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Restricted Sine-Gordon Theory in the Repulsive Regime as Perturbed Minimal CFTs

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ABSTRACT

We construct the restricted sine-Gordon theory by truncating the sine-Gordon multi-soliton Hilbert space for the repulsive coupling constant due to the quantum group symmetry $SL_q(2)$ which we identify from the Korepin's S -matrices. We connect this restricted sine-Gordon theory with the minimal ($c < 1$) conformal field theory $\mathcal{M}_{p/p+2}$ (p odd) perturbed by the least relevant primary field $\Phi_{1,3}$. The exact S -matrices are derived for the particle spectrum of a kink and neutral particles. As a consistency check, we compute the central charge of the restricted theory in the UV limit using the thermodynamic Bethe ansatz analysis and show that it is equal to that of $\mathcal{M}_{p/p+2}$.

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1. Introduction

Much progress have been made recently on integrable quantum field theories (QFTs) from the conformally invariant 2D field theories^[1] (CFTs) perturbed by the relevant operators.^[2] For certain CFTs, the integrability can be preserved under massive perturbations which breaks the conformal symmetry. The resulting integrable QFTs have been identified with various well-known integrable models as well as quite new ones.

There have been two major approaches on the integrable perturbations of CFTs. One is a direct method of constructing infinite number of conserved currents explicitly. After constructing the integrability, one derives the particle spectrum and exact S -matrices using the bootstrap conjecture.^[2] This method, however, has a limitation on its applicability in that it is not easy to construct the infinite number of conserved currents explicitly for general classes of the CFTs and perturbations.

The other approach is to construct new integrable field theories out of known one and identify the new theories with perturbed CFTs. New theories are constructed by restricting Hilbert space of the starting integrable QFT in such a manner that maintains the integrability. This is an QFT analogy of the ‘restricted solid-on-solid’ (RSOS) model in the integrable lattice model.^[3] Many new integrable QFTs with exact S -matrices have been obtained by this method and identified with perturbed CFTs.^[4–10] Most of these new integrable QFTs are obtained from the sine-Gordon (SG) theory as the starting integrable QFT. The restricted sine-Gordon (RSG) theory has been constructed and identified with minimal ($c < 1$) CFTs perturbed by the least relevant operator.^[4–6] This has been extended to the general coset CFTs perturbed by the least relevant operator and connected with many integrable QFTs with extra symmetries, such as supersymmetry.^[9,10] Recently, new restricted integrable QFTs have been constructed from Zhiber-Mikhailov-Shabat model ($A_2^{(2)}$ affine Toda model with imaginary coupling constant) and related to the minimal CFTs perturbed by $\Phi_{1,2}$ and $\Phi_{2,1}$ operators.^[11]

In this paper, we continue this approach and extend the RSG theory to a

different region of the SG coupling constant, the repulsive regime. All the previous results on the RSG theory^[4–10] were based on the SG theory in the attractive regime and special points in the repulsive regime. Outside of this region, the SG theory behaves quite differently. We will construct new RSG theory (particle spectrum and S -matrices) for this new region and identify this theory with minimal CFT ($c < 1$) $\mathcal{M}_{p/p+2}$ (p odd) perturbed by the least relevant field $\Phi_{1,3}$.

The RSG theory can be constructed in a similar way as minimal CFTs through the Feigin-Fuchs-Felder method. One can think of the SG theory as a massive perturbation of $c = 1$ free boson theory by the periodic potential. After introducing a background charge, one restricts the Fock space of the boson using the underlying BRST cohomology structure to get $c < 1$ minimal CFTs $\mathcal{M}_{p/q}$.^[12] The analogy of this procedure in the massive theory is the restriction of the multi-soliton Hilbert space based on the quantum group symmetry of the SG theory. The resulting theory, the RSG theory, is identified with the minimal CFTs $\mathcal{M}_{p/q}$ perturbed by $\Phi_{1,3}$. In a similar way as unitarity condition on the physical state further reduces the minimal CFTs to unitary CFTs $\mathcal{M}_{p/p+1}$,^[13] the unitarity condition on the S -matrices of the RSG theory allows only a few infinite series out of the minimal CFTs to yield the unitary S -matrices after the massive perturbation.^[6]

This paper is organized as follows: In sect.2, we review the major relevant results on the SG theory in the historical order. We will treat the SG theory and the massive Thirring model (MTM) as equivalent theories following S. Coleman's discovery.^[14] In terms of the MTM coupling constant, we classify the SG (MTM) theory into two regimes, attractive and repulsive regimes. We present semi-classical results on the bound state mass spectrum in the attractive regime,^[15] from which A.B. Zamolodchikov conjectured the familiar exact soliton S -matrix.^[17] The spectrum consists of the solitons, anti-solitons, and their bound states (the breathers). This result in the attractive regime has been confirmed by the Bethe ansatz.^[19,20]

The situation in the repulsive regime^{*} is quite different. The physical vacuum

^{*} By the repulsive regime, we mean the region except discrete first-order phase transition

constructed in the attractive regime turns out to be not true vacuum. It is V. Korepin who first constructed the true vacuum to get the physical spectrum of the MTM theory.^[21] The spectrum in this case consists of the solitons, anti-solitons, and neutral particles which are not bound states. Exact S -matrices of these particles have been also derived. Our construction of the RSG theory in the repulsive regime will be based on this result.

We explain the RSG theory in sect.3 starting with the $SL_q(2)$ quantum group symmetry in the SG theory.^[4-8] We derive the $SL_q(2)$ symmetry of the SG theory in the repulsive regime where the deformation parameter q becomes modified. Using this quantum group symmetry, we can restrict the multi-soliton Hilbert space to obtain the RSG theory in the repulsive regime. After imposing the unitarity condition on the S -matrices, we identify the RSG theory with perturbed minimal CFTs. The particle spectrum includes a kink and many neutral particles with all different masses.

In sect.4, we check these S -matrices of the new RSG theory using the thermodynamic Bethe ansatz.^[25-28] This proves the consistency of our construction of the new RSG theory as the minimal CFTs $\mathcal{M}_{p/p+2}$ perturbed by the $\Phi_{1,3}$ operator. We will conclude with a few comments and open questions in sect.5.

2. S -matrices of the sine-Gordon Theory

In this section, we review the results on the SG theory which will be used in later sections. In particular, we explain the Bethe ansatz which gives the exact results in the both attractive and repulsive regime.

points.

2.1. THE SINE-GORDON THEORY

We start with the SG Lagrangian

$$\mathcal{L}_{\text{SG}} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{M_0^2}{\beta^2}\cos(\beta\phi), \quad (2.1)$$

where M_0 is a mass scale and β is a coupling constant. Due to the periodic potential, the SG theory has classical soliton (and anti-soliton) solutions which connect two different vacua at $x = \pm\infty$. Each solitonic solution carries a conserved topological charge.

S. Coleman discovered^[14] that the quantum SG theory is equivalent to the massive Thirring model

$$\begin{aligned} \mathcal{L}_{\text{MTM}} &= i\bar{\psi}\gamma_\mu\partial_\mu\psi - m_0\bar{\psi}\psi - \frac{g}{2}(\bar{\psi}\gamma_\mu\psi)^2, \\ \frac{g}{\pi} &= \frac{4\pi}{\beta^2} - 1, \quad \text{for } 0 \leq \frac{\beta^2}{8\pi} < 1, \end{aligned} \quad (2.2)$$

and identified the SG soliton with the Thirring fermion. Therefore, we can consider the SG theory and MTM as two different realizations of the same theory.

