Finite-size effect of the dyonic giant magnons in $\mathcal{N} = 6$ super Chern-Simons theory

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We consider finite-size effects for the dyonic giant magnon of the type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ by applying the Lüscher μ-term formula which is derived from a recently proposed S matrix for the $\mathcal{N} = 6$ super Chern-Simons theory. We compute explicitly the effect for the case of a symmetric configuration where the two external bound states, each of A and B particles, have the same momentum p and spin J_2. We compare this with the classical string theory result which we computed by reducing it to the Neumann-Rosochatius system. The two results match perfectly.

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I. INTRODUCTION

The AdS/CFT correspondence between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills (SYM) theory [1–3] led to many exciting developments and to understanding nonperturbative structures of the string and gauge theories. Recently, an exciting possibility that the same type of duality does exist in three-dimensional gauge theory has been discovered. The promising candidate for the three-dimensional conformal field theory is $\mathcal{N} = 6$ super Chern-Simons (CS) theory with $SU(N) \times SU(N)$ gauge symmetry and level k. This model, which was first proposed by Aharony, Bergman, Jafferis, and Maldacena [4], is believed to be dual to M theory on $\mathrm{AdS}_4 \times S^7/Z_k$. Furthermore, in the planar limit of N, $k \to \infty$ with a fixed value of 't Hooft coupling $\lambda = N/k$, the $\mathcal{N} = 6$ CS is believed to be dual to type IIA superstring theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. This model contains two sets of scalar fields transforming in bifundamental representations of $SU(N) \times SU(N)$ along with respective superpartner fermions and nondynamic CS gauge fields.

The integrability of the planar $\mathcal{N} = 6$ CS theory, first discovered by Minahan and Zarembo [5] in the leading two-loop-order perturbative computation, is conjectured to exist in all-loop orders and corresponding all-loop Bethe ansatz equations were conjectured by Gromov and Vieira [6] based on the perturbative result [5] and the classical integrability in the large-coupling limit discovered in [7–9]. Recently, three groups [10–12] computed the one-loop correction to the energy of a folded spinning string, and seemed to find disagreement with the prediction of the all-loop Bethe ansatz equations (BAEs). This controversy may be resolved by a nonzero one-loop correction in the central interpolating function $h(\lambda)$ as suggested recently in [13]. (See also [14].)

On the other hand, based on the spectrum and symmetries of the model [5,15–17], Ahn and Nepomechie proposed an S matrix [18] which reproduces the all-loop BAEs. The S matrix has played an important role in AdS/CFT as noticed early in [19] and gets more so because it provides the only way of computing finite-size effect exactly. For example, one cannot reproduce this from the all-loop BAEs. Therefore, the finite-size effect can be a stringent check of the S matrix in integrable models if one can compare it with an independent result. In the AdS/CFT correspondence, there are alternative but approximate ways of computing finite-size effects in semiclassical ways [20] such as the algebraic curve method [21] or the Neumann-Rosochatius (NR) method [22–24]. For the $\mathcal{N} = 6$ CS theory, both methods have been implemented to compute the effect for a giant magnon (GM) which moves symmetrically in $SU(2) \times SU(2)$ subspace of $\mathbb{CP}^3$ [25]. See [26–30] for subsequent developments on the finite-size effects of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ from the string/membrane side.

The formalism to derive the finite-size effect from the S matrix is the Lüscher correction. This computes a shift in the energy due to the finite size of spatial length from the S matrix for all values of the 't Hooft coupling constant. This method has been successfully applied to the AdS/CFT duality in the $\mathcal{N} = 4$ SYM theory [31–35]. Recently two groups computed the finite-size corrections to the dispersion relation of GMs [36] from the $\mathcal{N} = 6$ CS theory side [37,38]. They showed that the results are consistent with the classical string theory, which strongly supports the validity of the S matrix proposed in [18]. Along this line of investigation, another interesting configuration is the classical string state with two angular momenta, usually called "two spin" solutions. The authors have computed the finite-size effect for the dyonic giant magnon (DGM) [39] in the classical limit by the Neumann-Rosochatius method [40]. It is further extended recently to a single DGM solution [41]. It is important to check this since the DGM maintains the BPS saturated form of the dispersion relation even in the classical limit. Therefore, it can check the finite-size effect in a most intact form.

