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Reflection amplitudes in non-simply laced Toda theories and thermodynamic Bethe ansatz

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Abstract

We study the ultraviolet asymptotics in non-simply laced affine Toda theories considering them as perturbed non-affine Toda theories, which possess the extended conformal symmetry. We calculate the reflection amplitudes, in non-affine Toda theories and use them to derive the quantization condition for the vacuum wave function, describing zero-mode dynamics. The solution of this quantization conditions for the ground state energy determines the UV asymptotics of the effective central charge. These asymptotics are in a good agreement with Thermodynamic Bethe Ansatz (TBA) results. To make the comparison with TBA possible, we give the exact relations between parameters of the action and masses of particles as well as the bulk free energies for non-simply laced affine Toda theories.

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1. Introduction

There is a large class of massive 2D integrable quantum field theories (IQFTs), which can be considered as perturbed conformal field theories (CFTs) [1]. The ultraviolet (UV) behavior of these IQFTs is encoded in the CFT data while their long distance properties are defined by the S-matrix data. If the basic CFT admits the representation of the primary fields of full symmetry algebra in terms of the exponential fields the CFT data include ''reflection amplitudes''. These functions define the linear transformations between different exponential fields, corresponding to the same primary field. Reflection amplitudes play the crucial role for the calculation of the one point functions [2] as well as for the description of the zero mode dynamics [3–5] in integrable perturbed CFTs. In particular, the zero mode dynamics determines the UV asymptotics of the ground state energy $E(R)$ (or effective central charge $c_{\text{eff}}(R)$) for the system on the circle of size R. The function $c_{\text{eff}}(R)$ admits in this case the UV series

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expansion in the inverse powers of $\log(1/R)$. The solution of the quantization condition for the vacuum wave function (which can be written in terms of the reflection amplitudes), supplemented with the exact relations between the parameters of the action and the masses of the particles determines all logarithmic terms in this UV expansion.

The effective central charge $c_{\rm eff}(R)$ in IQFT can be calculated independently from the S-matrix data using the TBA method [6,7]. At small R its asymptotics can be compared with that following from the CFT data. In the case when the basic CFT is known the agreement of both approaches can be considered as nontrivial test for the S-matrix amplitudes in IQFT. The corresponding analysis based on the both approaches was previously done for the sinh-Gordon [3], super-symmetric sinh-Gorgon, Bullough-Dodd [4] models and simply-laced affine Toda field theories (ATFTs) [5].

In this paper we study the UV behavior of the effective central charge in ATFTs associated with non-simply laced Lie algebras. These IQFTs have two different classical limits. Namely, the weak and strong coupling limits correspond to the dual pairs of affine Toda theories. As a result, the mass ratios in these IQFTs depend on the coupling constant and flow from the classical values characteristic for Lie algebra G [1] to the same values for the dual algebra $G^\vee$ [8]. The number of particles in ATFTs is equal to the rank r of G. For large r the numerical analysis of TBA equations, especially in the UV region, becomes rather complicated. The analytical approach to the TBA equations [10,11] does not give, at present, the regular UV expansion. So, it is useful to have the full logarithmic expansion for $c_{\rm eff}(R)$ following from CFT data. The agreement of this expansion with the TBA results confirms the S-matrix as well as the relations between the parameters of the action and masses of particles in non-simply laced ATFT.

The remarkable feature of ATFT is that effective central charge calculated from the CFT data with subtracted bulk free energy term (like in TBA approach) gives a good agreement with the TBA results even outside the UV region (at $R \sim \mathcal{O}(1)$). This ''experimental'' fact still needs the explanation.

The rest of the paper is organized as follows. After introduction of some basic notations we give the exact relations between the parameters of the action and masses of particles in non-simply laced ATFTs. Then following the procedure of Ref. [5], we obtain the reflection amplitudes and quantization conditions for the wave function, describing the vacuum zero mode dynamics. Using these results we calculate the UV asymptotics of the effective central charges for ATFTs and compare these asymptotics with numerical data following from TBA equations. We omit here the details, which can be found in Ref. [5], devoted to the analysis of UV asymptotics in simply laced ATFTs.

2. Mass-μ relations and reflection amplitudes

The ATFTs corresponding to Lie algebra G is described by the action

$$\mathcal{A} = \int d^2x \left[\frac{1}{8\pi} \left(\partial_\mu \varphi\right)^2 + \sum_{i=1}^{r} \mu_i e^{b\boldsymbol{e}_i \cdot \varphi} + \mu_0 e^{b\boldsymbol{e}_0 \cdot \varphi} \right], \tag{1}$$

where $\boldsymbol{e}_i$, $i = 1, \ldots, r$ are the simple roots of the Lie algebra G of rank r and $-\boldsymbol{e}_0$ is a maximal root, satisfying the relation:

$$\sum_{i=0}^{r} n_i \boldsymbol{e}_i = 0, \quad n_0 = 1. \tag{2}$$

Non-simply laced ATFTs have standard simple roots with $\boldsymbol{e}_i^2 = 2$ and nonstandard simple roots with $\boldsymbol{e}_i^2 \equiv \xi^2 (\neq 2)$. We choose the corresponding parameters μ_i as μ (for standard roots) and μ' (for nonstandard ones) respectively [2].

In the case of non-simply laced ATFTs, the exact mass ratios are different from the classical ones and

[1] Throughout the paper, we denote an untwisted algebra as G, while $G^\vee$ refers to a twisted one.