He also showed that if $\beta^2/8\pi \geq 1$ the energy density of the theory is unbounded from below and not well-defined. Throughout this paper, we will consider only the case of $\beta^2/8\pi < 1$. We define the following parameters for the later convenience:

$$P = \frac{\gamma}{8\pi} = \frac{\beta^2/8\pi}{1 - \beta^2/8\pi}, \quad \text{and} \quad \frac{\mu}{\pi} = 1 - \frac{\beta^2}{8\pi}. \quad (2.3)$$

The Thirring interaction becomes

$$\begin{aligned} \text{attractive } (g > 0) & \quad \text{for } \beta^2 < 4\pi, \quad \frac{\gamma}{8\pi} < 1, \quad \mu > \pi/2, \\ \text{repulsive } (g < 0) & \quad \text{for } \beta^2 > 4\pi, \quad \frac{\gamma}{8\pi} > 1, \quad \mu < \pi/2. \end{aligned} \quad (2.4)$$

The complete particle spectrum of the SG (MTM) theory was derived in the semi-classical (WKB) approximation and claimed to be exact even in the full quantum theory.^[15] In the attractive regime ($\gamma/8\pi < 1$), the spectrum consists of the SG

soliton (A^+), anti-soliton (A^-) with the mass m_f and the bound states of these, the breathers (B_n), with the masses

$$m_n = 2m_f \sin\left(\frac{n\gamma}{16}\right); \quad n = 1, 2, \dots < \frac{8\pi}{\gamma}. \quad (2.5)$$

The first exact S -matrix of the SG (MTM) theory was conjectured by Faddeev and Korepin^[16] for the special values in the attractive regime $\gamma/8\pi = 1/n$ with $n = 1, 2, \dots$ (the soliton sector),

$$S_{++,+}^{SG}\left(\theta, \frac{\gamma}{8\pi}\right) = e^{in\pi} \prod_{k=1}^n \left[\frac{\exp(\theta - i\pi k/n) + 1}{\exp \theta + \exp(-i\pi k/n)} \right]. \quad (2.6)$$

2.2. ZAMOLODCHIKOV'S S -MATRICES

The SG (MTM) theory is integrable in the sense that one can construct infinite number of commuting conserved currents in the both classical and quantum levels. In the quantum theory, this means that the S -matrices are elastic without any particle creation or annihilation. The multi-particle S -matrices are completely factorized into the products of two-particle S -matrices. These two-particle S -matrices should satisfy Yang-Baxter equations which determine the S -matrices completely along with unitarity and crossing symmetry upto the overall CDD factor. We will consider only the case where additional CDD factors are set to one (minimal solution).

A.B. Zamolodchikov derived the first exact S -matrix of the SG solitons and anti-solitons, assuming that the semi-classical masses in Eq.(2.5) are exact. This means that the soliton S -matrix should have poles in the physical strip which correspond to the bound states. Along with the unitarity, crossing symmetry, and the known exact results for the special attractive couplings in Eq.(2.6), the SG soliton S -matrix can be decided completely in the attractive regime.^[17] The S -matrix of the SG soliton (A^+) and anti-soliton (A^-) (or the Thirring fermion and

antifermion) is equal to

$$\begin{aligned}
S_{+++,++}^{SG} \left(\theta, \frac{\gamma}{8\pi} \right) &= S_{---,---}^{SG} \left(\theta, \frac{\gamma}{8\pi} \right) = \frac{U(\theta)}{i\pi} \sinh \left[\frac{8\pi}{\gamma} (i\pi - \theta) \right] \\
S^{SG} \left(\theta, \frac{\gamma}{8\pi} \right) &= \frac{U(\theta)}{i\pi} \begin{pmatrix} A^+ A^- & A^- A^+ \\ A^+ A^- \begin{pmatrix} \sinh \frac{8\pi\theta}{\gamma} & i \sin \frac{8\pi^2}{\gamma} \\ A^- A^+ \begin{pmatrix} i \sin \frac{8\pi^2}{\gamma} & \sinh \frac{8\pi\theta}{\gamma} \end{pmatrix} \end{pmatrix} \end{pmatrix}, \quad (2.7)
\end{aligned}$$

with $U(\theta)$, satisfying $U(\theta) = U(i\pi - \theta)$,

$$\begin{aligned}
U(\theta) &= \Gamma \left(\frac{8\pi}{\gamma} \right) \Gamma \left(1 + i \frac{8\theta}{\gamma} \right) \Gamma \left(1 - \frac{8\pi}{\gamma} - i \frac{8\theta}{\gamma} \right) \prod_{n=1}^{\infty} \frac{R_n(\theta) R_n(i\pi - \theta)}{R_n(0) R_n(i\pi)}, \\
R_n(\theta) &= \frac{\Gamma \left(2n \frac{8\pi}{\gamma} + i \frac{8\theta}{\gamma} \right) \Gamma \left(1 + 2n \frac{8\pi}{\gamma} + i \frac{8\theta}{\gamma} \right)}{\Gamma \left((2n+1) \frac{8\pi}{\gamma} + i \frac{8\theta}{\gamma} \right) \Gamma \left(1 + (2n-1) \frac{8\pi}{\gamma} + i \frac{8\theta}{\gamma} \right)}. \quad (2.8)
\end{aligned}$$

One can express these with an integral representation,

$$S_{+++,++}^{SG} \left(\theta, \frac{\gamma}{8\pi} \right) = \exp \left[- \int_0^{\infty} \frac{dx \sinh((4\pi x/\gamma) - x/2) \sinh(8i\theta x/\gamma)}{x \sinh(x/2) \cosh(4\pi x/\gamma)} \right]. \quad (2.9)$$

The soliton S -matrix (2.7) has been checked perturbatively based on the MTM.

The S -matrices between the breathers and solitons have been derived from three-solitons (and anti-solitons) scattering processes,^[18]

$$S_{\text{SG}}^{(n)} \left(\theta, \frac{\gamma}{8\pi} \right) = \frac{\sinh \theta + i \cos \frac{n\gamma}{16}}{\sinh \theta - i \cos \frac{n\gamma}{16}} \prod_{l=1}^{n-1} \frac{\sin^2 \left(\frac{n-2l}{32} \gamma - \frac{\pi}{4} + i \frac{\theta}{2} \right)}{\sin^2 \left(\frac{n-2l}{32} \gamma - \frac{\pi}{4} - i \frac{\theta}{2} \right)}, \quad (2.10)$$

for the process $A^\pm + B_n \rightarrow A^\pm + B_n$. The S -matrices between the breathers (four

(anti) solitons process) are equal to^[18]

$$S_{\text{SG}}^{(n,m)}\left(\theta, \frac{\gamma}{8\pi}\right) = \frac{\sinh \theta + i \sin\left(\frac{n+m}{16}\gamma\right)}{\sinh \theta - i \sin\left(\frac{n+m}{16}\gamma\right)} \frac{\sinh \theta + i \sin\left(\frac{n-m}{16}\gamma\right)}{\sinh \theta - i \sin\left(\frac{n-m}{16}\gamma\right)} \\ \times \prod_{l=1}^{\min(m,n)-1} \frac{\sin^2\left(\frac{m-n-2l}{32}\gamma + i\frac{\theta}{2}\right) \cos^2\left(\frac{m+n-2l}{32}\gamma + i\frac{\theta}{2}\right)}{\sin^2\left(\frac{m-n-2l}{32}\gamma + i\frac{\theta}{2}\right) \cos^2\left(\frac{m+n-2l}{32}\gamma + i\frac{\theta}{2}\right)}, \quad (2.11)$$

for the process $B_n(\theta_1) + B_m(\theta_2) \rightarrow B_n(\theta_1) + B_m(\theta_2)$ with $n \geq m$. One can check these S -matrices by comparing Eq.(2.11) for $m = n = 1$ with the perturbative computation based on the SG Lagrangian (2.1) because one can identify the lowest massive breather B_1 with the fundamental field ϕ in the SG Lagrangian.

We want to emphasize that Eqs.(2.7), (2.10) and (2.11) are based on the semi-classical mass spectrum and scattering amplitudes (2.6) in the attractive regime. Therefore, these results are valid in principle only in this regime. In the next subsection, we show that the Zamolodchikov's results are exact in the attractive regime while new results are obtained in the repulsive regime using the Bethe ansatz method.

2.3. BETHE ANSATZ APPROACH

The MTM theory has been solved exactly by the Bethe ansatz. The exact mass spectrum^[19] and S -matrices^[20] have been obtained in the attractive regime, which are consistent with the Zamolodchikov's results. Furthermore, this method has been extended to the repulsive regime^[21] to get the exact results which are quite different from the previous ones.*

The second quantized Hamiltonian of the MTM is from Eq.(2.2),

$$\hat{H} = \int dx \left[-i \left(\psi_1^\dagger \frac{\partial}{\partial x} \psi_1 - \psi_2^\dagger \frac{\partial}{\partial x} \psi_2 \right) + m_0 (\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) + 2g \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \right]. \quad (2.12)$$

Note that due to the negative sign of the second term in the kinetic energy, there are

* The Bethe ansatz solution of the SG theory regularized on the lattice has given consistent results with these.^[22]

negative energy modes. This means that the vacuum $|0\rangle$, defined by $\psi_{1,2}(x)|0\rangle = 0$, and states constructed upon it are not true physical states. We refer to these states created by the fermion modes acting on the false vacuum $|0\rangle$ as ‘pseudo-particles’. The true physical states and particles should be constructed out of the true vacuum which is a Dirac sea where all the negative energy states are completely filled.