The purpose of this note is to compute the finite-size effect of the DGM from the Lüscher formula and compare

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it with the previous result [40]. In Sec. II, we briefly review our derivation of the DGM in the $SU(2)\times SU(2)$ subspace of $\mathbb{CP}^3$ with $U(1)$ fiber dynamics and computation of the finite-size effect [40]. We generalize the Lüscher formula for multi-DGM particles in Sec. III. We also derive the S matrix elements between an elementary magnon and its bound state in Sec. III which will be used in the Lüscher formula. In the classical limit, we confirm that the Lüscher correction exactly matches with the classical string theory result. We conclude the paper with some remarks in Sec. IV.

II. CLASSICAL STRING ANALYSIS

Let us consider a classical string moving in $R_t\times\mathbb{CP}^3$. Using the complex coordinates

$$z = y^0 + iy^4, \qquad w_1 = x^1 + ix^2, \qquad w_2 = x^3 + ix^4, \qquad w_3 = x^5 + ix^6, \qquad w_4 = x^7 + ix^8,$$

we embed the string as follows [40]:

$$z = Z(\tau,\sigma) = \frac{R}{2}e^{it(\tau,\sigma)}, \qquad w_a = W_a(\tau,\sigma) = Rr_a(\tau,\sigma)e^{i\varphi_a(\tau,\sigma)}.$$

Here t is the AdS time. These complex coordinates should satisfy

$$\sum_{a=1}^4 W_a\bar{W}_a = R^2, \qquad \sum_{a=1}^4 (W_a\partial_m\bar{W}_a - \bar{W}_a\partial_m W_a) = 0,$$

or

$$\sum_{a=1}^4 r_a^2 = 1, \qquad \sum_{a=1}^4 r_a^2\partial_m\varphi_a = 0, \qquad m = 0, 1. \tag{2.1}$$

A. NR reduction

In order to reduce the string dynamics on $R_t\times\mathbb{CP}^3$ to the NR integrable system, we use the ansatz [22–24]

$$t(\tau,\sigma) = \kappa\tau, \qquad r_a(\tau,\sigma) = r_a(\xi), \qquad \varphi_a(\tau,\sigma) = \omega_a\tau + f_a(\xi), \qquad \xi = \alpha\sigma + \beta\tau, \qquad \kappa, \omega_a, \alpha, \beta = \text{constants}. \tag{2.2}$$

It can be shown [40] that, after integration of the equations of motion for f_a, which gives

$$f_a' = \frac{1}{\alpha^2-\beta^2}\left(\frac{C_a}{r_a^2} + \beta\omega_a\right), \qquad C_a = \text{constants}, \tag{2.3}$$

one ends up with the following effective Lagrangian for the coordinates r_a:

$$L_{\rm NR} = (\alpha^2-\beta^2)\sum_{a=1}^4\left[r_a'^2 - \frac{1}{(\alpha^2-\beta^2)^2}\left(\frac{C_a^2}{r_a^2} + \alpha^2\omega_a^2 r_a^2\right)\right] - \Lambda\left(\sum_{a=1}^4 r_a^2 - 1\right). \tag{2.4}$$

This is the Lagrangian for the NR integrable system [24]. In addition, the $\mathbb{CP}^3$ embedding conditions in (2.1) lead to

$$\sum_{a=1}^4 \omega_a r_a^2 = 0, \qquad \sum_{a=1}^4 C_a = 0. \tag{2.5}$$

The Virasoro constraints give the conserved Hamiltonian $H_{\rm NR}$ and a relation between the embedding parameters and the arbitrary constants C_a:

$$H_{\rm NR} = (\alpha^2-\beta^2)\sum_{a=1}^4\left[r_a'^2 + \frac{1}{(\alpha^2-\beta^2)^2}\times\left(\frac{C_a^2}{r_a^2} + \alpha^2\omega_a^2 r_a^2\right)\right] = \frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}\frac{\kappa^2}{4}, \tag{2.6}$$

$$\sum_{a=1}^4 C_a\omega_a + \beta(\kappa/2)^2 = 0. \tag{2.7}$$

For closed strings, r_a and f_a satisfy the following periodicity conditions:

$$r_a(\xi + 2\pi\alpha) = r_a(\xi), \qquad f_a(\xi + 2\pi\alpha) = f_a(\xi) + 2\pi n_a, \tag{2.8}$$

where n_a are integer winding numbers.

