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Finite-size dyonic giant magnons in TsT-transformed $AdS_5 \times S^5$

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Abstract: We investigate dyonic giant magnons propagating on γ-deformed $AdS_5 \times S^5$ by Neumann-Rosochatius reduction method with a twisted boundary condition. We compute finite-size effect of the dispersion relations of dyonic giant magnons which generalizes the previously known case of the giant magnons with one angular momentum found by Bykov and Frolov.

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Contents

1 Introduction

Investigations on AdS/CFT duality [1–3] for the cases with reduced or without supersymmetry is of obvious interest and importance. An interesting example of such correspondence between gauge and string theory models with reduced supersymmetry is provided by an exactly marginal deformation of $\mathcal{N} = 4$ super Yang-Mills theory [4] and string theory on a β-deformed $AdS_5 \times S^5$ background suggested in [5]. When $\beta \equiv \gamma$ is real, the deformed background can be obtained from $AdS_5 \times S^5$ by the so-called TsT transformation. It includes T-duality on one angle variable, a shift of another isometry variable, then a second T-duality on the first angle [5, 6]. Taking into account that the five-sphere has three isometric coordinates, one can consider generalization of the above procedure, consisting of chain of three TsT transformations. The result is a regular three-parameter deformation of $AdS_5 \times S^5$ string background, dual to a non-supersymmetric deformation of $\mathcal{N} = 4$ super Yang-Mills [6], which is conformal in the planar limit to any order of perturbation theory [7]. The action for this γ_i-deformed $(i = 1, 2, 3)$ gauge theory can be obtained from the initial one after replacement of the usual product with associative $*$-product [5, 6, 8].

An essential property of the TsT transformation is that it preserves the classical integrability of string theory on $AdS_5 \times S^5$ [6], which also implies that in the light-cone gauges of [9, 10] the string dynamics on both backgrounds is described by the same Hamiltonian density. The γ-dependence enters only through the *twisted* boundary conditions and the *level-matching* condition. The last one is modified since a closed string in the deformed background corresponds to an open string on $AdS_5 \times S^5$ in general.

The finite-size correction to the giant magnon [11] energy-charge relation, in the γ-deformed background, has been found in [12], by using conformal gauge and the string sigma model reduced to $R_t \times S^3$. For the deformed case, this is the smallest consistent reduction due to the *twisted* boundary conditions. It turns out that even for the three-parameter deformation, the reduced model depends only on one of them - γ_3. As far as there are two isometry angles ϕ_1, ϕ_2 on S^3, the solution can carry two non-vanishing angular momenta J_1, J_2. Then, the giant magnon is an open string solution with only one charge $J_1 \neq 0$. The momentum p of the magnon excitation in the corresponding spin chain is identified with the angular difference $\Delta\phi_1$ between the end-points of the string, since in the light-cone gauge $t = \tau$, $p_{\phi_1} = 1$, it is equal to the worldsheet momentum p_{ws} of a soliton [13]. The other angle satisfies the following *twisted* boundary conditions [12]

$$\Delta\phi_2 = 2\pi(n_2 - \gamma_3 J_1),$$

where n_2 is an integer winding number of the string in the second isometry direction of the deformed sphere S^3_γ.

An interesting extension of this study is the dyonic giant magnon. This state corresponds to bound states of the fundamental magnons and stable even in the deformed theory. Understanding its string theory analog in the strong coupling limit can be helpful to extend the AdS/CFT duality to the deformed theories.

The paper is organized as follows. In section 2 we consider in brief the γ-deformed giant magnon as described in the article by Bykov and Frolov [12], and give their result about the finite-size effect on the dispersion relation. In section 3 we introduce the classical string action on $R_t \times S^3$, the corresponding Neumann-Rosochatius (NR) integrable system and compute the conserved quantities and angular differences for the case at hand. In section 4 we provide our main result on the finite-size dyonic giant magnon. We conclude the paper with some remarks in section 5. appendix A contains information about the elliptic integrals appearing in the calculations, the ϵ-expansions used and the solutions for the parameters.

2 The γ-deformed giant magnon

Our aim here is to briefly describe the explanations and the main result derived in [12].