One can diagonalize the Hamiltonian with the Bethe ansatz wave function

$$|\Phi(\theta_1, \dots, \theta_N)\rangle = \int \prod_{i=1}^N (e^{im_0 x_i \sinh \theta_i} dx_i) \times \prod_{i < j \leq N} \exp \left[\frac{i}{2} \epsilon(x_j - x_i) \phi(\theta_j - \theta_i) \right] A^\dagger(\theta_1, x_1) \cdots A^\dagger(\theta_N, x_N) |0\rangle, \quad (2.13)$$

where $\epsilon(x)$ is the sign-function and $\phi(\theta)$ is the phase of the scattering amplitude $S(\theta)$ of the pseudo-particles,

$$S(\theta) = \exp[i\phi(\theta)] = -\frac{\sinh \frac{1}{2}(\theta - 2i\mu)}{\sinh \frac{1}{2}(\theta + 2i\mu)}, \quad (2.14)$$

in terms of the parameter μ defined in Eq.(2.3). By applying the Hamiltonian (2.12) to the state (2.13), one can find

$$\hat{H}|\Phi(\theta_1, \dots, \theta_N)\rangle = \left(\sum_{i=1}^N m_0 \cosh \theta_i \right) |\Phi(\theta_1, \dots, \theta_N)\rangle. \quad (2.15)$$

The operator $A^\dagger(\theta, x)$ is related to the fermion operators by

$$A^\dagger(\theta, x) = (2 \cos \theta)^{-1/2} \left[e^{\theta/2} \psi_1^\dagger(x) + e^{-\theta/2} \psi_2^\dagger(x) \right], \quad (2.16)$$

where the rapidity θ has complex values in general. In rapidity space, the Fourier transformed operator

$$A^\dagger(\theta) = \int dx e^{-ixm_0 \sinh \theta} A^\dagger(\theta, x), \quad (2.17)$$

creates positive energy modes for $\theta = \alpha$ and negative energy modes for $\theta = \alpha + i\pi$. We use α for a real value of the rapidity. The physical vacuum is the Dirac sea where all the rapidity states $\theta = \alpha + i\pi$ are completely filled.

The physical spectrum can be determined by imposing two conditions on the Bethe wave function (2.13). The first one is that the wave function (the integrand in Eq.(2.13)) should go to zero as $x_i \rightarrow \pm\infty$. If the rapidities have imaginary parts, Eq.(2.13) diverges unless the scattering amplitudes $S(\theta_i - \theta_j)$ vanish. This requires that $\theta_i - \theta_{i+1} = 2i(\pi - \mu)$ from Eq.(2.14). With the condition that the imaginary parts of all the allowed rapidities lie between $-i\pi$ and $i\pi$, the only allowed excited rapidity states are ‘string states’ ($S_n(\alpha)$ with $n = 1, 2, \dots$) which are composed of n pseudo-particles with the rapidities (in the attractive regime)

$$\begin{aligned} \theta_l &= \alpha + i(\pi - \mu)(n - 1 - 2l), \quad \text{for } \frac{\pi}{2} < \mu < \pi, \\ l &= 0, 1, \dots, n - 1 \quad \text{with } (\pi - \mu)(n - 1) < \pi. \end{aligned} \quad (2.18)$$

The other condition is the periodic boundary condition (PBC) in the large box with length L . From PBC on Eq.(2.13) ($x_i \rightarrow x_i + L$), one can find the following equation for each i :

$$e^{im_0 L \sinh \theta_i} \prod_{i \neq j} \exp [i\phi(\theta_j - \theta_i)] = 1. \quad (2.19)$$

Taking logarithms on both sides, one gets

$$-m_0 L \sinh \theta_i = \sum_{i \neq j} \phi(\theta_j - \theta_i) + 2\pi n_i, \quad (2.20)$$

with arbitrary integers n_i . As $L \rightarrow \infty$, in terms of the density of the rapidity states per unit length,

$$\rho(\theta) = \lim_{L \rightarrow \infty} \frac{1}{L(\theta_j - \theta_{j+1})}, \quad (2.21)$$

Eq.(2.20) becomes an integral equation for the density of the rapidity states in the Dirac sea at $\theta = \alpha + i\pi$:

$$2\pi\rho(\theta) = -m_0 \cosh \theta - \int_{-\Lambda+i\pi}^{\Lambda+i\pi} d\theta' \left[\frac{\partial}{\partial \theta} \phi(\theta - \theta') \right] \rho(\theta'). \quad (2.22)$$

Note that the density of the allowed states in the left-hand side (the density of n_i) are the same as that of the actual occupied states $\rho(\theta)$ because all the rapidity states

in the vacuum (the Dirac sea) are completely filled by definition. We introduced the rapidity cut-off parameter Λ to regularize the UV divergence. The cut-off dependence will be absorbed into the mass renormalization.

The physical excited states are the string states $S_n(\alpha)$ made of n pseudo-particles from the Dirac sea. Therefore, an physical excited state consists of a string state S_n and n ‘holes’ in the Dirac sea. Due to these holes, the pseudo-particle rapidity states in the Dirac sea are rearranged, which are determined by the following PBC equation:

$$\begin{aligned}
2\pi\widehat{\rho}(\theta) = & -m_0 \cosh \theta - \sum_{l=0}^{n-1} \phi(\theta - \theta_l) - n\phi(\theta - \theta^{(h)}) \\
& - \int_{-\Lambda+i\pi}^{\Lambda+i\pi} d\theta' \left[\frac{\partial}{\partial\theta} \phi(\theta - \theta') \right] \widehat{\rho}(\theta'),
\end{aligned} \tag{2.23}$$

with $\theta^{(h)} = \alpha_s + i\pi$ and θ_l in Eq.(2.18). The energy of this excited state relative to the vacuum is equal to

$$\begin{aligned}
E_n &= m_n \cos \alpha \\
&= \sum_{l=0}^{n-1} m_0 \cosh \theta_l - nm_0 \cosh \theta^{(h)} + m_0 \int_{-\Lambda+i\pi}^{\Lambda+i\pi} d\theta [\rho(\theta) - \widehat{\rho}(\theta)] \cosh \theta.
\end{aligned} \tag{2.24}$$

In the region

$$\frac{r\pi}{r+1} < \mu < \frac{(r+1)\pi}{r+2} \quad \text{with } r = 1, 2, \dots, \tag{2.25}$$

$r+2$ string states S_n ($n = 1, \dots, r+2$) are allowed from Eq.(2.18). The two longest strings S_{r+1}, S_{r+2} represent unbound fermion-antifermion pairs. The other r strings correspond to the bound states of a fermion-antifermion pair, i.e. the breathers (B_n). One can solve the integral equations (2.22) and (2.23) using the

Fourier transformation and evaluate the excited energy (2.24). From this, one can determine the physical particle spectrum,

$$\begin{aligned}
\cdot\text{soliton } A^\pm : \quad m_f &= m_0 \frac{\mu \tan \pi^2/2\mu}{\pi \pi - 2\mu} e^{\Lambda(1-\pi/2\mu)} \\
\cdot\text{breather } B_n : \quad m_n &= 2m_f \sin \left[\frac{n\pi}{2} \left(\frac{\pi}{\mu} - 1 \right) \right] \quad n = 1, \dots, r.
\end{aligned} \tag{2.26}$$

This is the mass spectrum derived in the semi-classical approximation in (2.5).

The S -matrices of the physical particles (the string states S_n) can be derived as the products of the scattering amplitudes of pseudo-particles which constitute the string states,

$$\begin{aligned}
S_m^n(\theta) &= \exp [\Phi_m^n(\theta)] \quad \text{for } n, m = 1, \dots, r+2 \\
&= \prod_{j=0}^{n-1} \prod_{k=0}^{m-1} \exp [\phi(\theta + i(\pi - \mu)(2j - 2k + m - n))].
\end{aligned} \tag{2.27}$$

These results give the exactly same S -matrices as Eqs.(2.7), (2.7) and (2.11).^[20]

The SG theory in the repulsive regime has been solved by V. Korepin.^[21] The essential difference in this regime arises from the fact that the allowed strings (S_n),

$$\begin{aligned}
\theta_l &= \alpha + i\pi + i\mu(n-1-2l), \quad \text{for } 0 < \mu < \frac{\pi}{2}, \\
l &= 0, 1, \dots, n-1, \quad \text{with } \mu(n-1) < \pi,
\end{aligned} \tag{2.28}$$

have negative energies from Eq.(2.15),

$$E[S_n] = -m_0 \frac{\sin(n\mu)}{\sin \mu} \cos \alpha, \tag{2.29}$$

with respect to the vacuum energy of a pseudo-particle ($E_0 = -m_0 \cos \alpha$) in the previous Dirac sea. This means one needs to construct new vacuum (Dirac sea) where the negative energy states created by the strings are completely filled.