The conserved charges can be defined by

$$E_s = -\int d\sigma\frac{\partial\mathcal{L}}{\partial(\partial_0 t)}, \qquad J_a = \int d\sigma\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi_a)}, \qquad a = 1, 2, 3, 4,$$

where $\mathcal{L}$ is the Polyakov string Lagrangian taken in conformal gauge. Using the ansatz (2.2) and (2.3), we can find

$$E_s = \frac{\kappa\sqrt{2\lambda}}{2\alpha}\int d\xi, \qquad J_a = \frac{2\sqrt{2\lambda}}{\alpha^2-\beta^2}\int d\xi\left(\frac{\beta}{\alpha}C_a + \alpha\omega_a r_a^2\right). \tag{2.9}$$

In view of (2.5), one obtains [17]

$$\sum_{a=1}^4 J_a = 0. \tag{2.10}$$

B. Dyonic giant magnon solution

We are interested in finding string configurations corresponding to the following particular solution of (2.5):

$$r_1 = r_3 = \frac{1}{\sqrt{2}}\sin\theta, \qquad r_2 = r_4 = \frac{1}{\sqrt{2}}\cos\theta,$$
$$\omega_1 = -\omega_3, \qquad \omega_2 = -\omega_4.$$

The two frequencies ω_1, ω_2 are independent and lead to strings moving in $\mathbb{CP}^3$ with two angular momenta. The special case $\omega_2 = 0$ corresponds to the solutions obtained in [17,27]. From the NR Hamiltonian (2.6) one finds

$$\theta'^2(\xi) = \frac{1}{(\alpha^2 - \beta^2)^2}\left[\frac{\kappa^2}{4}(\alpha^2 + \beta^2) - 2\left(\frac{C_1^2 + C_3^2}{\sin^2\theta} + \frac{C_2^2 + C_4^2}{\cos^2\theta}\right) - \alpha^2(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta)\right].$$

We further restrict ourselves to $C_2 = C_4 = 0$ to search for GM string configurations. Equations (2.5) and (2.7) give

$$C_1 = -C_3 = -\frac{\beta\kappa^2}{8\omega_1}.$$

In this case, the above equation for θ' can be rewritten in the form

$$(\cos\theta)' = \mp\frac{\alpha\sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2}\sqrt{(z_+^2 - \cos^2\theta)(\cos^2\theta - z_-^2)}, \tag{2.11}$$

where

$$z_\pm^2 = \frac{1}{2(1 - \frac{\omega_2^2}{\omega_1^2})}\left\{y_1 + y_2 - \frac{\omega_2^2}{\omega_1^2} \pm \sqrt{(y_1 - y_2)^2 - \left[2(y_1 + y_2 - 2y_1y_2) - \frac{\omega_2^2}{\omega_1^2}\right]\frac{\omega_2^2}{\omega_1^2}}\right\},$$
$$y_1 = 1 - \frac{\kappa^2}{4\omega_1^2}, \qquad y_2 = 1 - \frac{\beta^2}{\alpha^2}\frac{\kappa^2}{4\omega_1^2}.$$

The solution of (2.11) is given by

$$\cos\theta = z_+ dn(C\xi|m), \qquad C = \mp\frac{\alpha\sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2}z_+, \qquad m \equiv 1 - z_-^2/z_+^2, \tag{2.12}$$

where $dn(C\xi|m)$ is one of the elliptic functions.

To find the full string solution, we also need to obtain the explicit expressions for the functions f_a from (2.3)

$$f_a = \frac{1}{\alpha^2 - \beta^2}\int d\xi\left(\frac{C_a}{r_a^2} + \beta\omega_a\right).$$

Using the solution (2.12) for $\theta(\xi)$, we can find

$$f_1 = -f_3 = \frac{\beta/\alpha}{z_+\sqrt{1 - \omega_2^2/\omega_1^2}}\left[C\xi - \frac{2(\kappa/2)^2/\omega_1^2}{1 - z_+^2}\times\Pi\left(am(C\xi), -\frac{z_+^2 - z_-^2}{1 - z_+^2}|m\right)\right],$$
$$f_2 = -f_4 = \frac{\beta\omega_2}{\alpha^2 - \beta^2}\xi.$$