The bosonic part of the Green-Schwarz action for strings on the γ-deformed $AdS_5 \times S^5_\gamma$ [14] reduced to $R_t \times S^5_\gamma$ can be written as (the common radius R of AdS_5 and S^5_γ is set to 1)

$$\begin{aligned} S = -\frac{T}{2}\int d\tau d\sigma \Big\{ &\sqrt{-\gamma}\gamma^{ab}\left[-\partial_a t\partial_b t + \partial_a r_i \partial_b r_i + G r_i^2 \partial_a\varphi_i\partial_b\varphi_i \right. \\ &+ \left. G r_1^2 r_2^2 r_3^2 \left(\hat{\gamma}_i \partial_a \varphi_i\right)\left(\hat{\gamma}_j \partial_b\varphi_j\right)\right] \\ &-2G\,\epsilon^{ab}\left(\hat{\gamma}_3 r_1^2 r_2^2 \partial_a\varphi_1\partial_b\varphi_2 + \hat{\gamma}_1 r_2^2 r_3^2 \partial_a\varphi_2\partial_b\varphi_3 + \hat{\gamma}_2 r_3^2 r_1^2\partial_a\varphi_3\partial_b\varphi_1\right)\Big\}, \end{aligned} \tag{2.1}$$

where T is the string tension, γ^{ab} is the worldsheet metric, φ_i are the three isometry angles of the deformed S^5_γ, and

$$\sum_{i=1}^{3} r_i^2 = 1, G^{-1} = 1 + \hat{\gamma}_3 r_1^2 r_2^2 + \hat{\gamma}_1 r_2^2 r_3^2 + \hat{\gamma}_2 r_1^2 r_3^2. \tag{2.2}$$

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The deformation parameters $\hat{\gamma}_i$ are related to γ_i which appear in the dual gauge theory as follows

$$\hat{\gamma}_i = 2\pi T \gamma_i = \sqrt{\lambda}\gamma_i.$$

When $\hat{\gamma}_i = \hat{\gamma}$ this becomes the supersymmetric background of [5], and the deformation parameter γ enters the $\mathcal{N} = 1$ SYM superpotential in the following way

$$W \propto tr\left(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2\right).$$

By using the TsT transformations which map the string theory on $AdS_5 \times S^5$ to the γ_i-deformed theory, one can relate the angle variables ϕ_i on S^5 to the angles φ_i of the γ_i-deformed geometry [6]:

$$p_i = \pi_i, r_i^2\phi_i' = r_i^2\left(\varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k\right), i = 1, 2, 3, \tag{2.3}$$

where p_i, π_i are the momenta conjugated to ϕ_i, φ_i respectively, and the summation is over j, k. The equality $p_i = \pi_i$ implies that the charges

$$J_i = \int d\sigma p_i$$

are invariant under the TsT transformation.

If none of the variables r_i is vanishing on a given string solution, from (2.3) one gets

$$\phi_i' = \varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k.$$

Integrating the above equations and taking into account that for a closed string in the γ-deformed background

$$\Delta\varphi_i = \varphi_i(r) - \varphi_i(-r) = 2\pi n_i, n_i \in \mathbb{Z},$$

one finds the *twisted* boundary conditions for the angles ϕ_i on the original S^5 space

$$\Delta\phi_i = \phi_i(r) - \phi_i(-r) = 2\pi\left(n_i - \nu_i\right), \nu_i = \varepsilon_{ijk}\gamma_j J_k, J_i = \int_{-r}^{r} d\sigma p_i.$$

It is obvious that if the *twists* ν_i are not integer, then a closed string on the deformed background is mapped to an open string on $AdS_5 \times S^5$.