In the region of the repulsive regime

$$\frac{\pi}{r+2} < \mu < \frac{\pi}{r+1} \quad r = 1, 2, \dots, \quad (2.30)$$

there are r string ($S_n(\alpha)$ $n = 1, \dots, r$) states which should be filled.* In other words, the true vacuum is a r -component condensate in which all permitted rapidity states α of these composite particles are completely filled. The densities $\rho_n(\theta)$ of these strings S_n in the new Dirac sea can be determined as before by the PBC in the continuum limit

$$2\pi\rho_n(\alpha) = m_0 \frac{\sin(n\mu)}{\sin\mu} \cosh\alpha - \sum_{m=1}^r \int_{-\Lambda}^{\Lambda} d\alpha' \left[\frac{\partial}{\partial\alpha} \Phi_m^n(\alpha - \alpha') \right] \rho_m(\alpha'), \quad (2.31)$$

where the phase Φ_m^n of the scattering amplitude between two strings S_n and S_m is equal to

$$\Phi_m^n(\theta) = \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \phi(\theta + i\mu(2j - 2k + m - n)). \quad (2.32)$$

The physical excited states are created by removing strings S_n out of vacuum (or creating a ‘hole string’). If a string is removed, the vacuum is realigned and the densities of the strings in the new Dirac sea satisfy new integral equation

$$2\pi\hat{\rho}_n(\alpha) = m_0 \frac{\sin(n\mu)}{\sin\mu} \cosh\alpha + \Phi_h^n(\alpha - \bar{\alpha}) - \sum_{m=1}^r \int_{-\Lambda}^{\Lambda} d\alpha' \left[\frac{\partial}{\partial\alpha} \Phi_m^n(\alpha - \alpha') \right] \hat{\rho}_m(\alpha'). \quad (2.33)$$

* In fact, $r+2$ strings S_n are allowed from Eq.(2.28). But one can show using Eq.(2.29) that the energy of S_{r+2} is positive while that of S_{r+1} is positive relative to the energy of a pseudo-particle.

The excited energies can be expressed in terms of these densities as follows:

$$\begin{aligned}
E_n &= M_n \cosh \theta \\
&= m_0 \frac{\sin(n\mu)}{\sin \mu} \cosh \bar{\alpha} - m_0 \sum_{m=1}^r \int_{-\Lambda}^{\Lambda} d\alpha' [\rho_m(\alpha') - \hat{\rho}(\alpha')] \cosh \alpha'. \tag{2.34}
\end{aligned}$$

The string state S_r is interpreted as the fermion (the SG soliton). The other string states S_n ($n = 1, \dots, r-1$) correspond to new particles, called ‘neutral particles’ (N_n). One can find the mass spectrum from Eq.(2.34),^[21]

$$\begin{aligned}
\cdot \text{soliton } A^\pm : \quad m_f &= M \left[\frac{\sin(r-1)\mu}{\sin \mu} + \frac{\sin(r\mu)}{\sin \mu} \tan \frac{\pi}{2} \left(\frac{\pi}{\mu} - r - 1 \right) \right] \\
\cdot \text{neutrals } N_n : \quad M_n &= 2M \frac{\sin(n\mu)}{\tan \mu} \quad n = 1, \dots, r-1, \tag{2.35} \\
\text{with} \quad M &= m_0 \frac{\exp[(2\mu - \pi)\Lambda/2\mu]}{\pi - 2\mu} \quad \theta = \frac{\pi \bar{\alpha}}{2\mu}.
\end{aligned}$$

The scattering amplitudes of these physical particles can be computed from the products of those of each constituent pseudo-particles like Eq.(2.32). The S -matrices are given as follows:^[21]

$$S_{\text{rep}}^{SG} \left(\theta, \frac{\gamma}{8\pi} \right) = S^{SG} \left(\theta, \frac{\gamma^{\text{eff}}}{8\pi} \right), \tag{2.36}$$

for the SG soliton and anti-soliton (fermion and antifermion),

$$S_{\text{rep}}^{(n)}(\theta) = \left(\frac{i \exp(\theta) + 1}{\exp(\theta) + i} \right)^{\delta_{r-1}^n}, \tag{2.37}$$

for $A^\pm + N_n \rightarrow N_n + A^\pm$ ($n = 1, 2, \dots, r-1$), and

$$S_{\text{rep}}^{(n,m)}(\theta) = \left(\frac{i \exp(\theta) + 1}{\exp(\theta) + i} \right)^{\delta_{n-1}^m}, \tag{2.38}$$

for $N_n + N_m \rightarrow N_m + N_n$ ($n, m = 1, 2, \dots, r-1$). Note that the soliton S -matrix is the same as the Zamolodchikov’s $S^{SG}(\theta, \gamma/8\pi)$ (2.7) with a renormalized coupling

constant $\gamma_{\text{eff}}/8\pi$. For the coupling constant in the region (2.30), the renormalized coupling constant is equal to

$$\frac{\gamma_{\text{eff}}}{8\pi} = \frac{\gamma}{8\pi} - (r - 1), \quad \text{and} \quad 1 < \frac{\gamma_{\text{eff}}}{8\pi} < 2. \quad (2.39)$$

The neutral particles N_n are not the bound states of the fermion and anti-fermion because the soliton S -matrix in Eq.(2.36) has no poles in the physical strip. These are new excitation modes appearing only in the repulsive regime. In the region $\pi/3 < \mu < \pi/2$ ($r = 1$), since $\gamma_{\text{eff}} = \gamma$, the soliton S -matrix is equal to Eq.(2.7) without any additional particles. Therefore, the Zamolodchikov's results can be analytically continued to a part of the repulsive regime.

We conclude this section with remarks on the first-order phase transition points in the repulsive regime, $\mu = \pi/(r + 2)$. For these values, the Bethe ansatz analysis explained in sect.2 does not work. The reason is as follows: From Eq.(2.28), there are r strings S_n . (The other two strings S_{r+2} and S_{r+1} have positive and zero energies relative to that of a pseudo-particle.) The masses of these strings are degenerate, i.e. S_n has the same mass as S_{N-n} from Eq.(2.29). If there are several different particles with same masses, the S -matrices become non-diagonal, for which the Bethe ansatz analysis (the 'high-level Bethe ansatz') becomes very complicated. It has been claimed that these are the first-order phase transitions.^[22,23]

Taking the limit $\mu \rightarrow \pi/(r + 2)$ of Eq.(2.35), one can find the fermion (the SG soliton) mass becomes infinite. This would mean that one should have only neutral particles in the spectrum after removing the infinite mass solitons.^[26] On the other hand, however, one can choose an appropriate cut-off parameter in the mass renormalization such that the SG soliton mass remains finite. This regularization makes the masses of the neutral particles vanish. This seems to be more consistent with the results based on the lattice SG theory in which the fermion gets well-defined mass.^[22,23] After removing the massless sector from the massive theory, one obtains the SG theory with only solitons and anti-solitons in the spectrum. One can determine the soliton S -matrix from the Yang-Baxter equation which

should be the same form as Eq.(2.7). Therefore, it seems possible to extend the Zamolodchikov's results to the special first-order phase transition points in the repulsive regime. This S -matrix has been used to construct the RSG theory which is connected to the perturbed unitary CFTs.^[4,9,10] From now on, we will consider only the unambiguous region (2.30) in the repulsive regime.

3. Restricted sine-Gordon Theory

After reviewing the RSG theory based on the Zamolodchikov's results, we construct the RSG theory in the repulsive regime based on the Korepin's S -matrices by restricting the multi-soliton Hilbert space of the SG theory using the underlying quantum group symmetry. We connect this RSG theory with the perturbed minimal CFT $\mathcal{M}_{p/p+2}$.

3.1. THE QUANTUM GROUP SYMMETRY OF THE SG THEORY

The quantum group $SL_q(2)$ is defined by the universal enveloping algebra $\mathcal{U}_q[sl(2)]$ with the commutation relations^[24]

$$[J_+, J_-] = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad [h, J_{\pm}] = \pm 2J_{\pm}. \quad (3.1)$$

If the deformation parameter q goes to 1, Eq.(3.1) reduces to the ordinary $sl(2)$ commutation relations and the quantum group $SL_q(2)$ to the ordinary $SL(2)$ group. The $\mathcal{U}_q[sl(2)]$ forms the Hopf algebra with the 'comultiplication' Δ_q , defined by

$$\begin{aligned} \Delta_q(h) &= 1 \otimes h + h \otimes 1 \\ \Delta_q(J_{\pm}) &= q^{h/2} \otimes J_{\pm} + J_{\pm} \otimes q^{-h/2}. \end{aligned} \quad (3.2)$$

The irreducible representations of $SL_q(2)$ are generated by the comultiplication Δ_q which defines tensor product representations. Again, Eq.(3.2) reduces to the usual rule of the addition of angular momentum as $q \rightarrow 1$.