Here, Π is the elliptic integral of the third kind. As a consequence, the string solution can be written as

$$\begin{aligned} W_1 &= \frac{R}{\sqrt{2}}\sqrt{1 - z_+^2 dn^2(C\xi|m)}e^{i(\omega_1\tau + f_1)}, \\ W_2 &= \frac{R}{\sqrt{2}}z_+ dn(C\xi|m)e^{i(\omega_2\tau + f_2)}, \\ W_3 &= \frac{R}{\sqrt{2}}\sqrt{1 - z_+^2 dn^2(C\xi|m)}e^{-i(\omega_1\tau + f_1)}, \\ W_4 &= \frac{R}{\sqrt{2}}z_+ dn(C\xi|m)e^{-i(\omega_2\tau + f_2)}. \end{aligned} \tag{2.13}$$

The geometric meaning of the explicit solution (2.13) is as follows. Each pair of complex coordinates, (W_1, W_2) and (W_3, W_4), describes a spiky solutions in S^2 sphere geometry but with dynamics at opposite points in the $U(1)$ fiber. The two phases in (W_1, W_2) are exactly opposite to those of (W_3, W_4) which, together with the dynamics in $U(1)$, ensures the vanishing of the total momentum. This behavior has been also noticed for strings in $R_t \times S^2 \times S^2$ in [17].

The GM in infinite volume can be obtained by taking $z_- \to 0$. In this limit, the solution for θ reduces to

$$\cos\theta = \frac{\sin\frac{p}{2}}{\cosh(C\xi)},$$

where the constant $z_+ \equiv \sin p/2$ is given by

$$z_+^2 = \frac{y_2 - \omega_2^2/\omega_1^2}{1 - \omega_2^2/\omega_1^2}.$$

One spin solution corresponds to $\omega_2 = 0$. Inserting this into (2.9), one can find the energy-charge dispersion relation. For the *single* DGM, the energy and angular momentum J_1 become infinite but their difference remains finite:

$$E_s - J_1 = \sqrt{\frac{J_2^2}{4} + 2\lambda\sin^2\frac{p}{2}}. \tag{2.14}$$

C. Finite-size effects

Using the most general solutions (2.13), we can calculate the finite-size corrections to the energy-charge relation (2.14) in the limit when the string energy $E_s \to \infty$. Here we consider the case of $\alpha^2 > \beta^2$ only since it corresponds to the GM case. We obtain from (2.9) the following expressions for the conserved string energy E_s and the angular momenta J_a:

$$\mathcal{E} = \frac{2\kappa(1-\beta^2/\alpha^2)}{\omega_1 z_+\sqrt{1-\omega_2^2/\omega_1^2}}\mathbf{K}(1-z_-^2/z_+^2),$$
$$\mathcal{J}_1 = \frac{2z_+}{\sqrt{1-\omega_2^2/\omega_1^2}}\left[\frac{1-\beta^2(\kappa/2)^2/\alpha^2\omega_1^2}{z_+^2}\mathbf{K}(1-z_-^2/z_+^2) - \mathbf{E}(1-z_-^2/z_+^2)\right],$$
$$\mathcal{J}_2 = \frac{2z_+\omega_2/\omega_1}{\sqrt{1-\omega_2^2/\omega_1^2}}\mathbf{E}(1-z_-^2/z_+^2),$$
$$\mathcal{J}_3 = -\mathcal{J}_1, \qquad \mathcal{J}_4 = -\mathcal{J}_2. \tag{2.15}$$

As a result, the condition (2.10) is identically satisfied. Here, we introduced the notations

$$\mathcal{E} = \frac{E_s}{\sqrt{2\lambda}}, \qquad \mathcal{J}_a = \frac{J_a}{\sqrt{2\lambda}}. \tag{2.16}$$

The computation of $\Delta\varphi_1$ gives

$$p \equiv \Delta\varphi_1 = 2\int_{\theta_{\min}}^{\theta_{\max}}\frac{d\theta}{\theta'}f_1' = -\frac{2\beta/\alpha}{z_+\sqrt{1-\omega_2^2/\omega_1^2}}\left[\frac{(\kappa/2)^2/\omega_1^2}{1-z_+^2}\times\Pi\left(-\frac{z_+^2-z_-^2}{1-z_+^2}\middle|1-z_-^2/z_+^2\right) - \mathbf{K}(1-z_-^2/z_+^2)\right]. \tag{2.17}$$

In the above expressions, $\mathbf{K}(m)$, $\mathbf{E}(m)$, and $\Pi(n|m)$ are the complete elliptic integrals.