The particular case considered in [12] corresponds to $J_2 = J_3 = 0$, $\nu_1 = 0$, and as a result the angles $\phi_{1,2}$ of the undeformed S^3 satisfy the following *twisted* boundary conditions

$$p = \Delta\phi_1 = \phi_1(r) - \phi_1(-r), \delta = \Delta\phi_2 = \phi_2(r) - \phi_2(-r) = 2\pi\left(n_2 - \gamma_3 J_1\right),$$

where in fact δ plays the role of the deformation parameter. By using the ansatz

$$\begin{aligned}
\phi_1 &= \omega\tau + \frac{p}{2r}(\sigma - v\tau) + \phi(\sigma - v\tau), \\
\phi_2 &= \nu\tau + \frac{\delta}{2r}(\sigma - v\tau) + \alpha(\sigma - v\tau), \\
\chi &= \chi(\sigma - v\tau),
\end{aligned}$$

where ϕ, α and χ satisfy periodic boundary conditions, the authors of [12] found that the giant magnon string solution can be completely determined from the equations

$$\begin{aligned}\mathcal{E} \equiv \frac{E_s}{\frac{\sqrt{\lambda}}{2\pi}} &= 2\int_{-r}^{0} d\sigma = 2r,\\ \mathcal{J}_1 = \frac{J_1}{\frac{\sqrt{\lambda}}{2\pi}} &= \frac{2}{1-v^2}\left(rv^2A_1 + \omega\int_{\chi_{\min}}^{\chi_{\max}} d\chi \frac{1-\chi}{|\chi'|}\right),\\ \mathcal{J}_2 = \frac{J_1}{\frac{\sqrt{\lambda}}{2\pi}} &\propto rv^2A_2 + \nu\int_{\chi_{\min}}^{\chi_{\max}} d\chi\frac{\chi}{|\chi'|} = 0,\\ \frac{p}{2} + \frac{rv\omega}{1-v^2} &= -\frac{vA_1}{1-v^2}\int_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{(1-\chi)|\chi'|},\\ \delta + \frac{rv\nu}{1-v^2} &= -\frac{vA_2}{1-v^2}\int_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{\chi|\chi'|},\end{aligned} \tag{2.4}$$

where A_1 and A_2 are parameters related by $\omega A_1 + \nu A_2 + 1 = 0$, $\chi = 1 - r_1^2 = r_2^2$, and

$$|\chi'| = \frac{2\sqrt{\omega^2-\nu^2}}{1-v^2}\sqrt{(\chi_{\max}-\chi)(\chi-\chi_{\min})(\chi-\chi_n)},$$
$$0 < \chi_{\min} < \chi < \chi_{\max} < 1, \chi_n < 0.$$

The dispersion relation in the large $\mathcal{J}_1$ limit can be found from (2.4) as an expansion in

$$\exp\left(-\frac{\mathcal{J}_1}{\sin(p/2)}\right),$$

and up to the leading order it is [12]

$$E - J_1 = \frac{\sqrt{\lambda}}{\pi}\sin(p/2)\left[1 - \frac{4}{e^2}\sin^2(p/2)\cos(\Phi)\exp\left(-\frac{\mathcal{J}_1}{\sin(p/2)}\right)\right], \tag{2.5}$$

where

$$\Phi = \frac{\delta}{2^{3/2}\cos^3(p/4)}, -\pi \le \delta \le \pi, -\pi \le p \le \pi.$$

In the limit $\Phi \to 0$ the formula (2.5) reduces to the one obtained in [13].

3 Towards finite-size dyonic giant magnon

As explained in the previous section, instead of considering strings on the γ-deformed background $AdS_5 \times S^5_\gamma$, we can consider strings on the original $AdS_5 \times S^5$ space, but with *twisted* boundary conditions. Actually, here we are interested in string configurations living in the $R_t \times S^3$ subspace, which can be described by the NR integrable system [15].

3.1 Strings on $R_t \times S^3$ and the NR integrable system

We start with the Polyakov string action

$$S^P = -\frac{T}{2}\int d^2\xi\sqrt{-\gamma}\gamma^{ab}G_{ab}, G_{ab} = g_{MN}\partial_a X^M\partial_b X^N, \tag{3.1}$$
$$\partial_a = \partial/\partial\xi^a, a, b = (0,1), (\xi^0, \xi^1) = (\tau, \sigma), M, N = (0, 1, \dots, 9),$$

and choose *conformal gauge* $\gamma^{ab} = \eta^{ab} = diag(-1, 1)$, in which the Lagrangian and the Virasoro constraints take the form

$$\mathcal{L}_s = \frac{T}{2}(G_{00} - G_{11}) \tag{3.2}$$
$$G_{00} + G_{11} = 0, \qquad G_{01} = 0. \tag{3.3}$$