The representations of the $SL_q(2)$ are well-defined; there is one-to-one correspondence to the representation of the ordinary $SL(2)$ which is represented by half-integer spin j . Starting with the fundamental representation and using Eq.(3.2), one can generate all the irreducible representations with higher spins from the

relation

$$|J, M; j_1, j_2\rangle = \sum_{m_1, m_2} \begin{bmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{bmatrix}_q |j_1, m_1\rangle \otimes |j_2, m_2\rangle, \quad (3.3)$$

with the quantum group analogue of the Clebsch-Gordan (CG) coefficients. These quantum CG coefficients (and quantum $6j$ (the Wigner-Racah) symbols) are expressed with ‘ q -numbers’^[24]

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \text{and} \quad [n] \rightarrow n \quad \text{as} \quad q \rightarrow 1. \quad (3.4)$$

If the deformation parameter q is a root of unity, one can see from Eq.(3.4) that some q -CG coefficients (and q - $6j$ symbols) become singular. For this case, the sensible representation theory of the $SL_q(2)$ is possible by restricting the allowed spins to $\{0, 1/2, \dots, j_{\max}\}$. The j_{\max} is determined by the condition that $[2j_{\max} + 1]_q = 0$, which gives

$$j_{\max} = \frac{N}{2} - 1 \quad \text{for} \quad q^N = \pm 1. \quad (3.5)$$

This restriction on the allowed representations of the $SL_q(2)$ with q a root of unity leads to the truncation of the multi-soliton Hilbert space of the SG theory.

The quantum group $SL_q(2)$ can be realized by the R -matrix defined by

$$R(q)(g \otimes 1)(1 \otimes g) = (1 \otimes g)(g \otimes 1)R(q), \quad \text{for} \quad g \in SL_q(2). \quad (3.6)$$

With Eq.(3.2), this means $[R(q), \Delta_q(g)] = 0$ for any $g \in SL_q(2)$. For the fundamental representation (spin-1/2), g is 2×2 matrix with q -determinant 1 ($g_{11}g_{22} - qg_{12}g_{21} = 1$) and the R -matrix is given by

$$R(q) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \quad (3.7)$$

The solution of Yang-Baxter equation can be expressed in terms of the R -matrix

as follows:

$$\widehat{R}(x, q) = x\widehat{R}(q) - x^{-1}\widehat{R}^{-1}(q) \quad \text{with} \quad \widehat{R} = \mathcal{P}R, \quad (3.8)$$

where the permutation \mathcal{P} is defined by $\mathcal{P}(V_1 \otimes V_2) = V_2 \otimes V_1$.

The quantum group $SL_q(2)$ symmetry of the SG theory was discovered by simply noticing that the SG soliton S -matrix can be expressed in terms of the $R(x, q)$, which satisfies the Yang-Baxter equation.^[4-6] The Zamolodchikov's soliton S -matrix (2.7) can be written as

$$\begin{aligned} S^{SG} \left(\theta, \frac{\gamma}{8\pi} \right) &= \frac{U(\theta)}{2\pi i} R \left(x = e^\theta, q \right), \\ q &= -\exp \left(-\frac{i\pi}{P} \right) \quad \text{with} \quad P = \frac{\gamma}{8\pi}, \end{aligned} \quad (3.9)$$

upto a global gauge transformation. This result is rather expected because the Zamolodchikov's S -matrix was derived from the Yang-Baxter equation upto an overall factor.^[17]

Eq.(3.9) means the SG theory has the underlying quantum group symmetry $SL_q(2)$ where the deformation parameter q is related to the coupling constant. The soliton and anti-soliton pair forms the fundamental (spin-1/2) representation from the fact that $[S^{SG}, \Delta_q] = 0$. The multi-soliton states can be represented by the irreducible representations with higher spins which are generated by tensor products like Eq.(3.3). This quantum group symmetry of the SG theory can be derived directly from the SG Lagrangian.^[7]

3.2. THE RESTRICTED SINE-GORDON THEORY

As explained in the sect.1, the RSG theory is a modified SG theory which still preserves the integrability. This is a close analogy of the lattice RSOS model. It is well-known that the eight-vertex model is equivalent to the SOS model, which is obtained simply by changing from the vertex basis to the path basis.^[3] The SOS model is defined by the spins assigned on each face. (Fig.1) These two equivalent

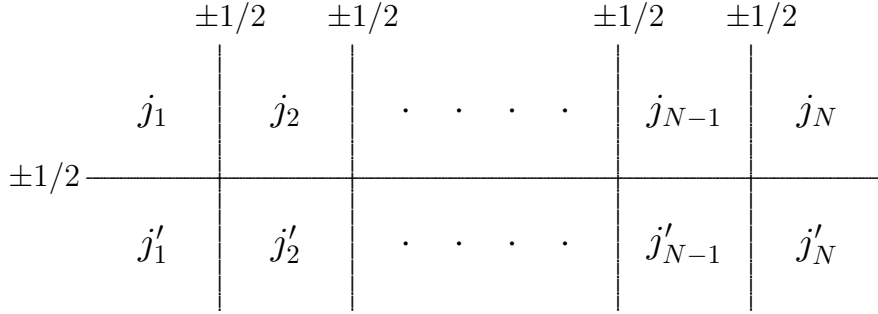


Figure 1. The change of basis from the vertex to the path form

models are described by $c = 1$ CFT at the critical point. A new integrable lattice model, the RSOS model, is constructed by restricting the spins upto a maximum spin. This RSOS model flows to $c < 1$ minimal CFTs in the UV limit (Regime III).

We do exactly the same manipulation for the SG theory to construct the RSG theory. Since a soliton-antisoliton pair forms a spin-1/2 representation $|1/2, \pm 1/2\rangle$ of $SL_q(2)$, we can decompose the multi-soliton Hilbert space into the irreducible spaces characterized by the higher spins (Fig.1):

$$\mathcal{H} = \sum_{m_i = \pm 1/2} |\frac{1}{2}, m_1\rangle \otimes |\frac{1}{2}, m_2\rangle \otimes \cdots \otimes |\frac{1}{2}, m_N\rangle = \sum_{\substack{0 \leq j_i \leq \infty \\ |j_i - j_{i+1}| = 1/2}} |j_1, \cdots, j_N\rangle, \quad (3.10)$$

with appropriate q -CG coefficients.

In this basis, the SG multi-soliton Hilbert space is spanned by the ‘kinks’ K_{ab} with a, b as SOS spins satisfying $|a - b| = 1/2$ and multi-kink states in Fig.1 $|K_{j_1 j_2} \cdots K_{j_{N-1} j_N}\rangle$. The S -matrices for the multi-kink scattering processes can be derived from the corresponding multi-soliton S -matrices with appropriate q -CG coefficients. Like S -matrices in the soliton basis, the multi-kink S -matrices can be decomposed into the products of two-kink S -matrices which satisfy the Yang-Baxter equation. One can write down these kink-kink S -matrices in terms of the SOS R -matrices which are given by the q -6j symbols.^[4-6]

If q is not a root of unity (P is not a rational), the SOS SG theory is well-defined and equivalent to the soliton SG theory. For q a root of unity (a rational P), however, the SOS SG theory is not well-defined. As we explained before, the allowed irreducible representations should be restricted to a maximum spin j_{\max} . This means the multi-soliton Hilbert space should be truncated according to the coupling constant of the SG theory. This is the RSG theory $RSG[P]$ with the kinks K_{ab} ,

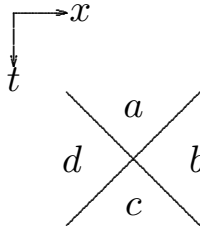
$$\begin{aligned} \text{For } P = \frac{p}{q-p} \text{ with coprime integers } p, q \text{ (} q > p \text{),} \\ K_{ab}, \text{ with } 0 \leq a, b \leq j_{\max} = \frac{p}{2} - 1 \quad |a - b| = \frac{1}{2}. \end{aligned} \quad (3.11)$$

The S -matrices of the RSG kinks are equal to^[4,6] (Fig.2)

$$\begin{aligned} S^{RSG} \begin{bmatrix} ab \\ dc \end{bmatrix} (\theta, P) &= \frac{U(\theta)}{2\pi i} \left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]} \right)^{-\theta/2\pi i} \mathcal{R}_{dc}^{ab}(\theta) \\ \mathcal{R}_{dc}^{ab}(\theta) &= \sinh\left(\frac{\theta}{P}\right) \delta_{db} \left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]} \right)^{1/2} + \sinh\left(\frac{i\pi - \theta}{P}\right) \delta_{ac}, \end{aligned} \quad (3.12)$$

for the process $|K_{da_1}(\theta_1)\rangle + |K_{a_2b}(\theta_2)\rangle \rightarrow |K_{dc_1}(\theta_2)\rangle + |K_{c_2b}(\theta_1)\rangle$ with $a_1 = a_2 = a$ and $c_1 = c_2 = c$. If this condition is not met, the scattering amplitude becomes zero. The explicit expressions of all non-vanishing amplitudes are given in Fig.3. We will discuss the unitarity and crossing symmetry of these S -matrices in the next subsection. In the attractive regime $P < 1$, the RSG S -matrix (3.12) has poles corresponding to the bound states. Since the breathers are singlets of the $SL_q(2)$, the restriction does not change the breather sector. The S -matrices of the breathers in the RSG theory are still given by Eqs.(2.10) and (2.11).

