(a_+b_-+a_-b_+)},\tag{6}$$

are independent of the rapidity. This model is equivalent to the general XY-model with a magnetic field,

*) The BWs in Eq. (4) become the S-matrix element if we adopt a convention that time flows from bottom-left to top-right(↗).

$$\mathcal{H}_{XY}=-J\sum_{j=1}^{N}[\sigma_j^+\sigma_{j+1}^-+\sigma_j^-\sigma_{j+1}^+ +\Gamma(\sigma_j^+\sigma_{j+1}^+ +\sigma_j^-\sigma_{j+1}^-)-h\sigma_j^z],$$

where $\sigma^\pm=(\sigma^x\pm i\sigma^y)/2$ with a conventional Pauli σ^i matrices.

The S-matrix of the SSG breathers satisfy the free fermion condition and

$$\Gamma=\frac{M_m}{2}=\sin\left(\frac{m\gamma}{16}\right), \quad h=-1.$$

This value of h corresponds to the critical point of the XY-model. In the parameter space spanned by (Γ, h), these supersymmetric models place on the line of $h=-1$. Field theories corresponding to other values of non-vanishing h are not clear.

§ 3. Thermodynamic Bethe ansatz

3.1. *Inversion relation*

For the nondiagonal theories, the PBC equation can be expressed as

$$e^{im_a L\sinh\theta}\mathcal{T}_{aa}(\theta|\theta_1, \cdots, \theta_N)=1, \quad \text{where}$$

$$\mathcal{T}_{ab}(\theta|\theta_1, \cdots, \theta_N)_{\{a_i\}}^{\{a_i'\}}=\sum_{\{\alpha_i\}}S_{aa_1}^{\alpha_2 a_1'}(\theta-\theta_1)S_{\alpha_2 a_2}^{\alpha_3 a_2'}(\theta-\theta_2)\cdots S_{\alpha_N a_N}^{b a_N'}(\theta-\theta_N). \tag{7}$$

If we add these equations for the index a, we can express it in terms of the transfer matrix

$$e^{imL\sinh\theta}T(\theta|\theta_1, \cdots, \theta_N)=N_c, \tag{8}$$

where the integer N_c is the number of colors and the transfer matrix $T\equiv\sum\mathcal{T}_{aa}$ acts on $V^{\otimes N}$. Precisely speaking, this is 'inhomogeneous' tranfer matrix because it depends on each rapidity of in-coming particle states.

To derive the TBA equations, one must diagonalize the transfer matrix. In this paper we achieve this using the 'inversion relation' following Felderhof.[12] Reexpressing the BWs in terms of the σ-matrices

$$S(\theta)=\begin{pmatrix} a_+\sigma^+\sigma^- + b_+\sigma^-\sigma^+ & d\sigma^+ + c\sigma^- \\ c\sigma^+ + d\sigma^- & b_-\sigma^+\sigma^- + a_-\sigma^-\sigma^+ \end{pmatrix},$$

the transfer matrix becomes $T(u|\theta_1, \cdots, \theta_N)=\mathrm{Tr}_2[\prod S(u-\theta_i)]$. Also defining new transfer matrix T_1 with new BWs $a_\pm{}^1=-b_\pm$, $b_\pm{}^1=a_\pm$, $c^1=c$, $d^1=-d$, $T_1(u|\theta_1, \cdots, \theta_N)=\mathrm{Tr}_2[\prod S_1(u-\theta_i)]$, with

$$S_1(\theta)=\begin{pmatrix} -b_+\sigma^+\sigma^- + a_+\sigma^-\sigma^+ & -d\sigma^+ + c\sigma^- \\ c\sigma^+ - d\sigma^- & a_-\sigma^+\sigma^- - b_-\sigma^-\sigma^+ \end{pmatrix}.$$

The next step is to show that $TT_1\propto 1$. From the relation

$$T(u)T_1(u)=\mathrm{Tr}_2\left[\prod_{i=1}^{N}S_i\right]\mathrm{Tr}_2\left[\prod_{i=1}^{N}S_{1,i}\right]=\mathrm{Tr}_{2\otimes 2}\left[\prod_{i=1}^{N}S_i\otimes S_{1,i}\right],$$

and the fact that the 4×4 matrix $S_i\otimes S_{1,i}$ can be transformed to upper triangular matrices by rapidity-independent similarity transformation, one can find