We embed the string in $R_t \times S^3$ subspace of $AdS_5 \times S^5$ as follows

$$Z_0 = Re^{it(\tau,\sigma)}, W_j = Rr_j(\tau,\sigma)e^{i\phi_j(\tau,\sigma)}, \sum_{j=1}^{2} W_j\bar{W}_j = R^2,$$

where R is the common radius of AdS_5 and S^5, and t is the AdS time. For this embedding, the metric induced on the string worldsheet is given by

$$G_{ab} = -\partial_{(a}Z_0\partial_{b)}\bar{Z}_0 + \sum_{j=1}^{2}\partial_{(a}W_j\partial_{b)}\bar{W}_j = R^2\left[-\partial_a t\partial_b t + \sum_{j=1}^{2}\left(\partial_a r_j\partial_b r_j + r_j^2\partial_a\phi_j\partial_b\phi_j\right)\right].$$

The corresponding string Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_s + \Lambda_s\left(\sum_{j=1}^{2} r_j^2 - 1\right),$$

where Λ_s is a Lagrange multiplier. In the case at hand, the background metric does not depend on t and ϕ_j. Therefore, the conserved quantities are the string energy E_s and two angular momenta J_j, given by

$$E_s = -\int d\sigma\frac{\partial\mathcal{L}_s}{\partial(\partial_0 t)}, J_j = \int d\sigma\frac{\partial\mathcal{L}_s}{\partial(\partial_0\phi_j)}. \tag{3.4}$$

In order to reduce the string dynamics to the NR integrable system, we use the ansatz [15]

$$t(\tau,\sigma) = \kappa\tau, r_j(\tau,\sigma) = r_j(\xi), \phi_j(\tau,\sigma) = \omega_j\tau + f_j(\xi), \tag{3.5}$$
$$\xi = \alpha\sigma + \beta\tau, \kappa, \omega_j, \alpha, \beta = constants.$$

It can be shown that after integrating once the equations of motion for f_a, which gives

$$f_a' = \frac{1}{\alpha^2 - \beta^2}\left(\frac{C_a}{r_a^2} + \beta\omega_a\right), C_a = constants, \tag{3.6}$$

one ends up with the following effective Lagrangian for the coordinates r_a (prime is used for $d/d\xi$)

$$L_{NR} = (\alpha^2 - \beta^2) \sum_{j=1}^{2} \left[r_j'^2 - \frac{1}{(\alpha^2-\beta^2)^2} \left(\frac{C_j^2}{r_j^2} + \alpha^2 \omega_j^2 r_j^2 \right) \right] + \Lambda_s \left(\sum_{j=1}^{2} r_j^2 - 1 \right). \quad (3.7)$$

This is the Lagrangian for the NR integrable system [15].

The Virasoro constraints (3.3) give the conserved Hamiltonian H_{NR} and a relation between the embedding parameters and the arbitrary constants C_j:

$$H_{NR} = (\alpha^2 - \beta^2) \sum_{j=1}^{2} \left[r_j'^2 + \frac{1}{(\alpha^2-\beta^2)^2} \left(\frac{C_j^2}{r_j^2} + \alpha^2 \omega_j^2 r_j^2 \right) \right] = \frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}\kappa^2, \quad (3.8)$$

$$\sum_{j=1}^{2} C_j \omega_j + \beta \kappa^2 = 0. \quad (3.9)$$

On the ansatz (3.5), E_s and J_j defined in (3.4) take the form

$$E_s = \frac{\sqrt{\lambda}}{2\pi} \frac{\kappa}{\alpha} \int d\xi, \; J_j = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{\alpha^2-\beta^2} \int d\xi \left(\frac{\beta}{\alpha} C_j + \alpha \omega_j r_j^2 \right), \quad (3.10)$$

where we have used that the string tension and the 't Hooft coupling constant λ are related by $TR^2 = \frac{\sqrt{\lambda}}{2\pi}$.