[2] We choose the convention that the length squared of the long roots are four for $C_r^{(1)}$ and two for the other untwisted algebras.

get quantum corrections [8,9]. To describe the spectrum it is convenient to introduce the notations:

$$B = \frac{b^2}{1+b^2}, \quad H = \frac{h + b^2 h^\vee}{1+b^2}, \tag{3}$$

where h and $h^\vee$ are Coxeter and dual Coxeter numbers of the algebra. Then the spectrum of ATFTs can be expressed in terms of one mass parameter $\overline{m}$ as:

$$\begin{aligned}
&B_r^{(1)}: \quad M_r = \overline{m}, \quad M_a = 2\overline{m}\sin(\pi a/H), \\
&\quad a = 1,2,\ldots,r-1 \\
&C_r^{(1)}: \quad M_a = 2\overline{m}\sin(\pi a/H), \quad a = 1,2,\ldots,r \\
&G_2^{(1)}: \quad M_1 = \overline{m}, \quad M_2 = 2\overline{m}\cos(\pi(1/3 - 1/H)) \\
&F_4^{(1)}: \quad M_1 = \overline{m}, \\
&\quad M_2 = 2\overline{m}\cos(\pi(1/3 - 1/H)), \\
&\quad M_3 = 2\overline{m}\cos(\pi(1/6 - 1/H)), \\
&\quad M_4 = 2M_2\cos(\pi/H).
\end{aligned} \tag{4}$$

The relation between the parameter $\overline{m}$ in the above spectra and the parameters μ_i in the action (1) can be obtained by Bethe Ansatz method (see for example [12,13]). The corresponding analysis gives:

$$\begin{aligned}
&\prod_{i=0}^{r} \left[-\pi\mu_i\gamma\left(1 + \boldsymbol{e}_i^2 b^2/2\right)\right]^{n_i} \\
&\quad = \left[\frac{\overline{m}k(G)}{2}\Gamma\left(\frac{1-B}{H}\right)\Gamma\left(1+\frac{B}{H}\right)\right]^{2H(1+b^2)},
\end{aligned} \tag{5}$$

where, as usual $\gamma(x) = \Gamma(x)/\Gamma(1-x)$, and $k(G)$ is a function depending on the algebra:

$$\begin{aligned}
&k\left(B_r^{(1)}\right) = \frac{2^{-2/H}}{\Gamma(1/H)}, \quad k\left(C_r^{(1)}\right) = \frac{2^{2B/H}}{\Gamma(1/H)}, \\
&k\left(G_2^{(1)}\right) = \frac{\Gamma(2/3)}{2\Gamma(1/2)\Gamma(1/6+1/H)}, \\
&k\left(F_4^{(1)}\right) = \frac{\Gamma(2/3)}{2\Gamma(1/2)\Gamma(1/6+1/H)}.
\end{aligned} \tag{6}$$

The similar relations for the dual ATFTs can be easily obtained from Eqs. (5) and (6) if we use the duality relations for the parameters μ_i and $\mu_i^\vee$ corresponding to the dual pairs of ATFTs:

$$\pi\mu_i\gamma\left(\frac{\boldsymbol{e}_i^2 b^2}{2}\right) = \left(\pi\mu_i^\vee\gamma\left(\frac{2}{\boldsymbol{e}_i^2 b^2}\right)\right)^{\boldsymbol{e}_i^2 b^2/2}. \tag{7}$$

The ATFTs can be considered as perturbed CFTs. Without the last term with the zeroth root $\boldsymbol{e}_0$, the action in Eq. (1) describes the non-affine Toda theory (NATT), which is conformal. To describe the generator of conformal symmetry we introduce the complex coordinates $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ and vector:

$$\begin{aligned}
&\boldsymbol{Q} = b\boldsymbol{\rho} + \frac{1}{b}\boldsymbol{\rho}^\vee, \quad \boldsymbol{\rho} = \frac{1}{2}\sum_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}, \\
&\boldsymbol{\rho}^\vee = \frac{1}{2}\sum_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}^\vee,
\end{aligned} \tag{8}$$

where the sum in definition of Weyl vector $\boldsymbol{\rho}$ ($\boldsymbol{\rho}^\vee$) runs over all positive roots $\boldsymbol{\alpha}$ (co-roots $\boldsymbol{\alpha}^\vee$) of G.

The holomorphic stress-energy tensor

$$T(z) = -\tfrac{1}{2}(\partial_z\boldsymbol{\varphi})^2 + \boldsymbol{Q}\cdot\partial_z^2\boldsymbol{\varphi} \tag{9}$$

ensures the local conformal invariance of the NATT with the central charge $c = r + 12\boldsymbol{Q}^2$.

Besides the conformal invariance the NATT possesses extended symmetry generated by $W(G)$-algebra. The full chiral $W(G)$-algebra contains r holomorphic fields $W_j(z)$ ($W_2(z) = T(z)$) with spins j which follows the exponents of Lie algebra G. The primary fields Φ_w of $W(G)$ algebra are classified by r eigenvalues w_j, $j = 1,\ldots,r$ of the operator $W_{j,0}$ (the zeroth Fourier component of the current $W_j(z)$):

$$W_{j,0}\Phi_w = w_j\Phi_w, \quad W_{j,n}\Phi_w = 0, \quad n > 0. \tag{10}$$

The exponential fields

$$V_{\boldsymbol{a}}(x) = e^{(\boldsymbol{Q}+\boldsymbol{a})\cdot\boldsymbol{\varphi}(x)} \tag{11}$$

are spinless conformal primary fields with dimensions $\Delta(\boldsymbol{a}) = w_2(\boldsymbol{a}) = (\boldsymbol{Q}^2 - \boldsymbol{a}^2)/2$. The fields $V_{\boldsymbol{a}}$ are also primary with respect to all chiral algebra $W(G)$ with the eigenvalues w_j depending on $\boldsymbol{a}$. The functions $w_j(\boldsymbol{a})$, which define the representation of $W(G)$-algebra possess the symmetry with respect to the Weyl group $\mathscr{W}$ of Lie algebra G [14,15], i.e.

$w_j(\hat{s}\boldsymbol{a}) = w_j(\boldsymbol{a})$; for any $\hat{s} \in \mathscr{W}$. It means that the fields $V_{\hat{s}\boldsymbol{a}}$ for different $\hat{s} \in \mathscr{W}$ are reflection images of each other and are related by the linear transformation:

$$V_{\boldsymbol{a}}(x) = R_{\hat{s}}(\boldsymbol{a}) V_{\hat{s}\boldsymbol{a}}(x), \tag{12}$$

where $R_{\hat{s}}(\boldsymbol{a})$ is the ''reflection amplitude''.