Expanding the elliptic integrals, we obtain

$$E - J_1 = 2\sqrt{\frac{J_2^2}{4}+2\lambda\sin^2\frac{p}{2}} - \frac{32\lambda\sin^4\frac{p}{2}}{\sqrt{J_2^2+8\lambda\sin^2\frac{p}{2}}}\exp\left[-\frac{2\sin^2\frac{p}{2}(J_1+\sqrt{J_2^2+8\lambda\sin^2\frac{p}{2}})\sqrt{J_2^2+8\lambda\sin^2\frac{p}{2}}}{J_2^2+8\lambda\sin^4\frac{p}{2}}\right]. \tag{2.18}$$

This also gives a finite-size effect for ordinary GM [20] by taking $J_2 \to 0$,

$$E - J_1 = 2\sqrt{2\lambda}\sin\frac{p}{2} - 16\sqrt{\frac{\lambda}{2}}\sin^3\frac{p}{2}\exp\left[-\frac{J_1}{\sqrt{2\lambda}\sin\frac{p}{2}} - 2\right]. \tag{2.19}$$

III. FINITE-SIZE EFFECTS FROM THE S MATRIX

The $\mathcal{N} = 6$ CS theory has two sets of excitations, namely A particles and B particles, each of which forms a four-dimensional representation of $SU(2|2)$ [16,18]. We propose an S matrix with the following structure:

$$S^{AA}(p_1,p_2) = S^{BB}(p_1,p_2) = S_0(p_1,p_2)\hat{S}(p_1,p_2),$$
$$S^{AB}(p_1,p_2) = S^{BA}(p_1,p_2) = \tilde{S}_0(p_1,p_2)\hat{S}(p_1,p_2),$$

where $\hat{S}$ is the matrix part determined by the $SU(2|2)$ symmetry, and is essentially the same as that found for $\mathcal{N} = 4$ YM in [42,43]. An important difference arises in the dressing phases S_0, $\tilde{S}_0$ due to the fact that the A and B particles are related by complex conjugation.

A. Lüscher μ-term formula

Here we want to generalize multiparticle Lüscher formula [34,35] to the case of the bound states. Consider M_A number of A-type DGMs, $|Q_1,\ldots Q_{M_A}\rangle$, and M_B number of B-type DGMs, $|\tilde{Q}_1,\ldots\tilde{Q}_{M_B}\rangle$. We use α_k for the $SU(2|2)$ quantum numbers carried by the DGMs and C_k for A or B, the two types of particles. Then we propose the multiparticle Lüscher formula for generic DGM states as follows:

$$\delta E_\mu = -i\sum_{b=1}^{4}\Bigg\{\sum_{l=1}^{M_A}(-1)^{F_b}\left(1-\frac{\epsilon_{Q_l}'(p_l)}{\epsilon_1'(\tilde{q}^*)}\right)e^{-i\tilde{q}^*L}\times\left[\mathrm{Res}_{q^*=\tilde{q}^*}S_{b\alpha_l}^{AAb\alpha_l}(q^*,p_l)\right]\prod_{k\neq l}^{M_A+M_B}S_{b\alpha_k}^{AC_kb\alpha_k}(q^*,p_k) + \sum_{l=1}^{M_B}(-1)^{F_b}\left(1-\frac{\epsilon_{\tilde{Q}_l}'(p_l)}{\epsilon_1'(\tilde{q}^*)}\right)e^{-i\tilde{q}^*L}\left[\mathrm{Res}_{q^*=\tilde{q}^*}S_{b\alpha_l}^{BBb\alpha_l}\times(q^*,p_l)\right]\prod_{k\neq l}^{M_A+M_B}S_{b\alpha_k}^{BC_kb\alpha_k}(q^*,p_k)\Bigg\}. \tag{3.1}$$

Here, the energy dispersion relation for the DGM is given by

$$\epsilon_Q(p) = \sqrt{\frac{Q^2}{4}+4g^2\sin^2\frac{p}{2}}. \tag{3.2}$$

Here the coupling constant $g = h(\lambda)$ is still an unknown function of λ which behaves as $h(\lambda) \sim \lambda$ for small λ, and $h(\lambda) \sim \sqrt{\lambda/2}$ for large λ.

B. S matrix elements for the dyonic GM

The S matrix elements for the DGM are in general complicated. However, we can consider a simplest case of the DGMs composed of only A-type ϕ_1's which are the first bosonic particle in the fundamental representation of $SU(2|2)$. It is obvious that these bound states do exist since the elementary S matrix element S_{11}^{AA11} does have a pole. The same holds for the B-type DGMs. However, the hybrid-type DGMs are not possible because the S^{AB} S matrix does not have any bound-state pole.