3.2 Conserved quantities and angular differences

If we introduce the variable

$$\chi = 1 - r_1^2 = r_2^2,$$

and use (3.9), the first integral (3.8) can be rewritten as

$$\begin{aligned}
\chi'^2 &= \frac{4\omega_1^2(1-u^2)}{\alpha^2(1-v^2)^2} \left\{ -\chi^3 + \frac{(1-w^2)+(1-v^2w^2)-u^2}{1-u^2}\chi^2 \right. \\
&\left. - \frac{1-(1+v^2)w^2+v^2[(w^2-u^2j)^2-j^2]}{1-u^2}\chi - \frac{v^2u^2j^2}{1-u^2} \right\} \\
&= \frac{4\omega_1^2(1-u^2)}{\alpha^2(1-v^2)^2} (\chi_{\max}-\chi)(\chi-\chi_{\min})(\chi-\chi_n), \quad (3.11)
\end{aligned}$$

where

$$v = -\frac{\beta}{\alpha}, u = \frac{\omega_2}{\omega_1}, w = \frac{\kappa}{\omega_1}, j = -\frac{C_2}{\beta\omega_2}.$$

Correspondingly, the conserved quantities (3.10) transform to

$$\begin{aligned}
\mathcal{E} &= \frac{\kappa}{\alpha} \int_{-r}^{r} d\xi = \frac{(1-v^2)w}{\sqrt{1-u^2}} \int_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{\sqrt{(\chi_{\max}-\chi)(\chi-\chi_{\min})(\chi-\chi_n)}}, \\
\mathcal{J}_1 &= \frac{1}{\sqrt{1-u^2}} \int_{\chi_{\min}}^{\chi_{\max}} \frac{\left[1-v^2\left(w^2-u^2j\right)-\chi\right] d\chi}{\sqrt{(\chi_{\max}-\chi)(\chi-\chi_{\min})(\chi-\chi_n)}}, \quad (3.12) \\
\mathcal{J}_2 &= \frac{u}{\sqrt{1-u^2}} \int_{\chi_{\min}}^{\chi_{\max}} \frac{\left(\chi-v^2j\right) d\chi}{\sqrt{(\chi_{\max}-\chi)(\chi-\chi_{\min})(\chi-\chi_n)}}.
\end{aligned}$$

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Now, let us compute the angular differences

$$p = \Delta\phi_1 = \phi_1(r) - \phi_1(-r), \delta = \Delta\phi_2 = \phi_2(r) - \phi_2(-r) = 2\pi\,(n_2 - \gamma_3 J_1)\,.$$

$$\begin{aligned} p &= \int_{-r}^{r} d\xi f_1' = \frac{\beta\omega_1}{\alpha^2(1-v^2)}\int_{-r}^{r}\left(1 - \frac{w^2 - u^2 j}{r_1^2}\right)d\xi \qquad (3.13)\\ &= \frac{v}{\sqrt{1-u^2}}\int_{\chi_{\min}}^{\chi_{\max}}\left(\frac{w^2-u^2 j}{1-\chi} - 1\right)\frac{d\chi}{\sqrt{(\chi_{\max}-\chi)(\chi-\chi_{\min})(\chi-\chi_n)}},\\ \delta &= \int_{-r}^{r} d\xi f_2' = \frac{\beta\omega_2}{\alpha^2(1-v^2)}\int_{-r}^{r}\left(1-\frac{j}{r_2^2}\right)d\xi \qquad (3.14)\\ &= \frac{uv}{\sqrt{1-u^2}}\int_{\chi_{\min}}^{\chi_{\max}}\left(\frac{j}{\chi}-1\right)\frac{d\chi}{\sqrt{(\chi_{\max}-\chi)(\chi-\chi_{\min})(\chi-\chi_n)}}. \end{aligned}$$

The elliptic integrals in (3.12), (3.13) and (3.14) are given in appendix A.