$$T\cdot T_1=\prod_{i=1}^{N}M_+(u-\theta_i)+\prod_{i=1}^{N}M_-(u-\theta_i)+F\Big(\prod_{i=1}^{N}F_+(u-\theta_i)+\prod_{i=1}^{N}F_-(u-\theta_i)\Big)$$

with $F=\prod\sigma_i^z=\pm1$ and

$$M_+=a_+a_- -d^2\,,\quad F_+=\sinh^2(\phi)a_+b_+ +\cosh^2(\phi)a_-b_- -2\sinh(2\phi)cd\,,$$

$$M_-=a_+a_- -c^2\,,\quad F_-=-\cosh^2(\phi)a_+b_+ -\sinh^2(\phi)a_-b_- +2\sinh(2\phi)cd\,,$$

where $\tanh(2\phi)=2cd/(a_+b_++a_-b_-)$. The angle ϕ parametrizing the similarity transformation can be computed from Eq. (3) to get $(1/2)\tanh^{-1}(M_n/2)$.

Now, consider a translation $u\to u+i\pi$. Under this the BWs change $a_\pm\to -a_\mp$, $b_\pm \to b_\mp$, $c\to d$, $d\to -c$ and the corresponding transfer matrix satisfies $T(u+i\pi|\theta_1,\cdots,\theta_N)=(-1)^N T_1(u|\theta_1,\cdots,\theta_N)$. Therefore, one can find the inversion relation

$$T(u|\theta_1,\cdots,\theta_N)T(u+i\pi|\theta_1,\cdots,\theta_N)$$
$$=(-1)^N\Big[\prod_{i=1}^{N}M_+(u-\theta_i)+\prod_{i=1}^{N}M_-(u-\theta_i)+F\Big(\prod_{i=1}^{N}F_+(u-\theta_i)+\prod_{i=1}^{N}F_-(u-\theta_i)\Big)\Big].$$

For simplicity, we will concentrate on the simplest case which involves the lowest breather only ($n=m=1$). This theory will be related to either the SShG or perturbed super Yang-Lee model. It is also possible to analyze the general supersymmetric models with TBA which will be published elsewhere due to its complicity. In this case one obtains (modulo the prefactor Y_{11})

$$M_+(u)=-2\coth(u/2)\sinh(u/2+i\alpha\pi)\sinh(u/2-i\alpha\pi)\,,$$

$$M_-(u)=-2\tanh(u/2)\cosh(u/2+i\alpha\pi)\cosh(u/2-i\alpha\pi)\,,$$

$$F_\pm(u)=-2\cosh(u/2\pm i\alpha\pi)\sinh(u/2\mp i\alpha\pi)\,,$$

where $\alpha=\gamma/16\pi$. From these expressions one can notice that, under the change $u\to u+i\pi$, $M_\pm\to M_\mp$ and $F_\pm\to F_\mp$. This means $T(u)=T(u+2\pi i)$ and the eigenvalues $\Lambda(u)$ of $T(u)$ are $2\pi i$ symmetric functions satisfying ($\beta_i=u-\theta_i$)

$$\Lambda(u|\theta_1,\cdots,\theta_N)\Lambda(u+\pi i|\theta_1,\cdots,\theta_N)$$
$$=2^{2N}\Big[\prod_{i=1}^{N}\sinh(\beta_i/2)\cosh(\beta_i/2+i|\alpha|\pi)+F\prod_{i=1}^{N}\cosh(\beta_i/2)\sinh(\beta_i/2+i|\alpha|\pi)\Big]$$
$$\times\Big[\prod_{i=1}^{N}\sinh(\beta_i/2)\cosh(\beta_i/2-i|\alpha|\pi)+F\prod_{i=1}^{N}\cosh(\beta_i/2)\sinh(\beta_i/2-i|\alpha|\pi)\Big].$$

Since $\Lambda(u)$ is $2\pi i$ symmetric entire function defined on the complex plane, it can be completely fixed by the location of zeroes and poles. With zeroes $x_k-i|\alpha|\pi$ and $x_k-i|\alpha|\pi+i\pi$ with real x_k which satisfies

$$\prod_{i=1}^{N}\frac{\tanh\Big(\dfrac{x_k-\theta_i}{2}-\dfrac{i|\alpha|\pi}{2}\Big)}{\tanh\Big(\dfrac{x_k-\theta_i}{2}+\dfrac{i|\alpha|\pi}{2}\Big)}=-F\,,$$

the 2^N number of eigenvalues are compactly expressed by

$$\Lambda(u)_{\epsilon_1,\cdots,\epsilon_N}=\text{const}\prod_{k=1}^{N}\lambda_{\epsilon_k}(u-x_k)\,,\quad \epsilon_k=\pm 1\,,$$

$$\lambda_\epsilon(\theta)=\sinh\left(\frac{\theta}{2}+\epsilon\frac{i|\alpha|\pi}{2}\right)\cosh\left(\frac{\theta}{2}-\epsilon\frac{i|\alpha|\pi}{2}\right). \tag{9}$$