The Lüscher correction needs only those S matrix elements which have the same incoming and outgoing $SU(2|2)$ quantum numbers after scattering with a virtual particle. In particular, we can easily compute the matrix elements between an elementary magnon and a the bound state made of only ϕ_1's (Q of them) denoted by $\mathbf{1}_Q$ [33]:

$$S_{b\mathbf{1}_Q}^{AAb\mathbf{1}_Q}(y, X^{(Q)}) = \prod_{k=1}^{Q} S_{b1}^{AAb1}(y, x_k) = \prod_{k=1}^{Q}\left[\frac{1-\frac{1}{y^+x_k^-}}{1-\frac{1}{y^-x_k^+}}\sigma_{\rm BES}(y, x_k)\tilde{a}_b(y, x_k)\right], \tag{3.3}$$

where $\tilde{a}_b$ are given by [42,43]

$$\tilde{a}_1(y, x) = a_1(y, x),$$
$$\tilde{a}_2(y, x) = a_1(y, x) + a_2(y, x),$$
$$\tilde{a}_3(y, x) = \tilde{a}_4(y, x) = a_6(y, x)$$
$$a_1(y, x) = \frac{x^- - y^+}{x^+ - y^-}\frac{\eta(x)\eta(y)}{\tilde{\eta}(x)\tilde{\eta}(y)}$$
$$a_2(y, x) = \frac{(y^- - y^+)(x^- - x^+)(x^- - y^+)}{(y^- - x^+)(x^-y^- - x^+y^+)}\frac{\eta(x)\eta(y)}{\tilde{\eta}(x)\tilde{\eta}(y)}$$
$$a_6(y, x) = \frac{y^+ - x^+}{y^- - x^+}\frac{\eta(y)}{\tilde{\eta}(y)}.$$

As noticed in [33], a_2/a_1 and a_6/a_1 are negligible $\mathcal{O}(1/g)$ corrections in the classical limit $g \gg 1$. Therefore, the S matrix with $b = 1$ is a most important factor for our computation which can be written as

$$\begin{aligned} S_{1\mathbf{1}_Q}^{AA1\mathbf{1}_Q}(y, X^{(Q)}) &= \sigma_{\rm BES}(y, X^{(Q)})\prod_{k=1}^{Q}\left[\frac{1-\frac{1}{y^+x_k^-}}{1-\frac{1}{y^-x_k^+}}\cdot\frac{x_k^- - y^+}{x_k^+ - y^-} \times \frac{\eta(x_k)\eta(y)}{\tilde{\eta}(x_k)\tilde{\eta}(y)}\right] \\ &= \sigma_{\rm BES}(y, X^{(Q)})S_{\rm BDS}(y, X^{(Q)})\frac{\eta(X^{(Q)})}{\tilde{\eta}(X^{(Q)})} \times \left(\frac{\eta(y)}{\tilde{\eta}(y)}\right)^Q, \end{aligned} \tag{3.4}$$

$$S_{1\mathbf{1}_Q}^{AB1\mathbf{1}_Q}(y, X^{(Q)}) = \sigma_{\rm BES}(y, X^{(Q)})\frac{\eta(X^{(Q)})}{\tilde{\eta}(X^{(Q)})}\left(\frac{\eta(y)}{\tilde{\eta}(y)}\right)^Q, \tag{3.5}$$

where $S_{\rm BDS}$ is defined by

$$S_{\rm BDS}(y, x) \equiv \frac{1-\frac{1}{y^+x^-}}{1-\frac{1}{y^-x^+}}\cdot\frac{x^- - y^+}{x^+ - y^-}. \tag{3.6}$$

The spectral parameter $X^{(Q)}$ for the DGM is defined by

$$X^{(Q)\pm} = \frac{e^{\pm ip/2}}{4g\sin\frac{p}{2}}\left(Q + \sqrt{Q^2 + 16g^2\sin^2\frac{p}{2}}\right) \equiv e^{(\theta\pm ip)/2}, \tag{3.7}$$

where we introduce θ defined by

$$\sinh\frac{\theta}{2} \equiv \frac{Q}{4g\sin\frac{p}{2}}. \tag{3.8}$$

The frame factors η and $\tilde{\eta}$ are given by [43]

$$\frac{\eta(x_1)}{\tilde{\eta}(x_1)} = \frac{\eta(x_2)}{\tilde{\eta}(x_2)} = 1 \tag{3.9}$$

for the spin-chain frame and

$$\frac{\eta(x_1)}{\tilde{\eta}(x_1)} = \sqrt{\frac{x_2^+}{x_2^-}}, \qquad \frac{\eta(x_2)}{\tilde{\eta}(x_2)} = \sqrt{\frac{x_1^-}{x_1^+}} \tag{3.10}$$

for the string frame.