This function plays an important role in the analysis of perturbed CFTs. It can be calculated by the CFT methods (exactly in the same way as it was done for the simply laced NATTs in [5]) and has the form:

$$R_{\hat{s}}(\boldsymbol{a}) = \frac{A_{\hat{s}\boldsymbol{a}}}{A_{\boldsymbol{a}}}, \tag{13}$$

where

$$A_{\boldsymbol{a}} = \prod_{i=1}^{r} \left[\pi\mu_i \gamma\left(\boldsymbol{e}_i^2 b^2/2\right)\right]^{\boldsymbol{\omega}_i^\vee \boldsymbol{a}/b} \times \prod_{\boldsymbol{\alpha}>0} \Gamma(1 - a_{\boldsymbol{\alpha}^\vee}/b)\, \Gamma(1 - a_{\boldsymbol{\alpha}} b), \tag{14}$$

here $a_{\boldsymbol{\alpha}} = \boldsymbol{a}\cdot\boldsymbol{\alpha}$, $a_{\boldsymbol{\alpha}^\vee} = \boldsymbol{a}\cdot\boldsymbol{\alpha}^\vee$ and vectors $\boldsymbol{\omega}_i^\vee$ are the co-weights of G, satisfying the condition $\boldsymbol{\omega}_i^\vee \cdot \boldsymbol{e}_j = \delta_{ij}$

In following we will be interested in the values of function $A(\boldsymbol{P}) = A_{i\boldsymbol{P}}$. We note that in the semiclassical limit ($b \to 0$ with $\boldsymbol{P}/b$ fixed) the functions $A(\hat{s}\boldsymbol{P})$ coincide with the amplitudes describing the asymptotics of the wave function of quantum mechanical non-affine Toda chain (16) (see for example [16]).

3. Quantization condition and UV expansion

Function $A(\boldsymbol{P})$ plays an important role in study of quantum mechanical problem for zero modes

$$\boldsymbol{\varphi}_0 = \int_0^{2\pi} \boldsymbol{\varphi}(x) \frac{dx_1}{2\pi}, \tag{15}$$

of the fields $\boldsymbol{\varphi}(x)$ defined on an infinite cylinder of circumference 2π with coordinate x_2 along the cylinder playing the role of imaginary time. In the semiclassical limit $b \to 0$, where one can neglect the oscillator modes of $\boldsymbol{\varphi}(x)$, the Schrödinger equation governing the zero-mode dynamics is given by:

$$\left[-\frac{r}{12} - \boldsymbol{\nabla}^2_{\boldsymbol{\varphi}_0} + \sum_{i=1}^{r} 2\pi\mu_i e^{b\boldsymbol{e}_i\cdot\boldsymbol{\varphi}_0}\right]\Psi_{\boldsymbol{P}}(\boldsymbol{\varphi}_0) = E_0 \Psi_{\boldsymbol{P}}(\boldsymbol{\varphi}_0) \tag{16}$$

with the energy

$$E_0 = -\frac{r}{12} + \boldsymbol{P}^2, \tag{17}$$

where the momentum $\boldsymbol{P}$ is a real vector. The full quantum effect can be implemented simply by introducing the exact reflection amplitudes which take into account also non-zero-mode contributions [3].

The wave function $\Psi_{\boldsymbol{P}}(\boldsymbol{\varphi}_0)$ in the asymptotic region (Weyl chamber) can be found by using the same arguments as was given in [5] for simply laced NATTs. The only difference is that there are now two kinds of roots with different lengths. Namely, each exponential term $\mu_i e^{b\boldsymbol{e}_i\boldsymbol{\varphi}_0}$ in the Hamiltonian can be considered as a potential wall normal to the $\boldsymbol{e}_i$ direction. An incident wave is reflected by this wall to the wave with the Weyl-reflected momentum. The phase change corresponding to this process should be the same as in Liouville field theory. By considering the reflections from all potential walls, we find that the wave function $\Psi_{\boldsymbol{P}}(\boldsymbol{\varphi}_0)$ can be written as a superposition of plane waves with the momenta forming the orbit of the Weyl group $\mathscr{W}$ of Lie algebra G,

$$\Psi_{\boldsymbol{P}}(\boldsymbol{\varphi}_0) \simeq \sum_{\hat{s}\in\mathscr{W}} A(\hat{s}\boldsymbol{P})\, e^{i\hat{s}\boldsymbol{P}\cdot\boldsymbol{\varphi}_0}, \tag{18}$$

where

$$A(\boldsymbol{P}) = \prod_{i=1}^{r} \left[\pi\mu_i \gamma\left(\boldsymbol{e}_i^2 b^2/2\right)\right]^{i\boldsymbol{\omega}_i^\vee \boldsymbol{P}/b} \times \prod_{\boldsymbol{\alpha}>0} \Gamma(1 - iP_{\boldsymbol{\alpha}} b)\, \Gamma(1 - iP_{\boldsymbol{\alpha}^\vee}/b). \tag{19}$$

For the Weyl element $\hat{s}_i$, associated with the simple root $\boldsymbol{e}_i$, the ratio $A(\hat{s}_i\boldsymbol{P})/A(\boldsymbol{P})$should be given by

the reflection amplitude $S_L(\boldsymbol{e}_i,\boldsymbol{P})$ [3] of the Liouville field theory

$$\frac{A(\hat{s}_i\boldsymbol{P})}{A(\boldsymbol{P})} = S_L(\boldsymbol{e}_i,\boldsymbol{P}) = \left[\pi\mu_i\gamma\left(\boldsymbol{e}_i^2 b^2/2\right)\right]^{-i\boldsymbol{P}\cdot\boldsymbol{e}_i^\vee/b} \times \frac{\Gamma(1+i\boldsymbol{P}\cdot\boldsymbol{e}_i b)\,\Gamma\left(1+i\boldsymbol{P}\cdot\boldsymbol{e}_i^\vee/b\right)}{\Gamma(1-i\boldsymbol{P}\cdot\boldsymbol{e}_i b)\,\Gamma\left(1-i\boldsymbol{P}\cdot\boldsymbol{e}_i^\vee/b\right)}. \tag{20}$$

One can easily check that function $A(\boldsymbol{P})$ satisfies this functional equation. With this function one can proceed to obtain the scaling functions in the UV region of the ATFTs defined on a cylinder with circumference $R\to 0$. The additional term in the ATFT Lagrangian corresponding to the zeroth root $\boldsymbol{e}_0$ introduces new potential wall in that direction. With this term the Weyl chamber is now closed and the momentum $\boldsymbol{P}$ of the wave function should be quantized. It depends on the size of the enclosed region, which is proportional to $\log(1/R)$. This quantized momentum $\boldsymbol{P}(R)$ defines the scaling function c_{eff} in the UV region by Eq. (17).