4 Finite-size dyonic giant magnon

First of all, for correspondence with the notations in [12], we fix $\kappa = \alpha = 1$, rename $\omega_1 \to \omega$, $\omega_2 \to \nu$, introduce the parameters A_1, A_2, and the functions $\phi(\xi)$, $\alpha(\xi)$ as follows

$$\begin{aligned} C_1 &= -vA_1, C_2 = -vA_2,\\ f_1(\xi) &= \frac{p}{2r}\xi + \phi(\xi), f_2(\xi) = \frac{\delta}{2r}\xi + \alpha(\xi). \end{aligned}$$

Then, from (3.12), (3.13) and (3.14) one finds

$$\begin{aligned} \mathcal{E} &= \frac{4\tilde{\kappa}}{\sqrt{(1-\chi_n)(1-\tilde{v}^2)}}\mathbf{K}(1-\epsilon),\\ \mathcal{J}_1 &= \frac{4\tilde{\kappa}}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}}\left[\left(\omega(1-\chi_n) - \frac{v^2}{\omega}(1+\nu A_2)\right)\mathbf{K}(1-\epsilon)\right.\\ &\quad \left. - \,\omega(1-\chi_n)(1-\tilde{v}^2)\mathbf{E}(1-\epsilon)\right],\\ \mathcal{J}_2 &= \frac{4\tilde{\kappa}}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}}\left[\left(v^2 A_2 + \nu\chi_n\right)\right)\mathbf{K}(1-\epsilon) \qquad (4.1)\\ &\quad \left. + \,\nu(1-\chi_n)(1-\tilde{v}^2)\mathbf{E}(1-\epsilon)\right],\\ p &= \frac{4\tilde{\kappa}v}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}}\left[\frac{1+\nu A_2}{\omega(1-\chi_n)\tilde{v}^2}\Pi\left(\frac{\tilde{v}^2-1}{\tilde{v}^2}(1-\epsilon)|1-\epsilon\right) - \omega\mathbf{K}(1-\epsilon)\right],\\ \delta &= -\frac{2\tilde{\kappa}v}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}}\left[\frac{A_2}{(1-\tilde{v}^2)\left(1+\chi_n\frac{\tilde{v}^2}{1-\tilde{v}^2}\right)}\Pi\left(\frac{1-\chi_n}{1+\chi_n\frac{\tilde{v}^2}{1-\tilde{v}^2}}(1-\epsilon)|1-\epsilon\right)\right.\\ &\quad \left. + \,\nu\mathbf{K}(1-\epsilon)\right],\\ \tilde{\kappa} &= \frac{1-v^2}{2\sqrt{\omega^2-\nu^2}}. \end{aligned}$$

In the above equalities we introduced the new parameters

$$\tilde{v}^2 = \frac{1-\chi_{\max}}{1-\chi_n}, \epsilon = \frac{\chi_{\min}-\chi_n}{\chi_{\max}-\chi_n}$$

instead of $\chi_{\max}$ and $\chi_{\min}$.

In order to obtain the finite-size correction to the energy-charge relation, we have to consider the limit $\epsilon \to 0$ in (4.1). The behavior of the complete elliptic integrals in this limit is given in appendix A. For the parameters in (4.1), we make the following ansatz

$$\begin{aligned} &v = v_0 + v_1\epsilon + v_2\epsilon\log(\epsilon), && \tilde{v} = \tilde{v}_0 + \tilde{v}_1\epsilon + \tilde{v}_2\epsilon\log(\epsilon), && \omega = 1 + \omega_1\epsilon, \\ &\nu = \nu_0 + \nu_1\epsilon + \nu_2\epsilon\log(\epsilon), && A_2 = A_{21}\epsilon, && \chi_n = \chi_{n1}\epsilon. \end{aligned} \tag{4.2}$$

We insert all these expansions into (4.1) and impose the conditions:

1. p - finite
2. $\mathcal{J}_2$ - finite
3. $\mathcal{E} - \mathcal{J}_1 = \frac{2\sqrt{1-v_0^2-\nu_0^2}}{1-\nu_0^2} - \frac{(1-v_0^2-\nu_0^2)^{3/2}}{2(1-\nu_0^2)}\cos(\Phi)\epsilon$