C. Symmetric DGM state

The classical two spins solution described in Sec. II is a symmetric DGM configuration for both of the S^2 subspaces. The corresponding Lüscher formula is given by Eq. (3.1) with $M_A = M_B = 1$, which can be much simplified as

$$\begin{aligned} \delta E_\mu = -i\sum_{b=1}^{4}(-1)^{F_b}e^{-i\tilde{q}^*L}&\left\{\left(1-\frac{\epsilon_Q'(p_1)}{\epsilon_1'(\tilde{q}^*)}\right) \times\left[\mathrm{Res}_{q^*=\tilde{q}^*}S_{b\mathbf{1}_Q}^{AAb\mathbf{1}_Q}(q^*, p_1)\right]S_{b\mathbf{1}_{\tilde{Q}}}^{ABb\mathbf{1}_{\tilde{Q}}}(q^*, p_2)\right. \\ &\left.+\left(1-\frac{\epsilon_{\tilde{Q}}'(p_2)}{\epsilon_1'(\tilde{q}^*)}\right)\left[\mathrm{Res}_{q^*=\tilde{q}^*}S_{b\mathbf{1}_{\tilde{Q}}}^{AAb\mathbf{1}_{\tilde{Q}}}(q^*, p_2)\right] \times S_{b\mathbf{1}_Q}^{ABb\mathbf{1}_Q}(q^*, p_1)\right\}. \end{aligned} \tag{3.11}$$

As mentioned earlier, only the two cases of $b = 1, 2$ contribute equally in the sum of Eq. (3.11) since these elements contain a_1. Instead of the summation, we can multiply a factor 2 for the case of $b = 1$. In that case, we can compute easily each term using the S matrix elements (3.4) and (3.5). Furthermore, we restrict ourselves for the case where the two DGMs are symmetric in both spheres, namely, $p_1 = p_2$ and $Q = \tilde{Q}$. This leads to

$$\delta E_\mu = -4ie^{-i\tilde{q}^*L}\left(1-\frac{\epsilon_Q'(p)}{\epsilon_1'(\tilde{q}^*)}\right) \times\left[\mathrm{Res}_{q^*=\tilde{q}^*}S_{1\mathbf{1}_Q}^{AA1\mathbf{1}_Q}(q^*, p)\right]S_{1\mathbf{1}_Q}^{AB1\mathbf{1}_Q}(q^*, p). \tag{3.12}$$

Explicit computations of each factor in (3.12) are exactly the same as those in [33]. There are two types of poles of $S_{\rm BDS}(y, X^{(Q)})$. The s-channel pole which describe $(Q+1)$ DGM arises at $y^- = X^{(Q)+}$ while the t-channel pole for $(Q-1)$ DGM (for $Q \ge 2$) at $y^+ = X^{(Q)+}$. We consider the s-channel pole first. Using the location of the pole, we can find

$$\tilde{q}^* = -\frac{i}{2g\sin(\frac{p-i\theta}{2})} \to e^{-i\tilde{q}^*L} \approx \exp\left[-\frac{L}{2g\sin(\frac{p-i\theta}{2})}\right]. \tag{3.13}$$

From Eq. (3.2), one can also obtain

$$1 - \frac{\epsilon'_Q(p)}{\epsilon'_1(\tilde{q}^*)} \approx \frac{\sin\frac{p}{2}\sin\frac{p-i\theta}{2}}{\cosh\frac{\theta}{2}}. \tag{3.14}$$