One can identify the RSG theory with the perturbed minimal CFTs under the conjecture that the truncation of the multi-soliton Hilbert space due to the quantum group symmetry is equivalent to adding a background charge to the SG theory.^[4,6] This conjecture can be justified by the argument that the truncation changes



$$K_{da}(\theta_1) + K_{ab}(\theta_2) \rightarrow K_{dc}(\theta_2) + K_{cb}(\theta_1)$$

$$: S^{RSG} \begin{bmatrix} ab \\ dc \end{bmatrix} \left(\theta, \frac{\gamma}{8\pi} \right) \quad (\theta = \theta_1 - \theta_2)$$

Figure 2. The RSOS S -matrices of the RSG theory

$$(1) \quad \begin{array}{c} \diagup l \pm \frac{1}{2} \diagdown \\ l \quad \quad \quad l \pm 1 \\ \diagdown l \pm \frac{1}{2} \diagup \end{array} : \mathcal{R}(\theta) = \sinh \left[\frac{i\pi - \theta}{P} \right]$$

$$(2) \quad \begin{array}{c} \diagup l + \frac{1}{2} \diagdown \\ l \quad \quad \quad l \\ \diagdown l + \frac{1}{2} \diagup \end{array} : \mathcal{R}(\theta) = \frac{\sin[\pi/P]}{\sin[(2l+1)\pi/P]} \sinh \left[\frac{i\pi(2l+1) + \theta}{P} \right]$$

$$(3) \quad \begin{array}{c} \diagup l - \frac{1}{2} \diagdown \\ l \quad \quad \quad l \\ \diagdown l - \frac{1}{2} \diagup \end{array} : \mathcal{R}(\theta) = \frac{\sin[\pi/P]}{\sin[(2l+1)\pi/P]} \sinh \left[\frac{i\pi(2l+1) - \theta}{P} \right]$$

$$(4) \quad \begin{array}{c} \diagup l \mp \frac{1}{2} \diagdown \\ l \quad \quad \quad l \\ \diagdown l \pm \frac{1}{2} \diagup \end{array} : \mathcal{R}(\theta) = \frac{\sqrt{\sin[2l\pi/P] \sin[(2l+2)\pi/P]}}{\sin[(2l+1)\pi/P]} \sinh \left[\frac{\theta}{P} \right]$$

Figure 3. The RSG S -matrices

the SG periodic potential to the Landau-Ginzburg potential which describes the minimal CFTs. The SG theory in Eq.(2.1) can be interpreted as the $c = 1$ free boson theory perturbed by the potential,^[8] $\cos \beta\phi = 1/2[\exp(i\beta\phi) + \exp(-i\beta\phi)]$. If one introduces a background charge, one of the perturbation is identified with the screening operator and the other with the perturbation. For the background charge $\alpha_0 = (p - q)/\sqrt{4pq}$, ($c = 1 - 6(p - q)^2/pq$), if $\Delta[\exp(i\beta\phi)] = 1$,

$$\Delta[\exp(-i\beta\phi)] = \frac{2p-q}{q} = \Delta[\Phi_{1,3}]. \quad (3.13)$$

This leads to the following identification:

$$RSG \left[\frac{\gamma}{8\pi} = \frac{p}{q-p} \right] = \mathcal{M}_{p/q} + \Phi_{1,3}. \quad (3.14)$$

This connection of the RSG theory to perturbed CFT is rather intuitive. The consistency check has been made by the thermodynamic Bethe ansatz.^[27,28]

3.3. THE UNITARITY CONDITION ON THE RSG S -MATRICES

It is easy to check the crossing symmetry of Eq.(3.12)

$$S^{RSG} \begin{bmatrix} ab \\ dc \end{bmatrix} (\theta) = S^{RSG} \begin{bmatrix} bc \\ ad \end{bmatrix} (i\pi - \theta) \quad \text{with} \quad C|K_{ab}\rangle = |K_{ba}\rangle, \quad (3.15)$$

where C is the charge conjugation operator for the kinks. From Eq.(3.11) there are $p-2$ kinks $K_{a,a+1/2}$ and their anti-kinks $K_{a+1/2,a}$ ($a = 0, 1/2, \dots, p/2-1$). A kink and an anti-kink are identified (Majorana) if they have the same scattering amplitude.^[10]

The unitarity of the RSG S -matrix (3.12) gives the condition on the deformation parameter. The unitarity of a S -matrix is

$$S(\theta)^\dagger S(\theta) = 1 \quad \Rightarrow \quad S(\theta)S^T(-\theta) = 1, \quad (3.16)$$

if $S^*(\theta) = S(-\theta)$. In Eq.(2.8), $U^*(\theta) = U(-\theta)$ and $U(\theta)U(-\theta) \propto 1$. Since all the q -numbers are real, the second prefactor also satisfies this. The R -matrices in the RSOS basis have been known to satisfy^[24]

$$\sum_e \mathcal{R}_{de}^{ab}(\theta) \mathcal{R}_{dc}^{eb}(-\theta) = \delta_{ac} \left[2 \cos \frac{2\pi}{P} - 2 \cosh \frac{2\theta}{P} \right]. \quad (3.17)$$

Combining these together, one can prove the relation

$$\sum_e S^{RSG} \begin{bmatrix} ab \\ ed \end{bmatrix}(\theta) S^{RSG} \begin{bmatrix} eb \\ cd \end{bmatrix}(-\theta) = \delta_{ac}. \quad (3.18)$$

Therefore, as far as the relation $\mathcal{R}_{dc}^{ab\star}(\theta) = -\mathcal{R}_{dc}^{ab}(-\theta)$ is satisfied,[★] the RSG S -matrix is unitary.

From the explicit expressions of \mathcal{R}_{dc}^{ab} in Fig.3, this is indeed the case except the last one (4) with the square root. If the factor inside the square root becomes negative, the \mathcal{R} becomes pure imaginary and gets an -1 by the complex conjugation, which breaks the unitarity. Therefore, the RSG S -matrices can be unitary only for P which makes the factor inside the square root positive for all $l = 0, 1/2, \dots, p/2 - 1$. This is possible only for the following values of $P(= \gamma/8\pi)$:^[6]

$$\frac{\gamma}{8\pi} = \frac{p}{q-p} = \frac{N}{Nk+1}, \quad \text{for } N \geq 2, \quad \text{and} \quad \frac{3}{3k+2}, \quad \text{where } k \geq 0. \quad (3.19)$$

This means that the perturbed minimal CFT $\mathcal{M}_{p/q}$ can have the unitary S -matrix only for p, q in Eq.(3.19). This reminds us of the unitarity condition of the minimal CFTs.^[13] In the massive case, the unitary S -matrices are possible not only for the unitary CFTs $\mathcal{M}_{p/p+1}$ but also for some of non-unitary CFTs.

3.4. RSG THEORY IN THE REPULSIVE REGIME

The previous RSG theory was based on the Zamolochikov's S -matrix in the attractive regime. We derive the RSG theory in the repulsive regime $1/(r+2) < \mu/\pi < 1/(r+1)$, $r = 1, 2, \dots$. We use a notation $\overline{RSG}[P]$ for this new RSG theory. In this range, the SG S -matrices are given by Eqs.(2.36)-(2.38). The spectrum is the SG soliton and anti-soliton (with renormalized mass) and $r-1$ neutral particles. The crucial observation is that the quantum group symmetry

[★] Note an extra negative sign which compensates another -1 arising from the complex conjugation of $2\pi i$ in Eq.(3.12).