3.2. *TBA equations*

From Eqs. (7) and (9), the PBC equation becomes

$$\frac{1}{2}e^{imsinh\,\theta}\prod_{i=1}^{N}Y_{11}(\theta-\theta_i)\prod_{k=1}^{N}\lambda_{\epsilon_k}(\theta|x_1,\cdots,x_N)=1\,,$$

and from the standard derivation of TBA equations, one can find the following set of integral equations:

$$mR\cosh\theta=\epsilon(\theta)+\left(\left[\Phi_{Y_{11}}-\frac{1}{2}\Phi*\Phi\right]*\ln[1+e^{-\epsilon}]\right)(\theta)+(\Phi*\ln[1+e^{-\mathcal{E}}])(\theta)\,,$$

$$0=\mathcal{E}(\theta)+(\Phi*\ln[1+e^{-\epsilon}])(\theta)\,, \tag{10}$$

where $*$ represents the integral convolution and pseudo-energies are introduced for the densities of occupied states. The kernels are defined by

$$\Phi_{Y_{11}}(\theta)=\frac{\partial}{\partial\theta}\text{Im}\ln[Y_{11}(\theta)]\,,\quad \Phi(\theta)=\frac{\partial}{\partial\theta}\text{Im}\ln\left[\frac{\lambda_+(\theta)}{\lambda_-(\theta)}\right].$$

In the UV limit, Eq. (10) reduces to simple algebraic equations of the variables $x=\exp[-\epsilon(0)]$, $X=\exp[-\mathcal{E}(0)]$ as argued before. For the SShG model, the algebraic equations become

$$x=(1+x)^a(1+X)^b\,,\quad X=(1+x)^b\,,$$

$$a=\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\left(\Phi_Y-\frac{1}{2}\Phi*\Phi\right)(\theta)\,,\quad b=\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\Phi(\theta)\,. \tag{11}$$

It is not difficult to compute these exponents as $a=0$, $b=1$ for the SShG model and $a=-1$, $b=1$ for the SYL model. Using these values, the solution of Eq. (11) can be found easily as $x=X=\infty$ and $x=\sqrt{2}$, $X=1+\sqrt{2}$, respectively.

One also needs the pseudo-energies as $\theta\to\infty$. Since the mass for $\epsilon(\theta)$ is non-zero, ϵ diverges as $\theta\to\infty$, and $y=\exp[-\epsilon(\infty)]=0$. Then, from the second equation of (10), $Y=\exp[-\mathcal{E}(\infty)]=1$.

The UV effective central charge is given by*)

$$C_{\text{eff}}=\frac{6}{\pi^2}\left[\mathcal{L}\left(\frac{x}{1+x}\right)+\mathcal{L}\left(\frac{X}{1+X}\right)-\mathcal{L}\left(\frac{y}{1+y}\right)-\mathcal{L}\left(\frac{Y}{1+Y}\right)\right],$$

in terms of Roger's dilogarithmic function $\mathcal{L}$. Using the values given in Eq. (11), one can correctly obtain the UV central charges,

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*) $C_{\text{eff}}=C-12(\Delta_{\text{min}}+\bar{\Delta}_{\text{min}})$, with minimum conformal dimension Δ_{min}.

$$C_{\rm eff}=\frac{3}{2}\ \text{for SShG}, \quad =\frac{3}{4}\ \text{for SYL model}.$$

§ 4. Remarks

So far we concentrated on the TBA analysis of the supersymmetric models which satisfy the free fermion condition. While we will refer the details to Ref. 7), we want to emphasize that the supersymmetry makes it easier to compute two-point form factors. The form-factors are expressed as products of two factors; one from the S-matrix of the supersymmetric part and the other from that of non-supersymmetric part. Using spectral representation of C-theorem, one can reproduce quite accurate UV central charges from these two-point functions.

Recent work shows that this tensor product representation of the form-factors for the scattering theory with tensor product S-matrices is true for arbitrary point form factors for large class of integrable field theories.[13)]

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