It is convenient to rescale back the size of the system from R to 2π. This leads to the following rescaling of the parameters μ_i in the action (1):

$$\mu_i \to \nu_i = \mu_i\left(\frac{R}{2\pi}\right)^{2+b^2\boldsymbol{e}_i^2}. \tag{21}$$

In the UV limit the size of enclosed region is rather big and we can neglect the subtleties of interaction (which give only exponential corrections) taking into account only the phase shifts coming from the reflections of the waves by the potential walls. Since the additional potential term is not different from the others, the amplitude $A(\hat{s}\boldsymbol{P})$ with the momentum $\hat{s}\boldsymbol{P}$ (where $\hat{s}$ is an arbitrary element of Weyl group) has to satisfy also the reflection relation (20) with respect to the zeroth root $\boldsymbol{e}_0$

$$\frac{A(\hat{s}_0\hat{s}\boldsymbol{P})}{A(\hat{s}\boldsymbol{P})} = S_L(\boldsymbol{e}_0,\hat{s}\boldsymbol{P}). \tag{22}$$

Inserting Eqs. (19) and (20) into Eq. (22), we obtain the condition for $\boldsymbol{P}$. After some transformations (see Ref. [5] for details), it can be written in the form:

$$\left[\prod_{i=0}^{r}\left(\pi\nu_i\gamma\left(\boldsymbol{e}_i^2 b^2/2\right)\right)^{n_i}\right]^{i\boldsymbol{P}\cdot\hat{s}\boldsymbol{e}_0^\vee/b} \times \prod_{\boldsymbol{\alpha}>0}\left[\frac{\mathcal{G}(\boldsymbol{\alpha},\boldsymbol{P})}{\mathcal{G}(\boldsymbol{\alpha},-\boldsymbol{P})}\right]^{\boldsymbol{\alpha}\cdot\hat{s}\boldsymbol{e}_0^\vee} = 1, \tag{23}$$

where ν_i are defined by Eq. (21) and

$$\mathcal{G}(\boldsymbol{\alpha},\boldsymbol{P}) = \Gamma(1-iP_{\boldsymbol{\alpha}}b)\,\Gamma(1-iP_{\boldsymbol{\alpha}^\vee}/b).$$

For the lowest energy state, Eq. (23) reduces to the following equation:

$$L\boldsymbol{P} = 2\pi\boldsymbol{\rho} - \sum_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}\,\delta(\boldsymbol{\alpha},\boldsymbol{P}), \tag{24}$$

where

$$L = -\frac{2}{b}\left(h+b^2h^\vee\right)\ln\frac{R}{2\pi} - \frac{1}{b}\ln\left[\prod_{i=0}^{r}\left(\pi\mu_i\gamma\left(\boldsymbol{e}_i^2 b^2/2\right)\right)^{n_i}\right], \tag{25}$$

and

$$\delta(\boldsymbol{\alpha},\boldsymbol{P}) = -i\log\frac{\Gamma(1+iP_{\boldsymbol{\alpha}}b)\,\Gamma(1+iP_{\boldsymbol{\alpha}^\vee}/b)}{\Gamma(1-iP_{\boldsymbol{\alpha}}b)\,\Gamma(1-iP_{\boldsymbol{\alpha}^\vee}/b)}. \tag{26}$$

This is the quantization condition for the momentum $\boldsymbol{P}$ in the UV region $R\to 0$. The ground state energy of the system on the circle of size R is then given by

$$E(R) = -\frac{\pi c_{\text{eff}}}{6R} \quad\text{with}\quad c_{\text{eff}} = r - 12\,\boldsymbol{P}^2, \tag{27}$$

where $\boldsymbol{P}$ is the solution of Eq. (24).

In the UV region we can solve Eq. (24) perturbatively by expanding $\delta(\boldsymbol{\alpha},\boldsymbol{P})$ in powers of $P_{\boldsymbol{\alpha}}$,

$$\delta(\boldsymbol{\alpha},\boldsymbol{P}) = \delta_1(\boldsymbol{\alpha},b)\,P_{\boldsymbol{\alpha}} + \delta_3(\boldsymbol{\alpha},b)\,P_{\boldsymbol{\alpha}}^3 + \delta_5(\boldsymbol{\alpha},b)\,P_{\boldsymbol{\alpha}}^5\cdots, \tag{28}$$

where the coefficients $\delta_1(\boldsymbol{\alpha},b)$ and $\delta_s(\boldsymbol{\alpha},b)$, $s=3,5$ are:

$$\delta_1(\boldsymbol{\alpha},b) = -2\gamma_E\left(b+\frac{2}{\boldsymbol{\alpha}^2 b}\right),$$

$$\delta_s(\boldsymbol{\alpha},b) = (-)^{\frac{s-3}{2}}\cdot\frac{2}{s}\zeta(s)\left(b^s+\left(\frac{2}{\boldsymbol{\alpha}^2 b}\right)^s\right). \tag{29}$$