$SL_q(2)$ still exists because the soliton S -matrix is the same as the Zamolodchikov's with the renormalized coupling constant in Eq.(2.36). Again, the (A^+, A^-) pair becomes the spin-1/2 representation of the quantum group $SL_q(2)$ from

$$\left[S_{\text{rep}}^{SG} \left(\theta, \frac{\gamma}{8\pi} \right), \Delta_{q_{\text{eff}}} \right] = 0, \quad (3.20)$$

with a modified deformation parameter

$$q_{\text{eff}} = -\exp \left(-\frac{i\pi}{P_{\text{eff}}} \right), \quad \text{with} \quad P_{\text{eff}} = \frac{\gamma}{8\pi} - (r - 1). \quad (3.21)$$

Using this quantum group $SL_{q_{\text{eff}}}(2)$ symmetry, one can restrict the multi-soliton Hilbert space for a rational value of P_{eff} . Since $1 < P_{\text{eff}} < 2$ from Eq.(2.39), we parametrize a rational P_{eff} with two coprime integers n, m ($n > m$)

$$P_{\text{eff}} = 1 + \frac{m}{n} \quad \Rightarrow \quad \frac{\gamma}{8\pi} = \frac{nr + m}{n}. \quad (3.22)$$

This means schematically the following relationship:

$$\overline{RSG} \left[\frac{nr + m}{n} \right] = RSG \left[\frac{n + m}{n} \right]. \quad (3.23)$$

Therefore, the kink S -matrices of the new RSG theory $\overline{RSG}[P]$ are given by $S^{RSG}(\theta, P_{\text{eff}})$ in Eq.(3.12) and Fig.3. The neutral particle sector, being singlets of the quantum group, is unchanged by the restriction. The S -matrices of neutrals with neutrals or kinks are Eqs.(2.37) and (2.38).

From the unitarity condition on the RSG S -matrices, P_{eff} should be one of the values given in Eq.(3.19). Since $P_{\text{eff}} > 1$, only solution is $P_{\text{eff}} = 3/2$. From Eq.(3.22), the only RSG theory with the unitary kink S -matrices in the repulsive regime is $\overline{RSG}[p/2]$ ($p = 2r + 1$). The connection of this RSG theory to the

$$\begin{array}{cc}
\begin{array}{c} \diagup \quad \frac{1}{2} \quad \diagdown \\ 0 \quad \quad \quad 0 \\ \diagdown \quad \frac{1}{2} \quad \diagup \end{array} & \begin{array}{c} \diagup \quad 0 \quad \diagdown \\ \frac{1}{2} \quad \quad \quad \frac{1}{2} \\ \diagdown \quad 0 \quad \diagup \end{array} \\
S_{K,\bar{K}} & S_{\bar{K},K}
\end{array}$$

Figure 4. $\overline{RSG}[(2r+1)/2]$ kink S -matrices

perturbed CFTs follows exactly the same way as before, which can be expressed schematically from Eq.(3.14),

$$\overline{RSG}\left[\frac{p}{2}\right] = \mathcal{M}_{p/p+2} + \Phi_{1,3}, \quad p = 2r + 1, \quad r = 1, 2, \dots \quad (3.24)$$

For $P_{\text{eff}} = 3/2$, there are two kinks $K = K_{0,1/2}$ and $\bar{K} = K_{1/2,0}$ and are related to each other by the charge conjugation $C(K) = \bar{K}$. The kink S -matrices $S^{RSG}(\theta, 3/2)$ are equal to (Fig.4)

$$S_{K,\bar{K}}(\theta) = -S_{\bar{K},K}(\theta) = -i \tanh\left[\frac{1}{2}\left(\theta - \frac{i\pi}{2}\right)\right]. \quad (3.25)$$

These two scattering amplitudes, related by the crossing symmetry, are same upto overall sign. This makes one to identify two kinks, $K = \bar{K}$. In this case, the crossing relation of Eq.(3.25) gets a factor -1 . This is the ‘ \star -violated’ phenomena.^[6] Other scattering amplitudes (kink-neutral, neutral-neutral) are given by Eqs.(2.37) and (2.38),

$$\begin{aligned}
S_{K,N}(\theta) &= \left(\frac{i \exp(\theta) + 1}{\exp(\theta) + i}\right)^{\delta_{r-1}^n}, \quad n = 1, 2, \dots, r-1 \\
S_{N,N}(\theta) &= \left(\frac{i \exp(\theta) + 1}{\exp(\theta) + i}\right)^{\delta_{n-1}^m}, \quad n, m = 1, 2, \dots, r-1.
\end{aligned} \quad (3.26)$$

All these scattering amplitudes are diagonal.

4. Thermodynamic Bethe Ansatz for the RSG theory

In this section, we use thermodynamic Bethe ansatz (TBA) to confirm the RSG S -matrices (3.25) and (3.26) to be those of the perturbed minimal CFT $\mathcal{M}_{p/p+2}$. The TBA^[25–28] is basically the same as the Bethe ansatz method except the fact that particles involved are the physical particles. We want to find out the wave functions of the physical particles to compute the energy spectrum. The ground state energy is connected with the central charge of the underlying UV CFT by the relation

$$\begin{aligned} E_0(R) &\sim -\frac{2\pi}{R} \left(\frac{c}{12} - 2\Delta_0 \right), \quad \text{as } R \rightarrow 0, \\ &= -\frac{2\pi}{R} \left(\frac{1}{12} - \frac{1}{2pq} \right) \quad \text{for } \mathcal{M}_{p/q}, \end{aligned} \quad (4.1)$$

where R is the inverse temperature ($R \rightarrow 0$ means the UV limit) and Δ_0 is the minimal conformal dimension allowed in the CFT (for unitary CFT, $\Delta_0 = 0$).

The integral equation for TBA from the PBC is

$$2\pi\rho(\theta) = -m \cosh \theta - \int_{-\infty}^{\infty} d\theta' \left[\frac{\partial}{\partial \theta} \Phi(\theta - \theta') \right] \rho_1(\theta'), \quad (4.2)$$

where ρ is the density of the allowed states and ρ_1 is the density of the occupied states by the physical particles. Note that in TBA the density of the allowed states is not necessarily the same as that of the occupied states. Φ is the phase of the physical scattering amplitude

$$\Phi(\theta) = -i \ln S(\theta), \quad (4.3)$$

if the S -matrices are diagonal. For non-diagonal S -matrices, one should diagonalize the transfer matrix. This has been succeeded only for part of the RSG theories.^[27,28]

Since the S -matrices of the new RSG theory are all diagonal, the diagonal TBA method is enough for our case.

One needs an extra equation to solve the integral equation involving two unknown variables like Eq.(4.2). In the high temperature limit, this extra equation comes from the minimization of the free energy. This is the TBA equation. If we introduce the ‘pseudo-energy’ defined by

$$\frac{\rho_1}{\rho} = \frac{e^{-\epsilon}}{1 + e^{-\epsilon}}, \quad (4.4)$$

the TBA equation is given by

$$Rm \cosh \theta = \epsilon(\theta) + \frac{1}{2\pi} [\phi' * \ln(1 + e^{-\epsilon})](\theta), \quad (4.5)$$

where the rapidity convolution is $[\phi' * f](\theta) = \int_{-\infty}^{\infty} \phi'(\theta - \theta') f(\theta') d\theta'$. The ground state energy is expressed with the pseudo-energy,

$$E_0(R) = \frac{m}{\pi} \int_0^{\infty} d\theta \cosh \theta \ln \left(1 + e^{-\epsilon(\theta)} \right). \quad (4.6)$$

The TBA equation can be solved in the $R \rightarrow 0$ limit and the ground state energy can be simply expressed in terms of the Rogers dilogarithmic functions.^[25–28]

The particle spectrum of the new RSG theory is one kink K ($a = r$) and $(r - 1)$ neutrals N_a ($a = 1, \dots, r - 1$) with all different masses where the S -matrices $S_{ab}(\theta)$ are given by Eqs.(3.25) and (3.26). Introducing r pseudo-energies $\epsilon_a(\theta)$, we find the following TBA equations:

$$Rm_a \cosh \theta = \epsilon_a(\theta) + \frac{1}{2\pi} \sum_{b=1}^r [\Phi'_{ab} * \ln(1 + e^{-\epsilon_b})](\theta), \quad (4.7)$$

$$\Phi'_{ab}(\theta) = -i \frac{\partial}{\partial \theta} \ln S_{ab}(\theta) \quad \text{for } a, b = 1, 2, \dots, r.$$