Furthermore, one can notice from Eqs. (3.4) and (3.5)

$$\begin{aligned}&[\mathrm{Res}_{q^*=\tilde{q}^*}S^{AA1\mathbf{1}_Q}_{1\mathbf{1}_Q}(q^*,p)]S^{AB1\mathbf{1}_Q}_{1\mathbf{1}_Q}(q^*,p)\\ &\quad = \mathrm{Res}_{q^*=\tilde{q}^*}S^{1\mathbf{1}_Q}_{\mathrm{SYM}1\mathbf{1}_Q}(q^*,p),\end{aligned} \tag{3.15}$$

where S_{SYM} is the S matrix of the $\mathcal{N}=4$ SYM theory. Explicit evaluation of the residue term becomes in the leading order

$$\begin{aligned}&-\frac{8ige^{-ip}\sin^2\frac{p}{2}}{\sin\frac{p-i\theta}{2}}\exp\left[-\frac{2e^{-\theta/2}\sin\frac{p}{2}}{\sin\frac{p-i\theta}{2}}\right]\left(\frac{\eta(X^{(Q)})}{\tilde{\eta}(X^{(Q)})}\right)^2\\ &\quad\times\left(\frac{\eta(y)}{\tilde{\eta}(y)}\right)^{2Q}.\end{aligned} \tag{3.16}$$

Combining all these together, we get

$$\begin{aligned}\delta E_\mu &= -\frac{8ge^{-ip}\sin^3\frac{p}{2}}{\cosh\frac{\theta}{2}}\exp\left[-\frac{2e^{-\theta/2}\sin\frac{p}{2}}{\sin\frac{p-i\theta}{2}} - \frac{L}{2g\sin(\frac{p-i\theta}{2})}\right]\left(\frac{\eta(X^{(Q)})}{\tilde{\eta}(X^{(Q)})}\right)^2\left(\frac{\eta(y)}{\tilde{\eta}(y)}\right)^{2Q}\\ &= -\frac{32g\sin^3\frac{p}{2}e^{i\alpha}}{\cosh\frac{\theta}{2}}\exp\left[-\frac{2\sin^2\frac{p}{2}\cosh^2\frac{\theta}{2}}{\sin^2\frac{p}{2}+\sinh^2\frac{\theta}{2}}\left(\frac{L-Q}{2g\sin\frac{p}{2}\cosh\frac{\theta}{2}}+1\right)\right]\\ &= -\frac{32g^2\sin^4\frac{p}{2}e^{i\alpha}}{\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}}\exp\left[-\frac{2\sin^2\frac{p}{2}(L+\sqrt{Q^2+16g^2\sin^2\frac{p}{2}})\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}}{Q^2+16g^2\sin^4\frac{p}{2}}\right].\end{aligned} \tag{3.17}$$

The phase factor $e^{i\alpha}$ includes various phases arising in the computation as well as the frame dependence of η. As argued in [33], we will drop this phase assuming that this cancels out with the appropriate prescription for the Lüscher formula.

The t-channel pole at $y^+ = X^{(Q)+}$ gives exactly the same contribution up to a phase factor. Therefore, combining together, we finally obtain the finite-size effect of the two symmetric DGM configuration as follows:

$$\delta E_\mu = -\frac{64g^2\sin^4\frac{p}{2}}{\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}}\exp\left[-\frac{2\sin^2\frac{p}{2}(L+\sqrt{Q^2+16g^2\sin^2\frac{p}{2}})\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}}{Q^2+16g^2\sin^4\frac{p}{2}}\right]. \tag{3.18}$$

This is exactly what we have derived in Eq. (2.18) if we identify $J_1 = L$, $J_2 = Q$, and $g = \sqrt{\lambda}/2$.

IV. CONCLUDING REMARKS

In this note we have proposed the Lüscher formula for μ-term correction of magnon bound states and computed explicitly the correction for the two symmetric DGMs. This result is compared with a classical string computation based on Neumann-Rosochatius reduction. We showed that the two results match exactly. This provides another confirmation for the S matrix of the $\mathcal{N}=6$ CS theory [18] in addition to those already investigated [37,38]. It is interesting to apply a similar analysis to asymmetric GM and DGM configurations on the two S^2 spheres. If the A and B particles are introduced asymmetrically, the S matrix elements entering into the Lüscher formula become quite different from those of $\mathcal{N}=4$ SYM theory. A similar analysis for "small GM" has been performed for one spin case in [37] which contains an imaginary value in the correction. One way of clarifying the unusual result is to do a similar computation for DGMs which have two spins. Finally, we emphasize that we have computed only the μ-term in this paper which gives the leading classical limit. It would be important to extend this result to one-loop order in semiclassical string theory and compare with the S matrix computation.

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