Using the relations: $\sum_{\boldsymbol{\alpha}>0}(\boldsymbol{\alpha})^a(\boldsymbol{\alpha})^b = h^\vee\delta^{ab}$, and $\sum_{\boldsymbol{\alpha}>0}(\boldsymbol{\alpha})^a(\boldsymbol{\alpha}^\vee)^b = h\delta^{ab}$, we obtain that:

$$l\boldsymbol{P} = 2\pi\boldsymbol{\rho} - \sum_{\boldsymbol{\alpha}>0}\delta_3(\boldsymbol{\alpha},b)\,\boldsymbol{\alpha}P_{\boldsymbol{\alpha}}^3 - \sum_{\boldsymbol{\alpha}>0}\delta_5(\boldsymbol{\alpha},b)\,\boldsymbol{\alpha}P_{\boldsymbol{\alpha}}^5 - \cdots,$$

with

$$l = L - 2\gamma_E\left(bh^\vee + h/b\right) \equiv L - L_0. \tag{30}$$

The above equation can be solved iteratively in powers of $1/l$. Inserting the solution into Eq. (27), we find:

$$c_{\text{eff}} = r - r(h+1)h^\vee\left(\frac{2\pi}{l}\right)^2 + \frac{8}{\pi}\zeta(3)\left[C_4(G^\vee)b^3 + C_4(G)/b^3\right]\left(\frac{2\pi}{l}\right)^5 - \frac{24}{5\pi}\zeta(5)\left[C_6(G^\vee)b^5 + C_6(G)/b^5\right]\times\left(\frac{2\pi}{l}\right)^7 + \mathcal{O}(l^{-8}), \tag{31}$$

where the coefficients $C(G)$ are defined as:

$$C_4(G) = \sum_{\boldsymbol{\alpha}>0}\rho_{\boldsymbol{\alpha}}\,\rho_{\boldsymbol{\alpha}^\vee}^3, \quad C_4(G^\vee) = \sum_{\boldsymbol{\alpha}>0}\rho_{\boldsymbol{\alpha}}^4,$$

$$C_6(G) = \sum_{\boldsymbol{\alpha}>0}\rho_{\boldsymbol{\alpha}}\,\rho_{\boldsymbol{\alpha}^\vee}^5, \quad C_6(G^\vee) = \sum_{\boldsymbol{\alpha}>0}\rho_{\boldsymbol{\alpha}}^6.$$

For simply laced algebras, these coefficients were calculated in [5] and have the values:

$$\begin{aligned}
C_4\left(A_{n-1}^{(1)}\right) &= \tfrac{1}{60}n^2(n^2-1)(2n^2-3),\\
C_6\left(A_{n-1}^{(1)}\right) &= \tfrac{1}{168}n^2(n^2-1)(n^2-2)(3n^2-5),\\
C_4\left(D_n^{(1)}\right) &= \tfrac{1}{30}(16n^3-45n^2+27n+8)\\
&\quad\times n(n-1)(2n-1),\\
C_6\left(D_n^{(1)}\right) &= \tfrac{1}{42}(48n^5-213n^4+262n^3\\
&\quad+6n^2-101n-32)\,n(n-1)(2n-1).
\end{aligned} \tag{32}$$

For the non-simply laced algebras $B_n^{(1)}$ and $C_n^{(1)}$, we can express the results through these values. Namely, we find:

$$\begin{aligned}
C_i\left(B_n^{(1)}\right) &= \tfrac{1}{2}C_i\left(A_{2n-1}^{(1)}\right),\\
C_i\left(B_n^{(1)\,\vee}\right) &= C_i\left(D_{n+1/2}^{(1)}\right),\\
C_i\left(C_n^{(1)}\right) &= C_i\left(D_{n+1}^{(1)}\right),\\
C_i\left(C_n^{(1)\,\vee}\right) &= C_i\left(D_{-n}^{(1)}\right), \quad (i=4,6).
\end{aligned} \tag{33}$$

For exceptional algebras $G_2^{(1)}$ and $F_4^{(1)}$, we obtain:

$$\begin{aligned}
C_4\left(G_2^{(1)}\right) &= \tfrac{1}{3}C_4\left(D_4^{(1)}\right) = 392,\\
C_4\left(G_2^{(1)\,\vee}\right) &= \tfrac{980}{9},\\
C_6\left(G_2^{(1)}\right) &= \tfrac{1}{3}C_6\left(D_4^{(1)}\right) = 7386,\\
C_6\left(G_2^{(1)\,\vee}\right) &= \tfrac{199516}{243},\\
C_4\left(F_4^{(1)}\right) &= \tfrac{1}{2}C_4\left(E_6^{(1)}\right) = 27378,\\
C_4\left(F_4^{(1)\,\vee}\right) &= \tfrac{22815}{2},\\
C_6\left(F_4^{(1)}\right) &= \tfrac{1}{2}C_6\left(E_6^{(1)}\right) = 2203578,\\
C_6\left(F_4^{(1)\,\vee}\right) &= \tfrac{4052763}{8}.
\end{aligned} \tag{34}$$

We note that above equations relating coefficients $C_i(G)$ for different Lie algebras follow from the similar exact relations between the ground state energies $e(G)$ of quantum affine Toda chains associated

with these Lie algebras. These exact relations are valid if the parameters μ, μ' for non-simply laced Lie algebras and corresponding parameter μ_{sl} for simply laced ones satisfy the condition: $\mu^{h-z}(2\mu'/\xi^2)^z = \mu_{sl}^h$, where $z = \frac{2(h-h^\vee)}{2-\xi^2}$.

4. Comparison with TBA results

The effective central charge calculated above from the CFT data (reflection amplitudes) can be compared with the same function determined from numerical solution of the TBA equations for ATFTs. Namely:

$$c_{\rm eff}^{(\rm TBA)}(R) = \sum_{i=1}^{r} \frac{3Rm_i}{\pi^2} \times \int \cosh\theta \log\left(1 + e^{-\epsilon_i(\theta,R)}\right) d\theta, \quad (35)$$

where functions $\epsilon_i(\theta,R)$ $(i = 1,\ldots,r)$ satisfy the system of r coupled integral equations:

$$m_i R\cosh\theta = \epsilon_i(\theta,R) + \sum_{j=1}^{r} \int \varphi_{ij}(\theta-\theta') \times \log\left(1 + e^{-\epsilon_i(\theta',R)}\right) \frac{d\theta'}{2\pi}, \quad (36)$$

with the kernels φ_{ij}, equal to the logarithmic derivatives of the S-matrices $S_{ij}(\theta)$ of ATFTs, conjectured in [8,9].