From the first two conditions, we obtain the relations

$$p = \arcsin\left(\frac{2v_0\sqrt{1-v_0^2-\nu_0^2}}{1-\nu_0^2}\right), \tilde{v}_0 = \frac{v_0}{\sqrt{1-\nu_0^2}}, \mathcal{J}_2 = \frac{2\nu_0\sqrt{1-v_0^2-\nu_0^2}}{1-\nu_0^2}, \tag{4.3}$$

as well as six more equations. The third condition gives another two equations for the coefficients in (4.2). Thus, we have a system of eight equations, from which we can find all remaining coefficients in (4.2), except A_{21}. A_{21} can be found from the equation for δ to be

$$A_{21} = -\Lambda\frac{(1-v_0^2-\nu_0^2)^{3/2}}{v_0(1-\nu_0^2)}\sin(\Phi),$$

where Λ is constant with respect to Φ. The equations (4.3) are solved by

$$v_0 = \frac{\sin(p)}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}, \tilde{v}_0 = \cos(p/2), \nu_0 = \frac{\mathcal{J}_2}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}. \tag{4.4}$$

Replacing (4.4) into the solutions for the other coefficients, one obtains the expressions given in appendix A.

To the leading order, the equation for $\mathcal{J}_1$ gives

$$\epsilon = 16\exp\left[-\frac{2\left(\mathcal{J}_1 + \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right)\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\sin^2(p/2)}{\mathcal{J}_2^2 + 4\sin^4(p/2)}\right].$$

Accordingly, to the leading order again, the equation for δ reads

$$2\pi\left(n_2 - \gamma_3\frac{\sqrt{\lambda}}{2\pi}\mathcal{J}_1\right) + \frac{1}{2}\mathcal{J}_2\frac{\mathcal{J}_1 + \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}{\mathcal{J}_2^2 + 4\sin^4(p/2)}\sin(p) = \Lambda\Phi. \tag{4.5}$$

Finally, the dispersion relation, including the leading finite-size correction, takes the form

$$\begin{aligned} \mathcal{E} - \mathcal{J}_1 = &\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)} - \frac{16\sin^4(p/2)}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}\cos(\Phi) \\ &\exp\left[-\frac{2\left(\mathcal{J}_1 + \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right)\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\sin^2(p/2)}{\mathcal{J}_2^2 + 4\sin^4(p/2)}\right]. \end{aligned} \tag{4.6}$$

For $\mathcal{J}_2 = 0$, (4.6) reduces to the result found in [12].

5 Concluding remarks

In this paper we considered giant magnons with two angular momenta, or dyonic giant magnons, propagating on γ-deformed $AdS_5 \times S^5_\gamma$, obtained from $AdS_5 \times S^5$ by means of a chain of TsT transformations. In the framework of the approach used in [12], instead of considering strings on the γ-deformed background $AdS_5 \times S^5_\gamma$, we considered strings on the original $AdS_5 \times S^5$ space, but with *twisted* boundary conditions. Restricting ourselves to the $R_t \times S^3$ subspace, we determined the leading finite-size effect on the dispersion relation. The obtained dispersion relation is a generalization of the previously known one for the giant magnons with one angular momentum, found by Bykov and Frolov in [12].

It would be interesting to reproduce the energy-charge relation (4.6) by using the Lüscher's approach [16]. To this end, we need a generalization of the Lüscher's formulas for the case of nontrivial *twisted* boundary conditions.

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A Elliptic integrals, ϵ-expansions and solutions for the parameters

The elliptic integrals appearing in the main text are given by

$$\int_{\chi_{\rm min}}^{\chi_{\rm max}} \frac{d\chi}{\sqrt{(\chi_{\rm max}-\chi)(\chi-\chi_{\rm min})(\chi-\chi_n)}} = \frac{2}{\sqrt{\chi_{\rm max}-\chi_n}}\mathbf{K}(1-\epsilon),$$

$$\int_{\chi_{\rm min}}^{\chi_{\rm max}} \frac{\chi d\chi}{\sqrt{(\chi_{\rm max}-\chi)(\chi-\chi_{\rm min})(\chi-\chi_n)}} = \frac{2\chi_n}{\sqrt{\chi_{\rm max}-\chi_n}}\mathbf{K}(1-\epsilon)+2\sqrt{\chi_{\rm max}-\chi_n}\mathbf{E}(1-\epsilon),$$

$$\int_{\chi_{\rm min}}^{\chi_{\rm max}} \frac{d\chi}{\chi\sqrt{(\chi_{\rm max}-\chi)(\chi-\chi_{\rm min})(\chi-\chi_n)}} = \frac{2}{\chi_{\rm max}\sqrt{\chi_{\rm max}-\chi_n}}\Pi\left(1-\frac{\chi_{\rm min}}{\chi_{\rm max}}|1-\epsilon\right),$$