In terms of these pseudo-energies, the ground state energy becomes

$$E_0(R) = \frac{m}{\pi} \sum_{a=1}^r \int_0^{\infty} d\theta \cosh \theta \ln \left(1 + e^{-\epsilon_a(\theta)} \right). \quad (4.8)$$

In the UV limit ($R \rightarrow 0$), the pseudo-energies $\epsilon_a(\theta)$ have constant values $\epsilon_a(0)$ in the region $|\theta| \lesssim 1/R$. It is straightforward to express the ground state energy

$E_0(R)$ as an integral over the pseudo-energy,^[25–28]

$$\begin{aligned} E_0(R) &\sim -\frac{1}{4\pi R} \sum_{a=1}^r \int_{\epsilon_a(0)}^{\epsilon_a(\infty)} d\epsilon \left[\ln(1 + e^{-\epsilon}) + \frac{\epsilon e^{-\epsilon}}{1 + e^{-\epsilon}} \right], \\ &= -\frac{1}{\pi R} \sum_{a=1}^r \left[\mathcal{L} \left(\frac{x_a}{1 + x_a} \right) - \mathcal{L} \left(\frac{y_a}{1 + y_a} \right) \right], \end{aligned} \quad (4.9)$$

where $x_a = \exp[-\epsilon_a(0)]$ and $y_a = \exp[-\epsilon_a(\infty)]$. The Rogers dilogarithmic function $\mathcal{L}(x)$ is defined by

$$\mathcal{L}(x) = -\frac{1}{2} \int_0^x dt \left[\frac{\ln(1-t)}{t} + \frac{\ln t}{(1-t)} \right]. \quad (4.10)$$

The pseudo-energies $\epsilon_a(0)$ can be determined by algebraic equations. As $\theta \rightarrow 0$ in Eq.(4.7), x_a 's satisfy the following algebraic equations:

$$x_a = \prod_{b=1}^r (1 + x_b)^{N_{ab}}, \quad \text{for } a = 1, \dots, r, \quad (4.11)$$

where $N_{ab} = [\Phi_{ab}(\infty) - \Phi_{ab}(-\infty)]/2\pi$ can be easily computed from Eqs.(3.25) and (3.26):

$$N = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \dots & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \quad (4.12)$$

We have solved these coupled equations to get the solutions

$$x_a = \frac{\sin \left(\frac{(a+2)\pi}{2r+3} \right) \sin \left(\frac{a\pi}{2r+3} \right)}{\sin^2 \left(\frac{\pi}{2r+3} \right)} \quad \text{for } a = 1, 2, \dots, r. \quad (4.13)$$

One can check that these solutions satisfy Eq.(4.11). In terms of these solutions,

we have found the following sum rule for the Rogers dilogarithmic functions numerically:

$$\frac{1}{2\pi^2} \sum_{a=1}^r \mathcal{L} \left(\frac{x_a}{1+x_a} \right) = \frac{r(2r+1)}{12(2r+3)}. \quad (4.14)$$

The $\theta \rightarrow \infty$ limit of Eq.(4.7) is more subtle because one needs to know the mass renormalization for the finite temperature. The mass renormalization will be again given by Eq.(2.35) with a possible temperature dependent correction. Without this correction, the dimensionless combination RM_a in the left-hand side of Eq.(4.7) becomes $m_0 R \exp[-\Lambda]$ from Eq.(2.35), which vanishes as $R \rightarrow 0$ and $\Lambda \rightarrow \infty$.^{*} However, one may get additional finite mass corrections due to the finite temperature effect. From the dimensional analysis, the correction should be $\delta M_a \propto 1/R$. We conjecture that this happens only for the lowest mass term allowed in the spectrum, that is, M_1 of the first neutral particle N_1 . This is a natural guess because the lowest mass particle will be most easily excited by the thermal effect. This leads to the conjecture that the left-hand side of Eq.(4.7) vanishes for $a = 2, 3, \dots, r$ and diverges for $a = 1$ as $\theta \rightarrow \infty$. Similar conjecture has been made for TBA computation in ref.[26].

According to this speculation, the algebraic equations for $y_a = \exp[-\epsilon_a(\infty)]$ are given by Eq.(4.11) with $y_1 = 0$

$$y_a = \prod_{b=2}^r (1 + y_b)^{N_{ab}}, \quad \text{for } a = 2, \dots, r. \quad (4.15)$$

The solutions are given by Eq.(4.13) with r replaced by $r - 1$,

$$y_a = \frac{\sin \left(\frac{(a+1)\pi}{2r+1} \right) \sin \left(\frac{(a-1)\pi}{2r+1} \right)}{\sin^2 \left(\frac{\pi}{2r+1} \right)}, \quad \text{for } a = 1, 2, \dots, r. \quad (4.16)$$

Therefore, from the sum rule Eq.(4.14) with $y_r = 0$ one can evaluate the ground

^{*} In the attractive regime, from Eq.(2.26), this term is equal to $m_0 R \exp[\Lambda]$, which is non-zero in the double limit. Therefore, the pseudo-energies $\epsilon_a(\Lambda)$ in the attractive regime diverge.

state energy E_0 as follows:

$$\begin{aligned}
E_0(R) &\sim -\frac{2\pi}{R} \left[\frac{r(2r+1)}{12(2r+3)} - \frac{(r-1)(2r-1)}{12(2r+1)} \right], \\
&= -\frac{2\pi}{R} \left[\frac{1}{12} - \frac{1}{2(2r+1)(2r+3)} \right].
\end{aligned} \tag{4.17}$$

Comparing this with Eq.(4.1), one can confirm that the $RSG[(2r+1)/2]$ theory is indeed the perturbed minimal CFT $\mathcal{M}_{2r+1/2r+3}$ by the $\Phi_{1,3}$.

5. Discussions

In this paper, we studied the SG (MTM) theory and its restricted version (the RSG theory) in the repulsive regime and identified the RSG theory with the perturbed minimal CFT $\mathcal{M}_{p/p+2}$ by the least relevant operator. The complete S -matrices, satisfying the unitarity and crossing symmetry, and the particle spectrum have been derived and checked using the TBA analysis. We discovered the quantum group $SL_q(2)$ of the SG theory in the repulsive regime where the deformation parameter is modified. It would be nice to understand this quantum group symmetry in the repulsive regime starting with the SG lagrangian.^[7]

The RSG theory is a building block for a wide class of two-dimensional integrable QFTs.^[9,10] Many new integrable models have been identified with the general CFTs perturbed by the massive operators along with exact S -matrices. In particular, the complete S -matrices of the supersymmetric sine-Gordon (SSG) theory has been derived by ‘unrestricting’ the restricted SSG theory.^[9] The S -matrices of the SSG theory have the factorized form $S^{SSG}(\theta) = S^{RSG}(\theta) \otimes S^{SG}(\theta)$, where the first factor is the RSG S -matrix for $\gamma/8\pi = 4$ which commutes with the supersymmetric charge and the second one is the Zamolodchikov’s S -matrices. This leads us to suggest the exact S -matrices of the SSG theory in the repulsive regime have the following form:

$$S_{\text{rep}}^{SSG}(\theta) = S^{RSG}(\theta) \otimes S_{\text{rep}}^{SG}(\theta). \quad (5.1)$$

Since the first factor carries the supersymmetric charge, the particle spectrum should be not only the solitons, anti-solitons, and neutrals but also their supersymmetric partners. Furthermore, by restricting the multi-soliton (SSG) Hilbert space, one can derive the S -matrices of the perturbed superminimal CFTs $SU(2)_2 \otimes SU(2)_L/SU(2)_{L+2}$ (L a half-integer) by the least relevant operator to be $S^{RSG} \otimes S_{\text{rep}}^{RSG}$ where S_{rep}^{RSG} ’s are given by Eqs.(3.25),(3.26). The extension to the CFTs with fractional supersymmetry is straightforward by considering higher level ($K > 2$) for the first $SU(2)$. We will report the details elsewhere.

We finish this paper with some open questions. One is an ambiguity in the first-order phase transition points which we have discussed at the end of the sect.2; there exist two different pictures of the SG theory with $\gamma/8\pi =$ integers in the literature (refs.[4] and [26]). One may need the high-level Bethe ansatz analysis to resolve the ambiguity. Another is about our conjecture for the delicate limit of the infinite rapidity and $R \rightarrow 0$ in the TBA computation. One needs to understand how the mass renormalization in the Bethe ansatz approach is affected by the finite temperature effect. Although we could not give the answer to this, our TBA computation is still a non-trivial check in that the complicated coupled integral equations satisfy new sum rule of the Rogers Dilogarithmic functions and correctly reproduce the central charge.

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