The function $E^{(\rm TBA)}(R)$ defined from the TBA equations differs from the ground state energy $E(R)$ of the system on the circle of size R by the bulk term: $E^{(\rm TBA)}(R) = E(R) - fR$, where f is a specific bulk free energy [7]. To compare the same functions we should subtract this term from the function $E(R)$ defined by Eq. (31) i.e.

$$c_{\rm eff}^{(\rm TBA)}(R) = c_{\rm eff}^{(\rm RA)}(R) + \frac{6R^2}{\pi} f(G). \quad (37)$$

The specific bulk free energy $f(G)$ for non-simply laced ATFTs can be calculated by the Bethe Ansatz method with the result:

$$f(G) = \frac{\overline{m}^2 \sin(\pi/H)}{8\sin(\pi B/H)\sin(\pi(1-B)/H)}, \quad G = B_r^{(1)}, C_r^{(1)},$$

$$f(G) = \frac{\overline{m}^2 \cos(\pi(1/3 - 1/H))}{16\cos(\pi/6)\sin(\pi B/H)\sin(\pi(1-B)/H)}, \quad G = G_2^{(1)}, F_4^{(1)}. \quad (38)$$

The contribution of bulk term $f(G)$ becomes quite essential at $R \sim \mathcal{O}(1)$.

The TBA equations (36) were solved numerically for non-simply laced algebras, $B_2^{(1)}$ $(= C_2^{(1)})$, $B_3^{(1)}$, $B_4^{(1)}$, $C_3^{(1)}$, $C_4^{(1)}$, $G_2^{(1)}$ and $F_4^{(1)}$. The effective central charge $c_{\rm eff}^{(\rm TBA)}(R)$ was then computed from Eq. (35) for many different values of parameter $\overline{m}R$. After taking into account the bulk term, the numerical

Table 1
$L_0^{(\rm RA)}$ versus $L_0^{(\rm TBA)}$ for non-simply laced ATFTs

	B					
	0.3	0.4	0.5	0.6	0.7	0.8
$L_0^{(\rm RA)}(C_2^{(1)})$	11.5882	11.3111	11.5443	12.2537	13.6035	16.162
$L_0^{(\rm TBA)}(C_2^{(1)})$	11.5882	11.3111	11.5443	12.2537	13.6035	16.162
$L_0^{(\rm RA)}(C_3^{(1)})$	16.6266	16.0240	16.1620	16.9666	18.6419	21.9342
$L_0^{(\rm TBA)}(C_3^{(1)})$	16.6266	16.0240	16.1620	16.9666	18.6419	21.9319
$L_0^{(\rm RA)}(C_4^{(1)})$	21.6649	20.7370	20.7798	21.6796	23.6803	27.7064
$L_0^{(\rm TBA)}(C_4^{(1)})$	21.6649	20.7370	20.7798	21.6795	23.6802	27.5
$L_0^{(\rm RA)}(B_3^{(1)})$	14.3593	13.1962	12.6987	12.7250	13.3516	15.0076
$L_0^{(\rm TBA)}(B_3^{(1)})$	14.3589	13.1962	12.6987	12.7250	13.3516	15.0076
$L_0^{(\rm RA)}(B_4^{(1)})$	19.3977	17.9092	17.3165	17.4379	18.3900	20.7798
$L_0^{(\rm TBA)}(B_4^{(1)})$	19.32	17.9089	17.3165	17.4379	18.3900	20.7792
$L_0^{(\rm RA)}(G_2^{(1)})$	13.6035	12.2537	11.5443	11.3111	11.5882	12.6987
$L_0^{(\rm TBA)}(G_2^{(1)})$	13.3	12.2529	11.5443	11.3111	11.5882	12.6987
$L_0^{(\rm RA)}(F_4^{(1)})$	27.9629	25.4499	24.2431	24.0360	24.9398	27.7064
$L_0^{(\rm TBA)}(F_4^{(1)})$		25.1	24.238	24.0360	24.9398	27.5

Table 2
$c_5^{(RA)}$ versus $c_5^{(TBA)}$ for non-simply laced ATFTs

	B					
	0.3	0.4	0.5	0.6	0.7	0.8
$c_5^{(RA)}(C_2^{(1)})$	1569.60	124216	1438.68	2183.98	3961.81	8713.17
$c_5^{(TBA)}(C_2^{(1)})$	1567.	1240.	1437.	2182.	3959.	8708.
$c_5^{(RA)}(C_3^{(1)})$	15018.6	10858.7	11399.2	16288.0	28809.0	62845.7
$c_5^{(TBA)}(C_3^{(1)})$	15000	10840	11380	16270	28790	68000
$c_5^{(RA)}(C_4^{(1)})$	76141.1	52782.9	52563.7	72394.9	125955.	273273.
$c_5^{(TBA)}(C_4^{(1)})$	76100	52700	52500	72300	125990	
$c_5^{(RA)}(B_3^{(1)})$	8260.97	4765.97	3488.60	3541.95	5151.97	10444.4
$c_5^{(TBA)}(B_3^{(1)})$	8500	4761.	3484.	3538.	5147.	10439.
$c_5^{(RA)}(B_4^{(1)})$	48261.8	28350.9	21550.3	22970.0	34594.6	71159.4
$c_5^{(TBA)}(B_4^{(1)})$		29000	21530	22940	34560	73000
$c_5^{(RA)}(G_2^{(1)})$	4370.29	2385.82	1533.23	1265.48	1524.65	2816.47
$c_5^{(TBA)}(G_2^{(1)})$		2600	1532.	1264.	1523.	2814.
$c_5^{(RA)}(F_4^{(1)})$	308495.	172966.	118723.	109767.	147970.	289824.
$c_5^{(TBA)}(F_4^{(1)})$			160000	110055.	148248.	

solution for $c_{\text{eff}}^{(TBA)}(R)$ was fitted with the expansion (31) (neglecting higher order terms in $1/l$):

$$c_{\text{eff}}^{(RA)}(R) = r - r(h+1)h^{\vee}\left(\frac{2\pi}{l}\right)^2 + c_5\left(\frac{2\pi}{l}\right)^5 + c_7\left(\frac{2\pi}{l}\right)^7, \tag{39}$$

with fitting parameters L_0, c_5 and c_7, where parameter L_0 is defined by Eq. (30). The exact values of these parameters can be easily identified from Eqs. (30) and (31). To compare the expansion (39) with TBA results we use the relations (5) between parameters μ_i in the action and the parameter $\overline{m}$ characterizing the spectrum of particles. It gives the following expression for function $L(R)$ in Eqs. (30):

$$L = -\frac{2}{b}(h + b^2 h^{\vee}) \times \ln\left[\frac{\overline{m}R}{4\pi}k(G)\,\Gamma\left(\frac{1-B}{H}\right)\Gamma\left(1+\frac{B}{H}\right)\right] + \frac{2}{b}\ln\left(b^{2h}(\xi^2/2)^z\right). \tag{40}$$

Table 3
$c_7^{(RA)}$ versus $c_7^{(TBA)}$ for non-simply laced ATFTs

	B					
	0.3	0.4	0.5	0.6	0.7	0.8
$c_7^{(RA)}(C_2^{(1)})$	−12262.9	−6566.01	−9109.78	−21843.4	−64594.2	−247955.
$c_7^{(TBA)}(C_2^{(1)})$	−14000	−8000	−10600	−23800	−68000	−256000
$c_7^{(RA)}(C_3^{(1)})$	−304830.	−137513.	−148031.	−324132.	−944077.	−3617480
$c_7^{(TBA)}(C_3^{(1)})$	−330000	−160000	−170000	−350000	−990000	
$c_7^{(RA)}(C_4^{(1)})$	−2878340	−1216910	−1148970	−2366240	−6815270	−26079300
$c_7^{(TBA)}(C_4^{(1)})$	−3000000	−1400000	−1300000	−2600000	−7000000	
$c_7^{(RA)}(B_3^{(1)})$	−174026.	−60330.1	−29061.1	−30298.6	−76073.1	−264814.
$c_7^{(TBA)}(B_3^{(1)})$		−65000	−33000	−34000	−76000	−270000
$c_7^{(RA)}(B_4^{(1)})$	−1847280	−650082.	−338009.	−404020.	−1004420	−3769680
$c_7^{(TBA)}(B_4^{(1)})$			−380000	−450000	−1100000	−2200000
$c_7^{(RA)}(G_2^{(1)})$	−97474.0	−32718.2	−13002.5	−7830.99	−12225.2	−41991.4
$c_7^{(TBA)}(G_2^{(1)})$			−14000	−9000	−14000	−44000
$c_7^{(RA)}(F_4^{(1)})$	−2913070	−9911730	−4293750	−3478610	−7094660	−25792400
$c_7^{(TBA)}(F_4^{(1)})$				−3410000	−7090000	

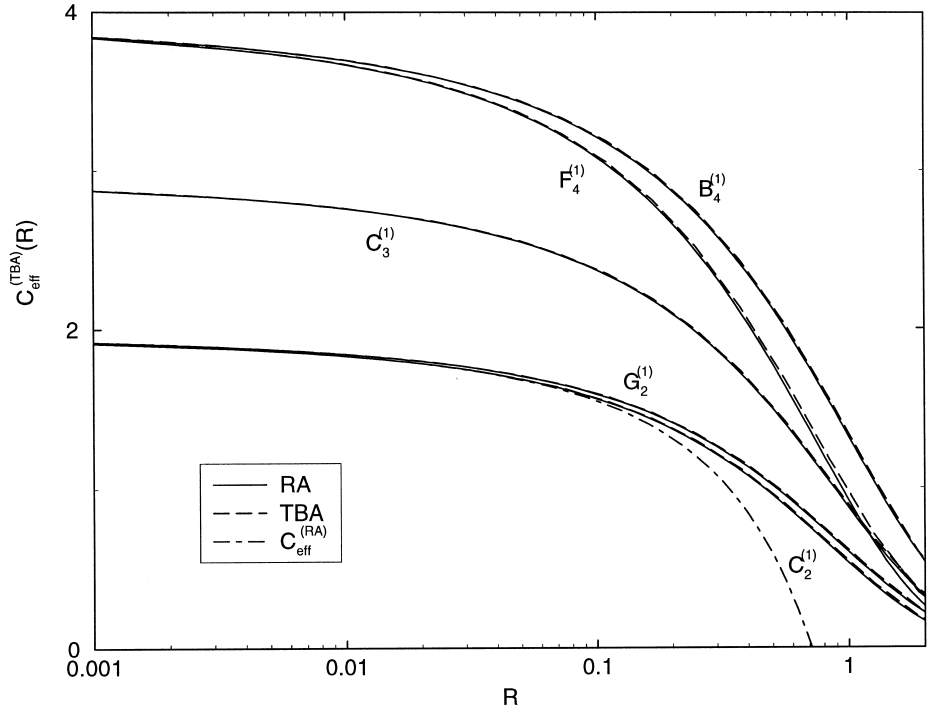

Fig. 1. Plot of $c_{\text{eff}}^{(\text{TBA})}$ for $C_2^{(1)}$, $C_3^{(1)}$, $B_4^{(1)}$, $G_2^{(1)}$ and $F_4^{(1)}$ ATFTs at $B = 0.5$. (We omit in the figure $C_4^{(1)}$ and $B_3^{(1)}$ cases not to make it too complicated.) As an example, we also display $c_{\text{eff}}^{(\text{RA})}$ for $C_2^{(1)}$ calculated without taking into account the bulk term. The difference between this function and $c_{\text{eff}}^{(\text{TBA})}$ gives the bulk free energy of $C_2^{(1)}$ ATFT according to Eq. (37).

Tables 1–3 show the values of parameters L_0, c_5 and c_7 obtained numerically from TBA equations (denoted with the superscript (TBA)) and those obtained analytically (Eqs. (30) and (31)) from reflection amplitudes (denoted with the superscript (RA)) for $C_2^{(1)}$, $C_3^{(1)}$, $C_4^{(1)}$, $B_3^{(1)}$, $B_4^{(1)}$, $G_2^{(1)}$ and $F_4^{(1)}$ ATFTs with different values of the parameter B. We see that both data are in excellent agreement. (Relatively poor accuracy for c_7 is mainly due to the limitation of numerical accuracy and the influence of higher order term ($\mathscr{O}(l^{-8})$) in the expansion (39).) This agreement supports the approach based on the reflection amplitudes, μ-$\overline{m}$ relations and quantization conditions as well as the S-matrices for non-simply laced ATFTs.

In Fig. 1, we plot the functions $c_{\text{eff}}^{(\text{TBA})}(R)$ and $c_{\text{eff}}^{(\text{RA})}(R)$ for different ATFTs setting $\overline{m} = 1$. The first function is computed numerically from TBA equations. The second one is calculated using Eqs. (24) and (27), based on the reflection amplitudes, with taking into account the bulk free energy term according to Eq. (37). For all models, the two curves are almost identical without essential difference in the graphs even at $R \sim \mathscr{O}(1)$. This good agreement outside the UV region looks not to be accidental. However, at present, we have no satisfactory explanation of this interesting phenomena in ATFTs.

5. Concluding remarks

In the main part of this paper we considered the UV asymptotics of the effective central charges in ATFTs. The most important CFT data, which we used for this analysis were the reflection amplitudes (13) of NATTs. It was mentioned in Introduction, that these functions play also a crucial role in the calculation of the one point functions in perturbed

CFT. The one point functions of the exponential fields in ATFTs:

$$\mathscr{T}(\boldsymbol{a}) = \langle \exp \boldsymbol{a}\cdot\boldsymbol{\varphi}\rangle \tag{41}$$

can be reconstructed from from the same reflection amplitudes. It follows from the results of the paper [2] that functions (41) satisfy the functional equations similar to the relations (12) for the vertex operators. These equations together with analyticity and symmetry conditions fix one point functions in perturbed CFTs. One can find the solution of these functional equations with proper analyticity properties and respecting all symmetries of extended Dynkin diagram of Lie algebra G. This solution is a natural generalization to the non-simply laced case of the one point function for *ADE* series of ATFTs calculated in [17] and can be written in the form:

$$\begin{aligned}\mathscr{T}(\boldsymbol{a}) = {} & \left[\frac{\overline{m}k(G)}{2}\Gamma\left(\frac{1-B}{H}\right)\Gamma\left(1+\frac{B}{H}\right)\right]^{2\boldsymbol{Q}\cdot\boldsymbol{a}-\boldsymbol{a}^2} \\ & \times\prod_{i=1}^{r}\left[-\pi\mu_i\gamma\left(1+\boldsymbol{e}_i^2 b^2/2\right)\right]^{-\boldsymbol{\omega}_i^{\vee}\boldsymbol{a}/b} \\ & \times\exp\int\frac{dt}{t}\left[\boldsymbol{a}^2 e^{-2t}-\mathscr{F}(\boldsymbol{a},t)\right],\end{aligned} \tag{42}$$

where

$$\begin{aligned}\mathscr{F}(\boldsymbol{a},t) = {} & \sum_{\boldsymbol{\alpha}>0}\frac{\sinh(a_{\boldsymbol{\alpha}}bt)\sinh\left(\left(ba_{\boldsymbol{\alpha}}-2bQ_{\boldsymbol{\alpha}}+(1+b^2)H\right)t\right)\sinh\left(\left(b^2\boldsymbol{\alpha}^2/2+1\right)t\right)}{\sinh t\sinh\left(b^2\boldsymbol{\alpha}^2 t/2\right)\sinh\left((1+b^2)Ht\right)}.\end{aligned} \tag{43}$$

The one point function $\mathscr{T}(\boldsymbol{a})$ can be used for the analysis of ATFTs. In particular, it contains the information about the bulk free energy $f(G)$, which was calculated independently by Bethe Ansatz method. One can easily derive from Eqs. (1) and (5) that:

$$\frac{n_i f(G)}{H(1+b^2)} = \mu_i\mathscr{T}(b\boldsymbol{e}_i). \tag{44}$$

Using Eq. (42) for function $\mathscr{T}(\boldsymbol{a})$ one finds:

$$\begin{aligned}&\frac{-4\pi\gamma\left(1+\boldsymbol{e}_i^2 b^2/2\right)n_i f(G)}{H(1+b^2)\left(\overline{m}k(G)\right)^2} \\ &\quad=\left[\Gamma\left(\frac{1-B}{H}\right)\Gamma\left(1+\frac{B}{H}\right)\right]^2 \\ &\qquad\times\exp\int\frac{dt}{t}\left[(b\boldsymbol{e}_i)^2 e^{-2t}-\mathscr{F}(b\boldsymbol{e}_i,t)\right].\end{aligned} \tag{45}$$

The integral in the exponent can be calculated and results coincides with Eq. (38). This gives the non-perturbative test to the one point function $\mathscr{T}(\boldsymbol{a})$. In particular, taking the limit $b\to 0$ in Eq. (45) (and the dual limit) one can derive the amusing relations for gamma-functions associated with Lie algebras G. It is convenient to introduce the integers $n_i^{\vee} = n_i\boldsymbol{e}_i^2/2$ $i=0,\ldots,r$. Then these relations can be written as:

$$\begin{aligned}&\prod_{\boldsymbol{\alpha}>0}\left(\gamma(\boldsymbol{\alpha}\cdot\boldsymbol{\rho}^{\vee}/h)\right)^{-\boldsymbol{e}_i\cdot\boldsymbol{\alpha}^{\vee}} \\ &\quad= n_i^{\vee}\left(\prod_{i=0}^{r}\left(n_i^{\vee}\right)^{n_i}\right)^{-1/h},\end{aligned} \tag{46}$$

and

$$\begin{aligned}&\prod_{\boldsymbol{\alpha}>0}\left(\gamma(\boldsymbol{\alpha}\cdot\boldsymbol{\rho}/h^{\vee})\right)^{-\boldsymbol{e}_i^{\vee}\cdot\boldsymbol{\alpha}} \\ &\quad= n_i^{\vee}\boldsymbol{e}_i^2/2\left(\prod_{i=0}^{r}\left(n_i^{\vee}\boldsymbol{e}_i^2/2\right)^{n_i^{\vee}}\right)^{-1/h^{\vee}}.\end{aligned} \tag{47}$$

More detailed consideration of one point functions in ATFTs we suppose to give in another publication.

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