$$\int_{\chi_{\rm min}}^{\chi_{\rm max}} \frac{d\chi}{(1-\chi)\sqrt{(\chi_{\rm max}-\chi)(\chi-\chi_{\rm min})(\chi-\chi_n)}} = \frac{2}{(1-\chi_{\rm max})\sqrt{\chi_{\rm max}-\chi_n}}\Pi\left(-\frac{\chi_{\rm max}-\chi_{\rm min}}{1-\chi_{\rm max}}|1-\epsilon\right),$$

where

$$\epsilon = \frac{\chi_{\rm min}-\chi_n}{\chi_{\rm max}-\chi_n}.$$

We use the following expansions for the elliptic integrals [17]

$$\begin{aligned}\mathbf{K}(1-\epsilon) &= -\frac{1}{2}\log(\epsilon)\left(1+\frac{\epsilon}{4}+O(\epsilon^2)\right)+\log(4)-\frac{1}{4}\left(1-\log(4)\right)\epsilon+O(\epsilon^2),\\ \mathbf{E}(1-\epsilon) &= 1-\epsilon\left(\frac{1}{4}-\log(2)\right)(1+O(\epsilon))-\frac{\epsilon}{4}\log(\epsilon)\left(1+O(\epsilon)\right),\\ \Pi(n|1-\epsilon) &= \frac{1}{2}\left(\frac{\log(\epsilon)}{n-1}\left(1-\frac{n+1}{4(n-1)}\epsilon+O(\epsilon^2)\right)\right.\\ &\quad\left.+\frac{\sqrt{n}\log\left(\frac{1+\sqrt{n}}{1-\sqrt{n}}\right)-\log(16)}{n-1}-\frac{\sqrt{n}\log\left(\frac{1+\sqrt{n}}{1-\sqrt{n}}\right)-(n+1)\log(4)+1}{2(n-1)^2}\epsilon+O(\epsilon^2)\right).\end{aligned}$$

By using the equality [18]

$$\Pi(\nu|m) = \frac{q_1}{q}\Pi(\nu_1|m) - \frac{m}{q\sqrt{-\nu\nu_1}}\mathbf{K}(m),$$

where

$$\begin{aligned}q &= \sqrt{(1-\nu)\left(1-\frac{m}{\nu}\right)}, q_1 = \sqrt{(1-\nu_1)\left(1-\frac{m}{\nu_1}\right)},\\ \nu &= \frac{\nu_1-m}{\nu_1-1}, \nu_1<0, m<\nu<1,\end{aligned}$$

and the above expansion for $\Pi(n|1-\epsilon)$, one can find the following expansion

$$\begin{aligned}\Pi(1-\alpha\epsilon|1-\epsilon) &= \frac{\arctan\left(\sqrt{\frac{1}{\alpha}-1}\right)}{\sqrt{\frac{1}{\alpha}-1}\alpha\epsilon}\\ &+\frac{1}{4}\left(1-\frac{2\arctan\left(\sqrt{\frac{1}{\alpha}-1}\right)}{\sqrt{\frac{1}{\alpha}-1}}-\log\left(\frac{\epsilon}{16}\right)\right)\\ &-\frac{4\alpha^2\sqrt{\frac{1}{\alpha}-1}\arctan\left(\sqrt{\frac{1}{\alpha}-1}\right)+(1-\alpha)\left(2(1+\alpha)+(1+2\alpha)\log\left(\frac{\epsilon}{16}\right)\right)}{8(1-\alpha)}\epsilon+O(\epsilon^2),\end{aligned}$$

where $0<\alpha<1$.

We use the following expansions for the parameters

$$\begin{aligned}v &= v_0+v_1\epsilon+v_2\epsilon\log(\epsilon), \tilde{v}=\tilde{v}_0+\tilde{v}_1\epsilon+\tilde{v}_2\epsilon\log(\epsilon), \omega=1+\omega_1\epsilon,\\ \nu &= \nu_0+\nu_1\epsilon+\nu_2\epsilon\log(\epsilon), A_2=A_{21}\epsilon, \chi_n=\chi_{n1}\epsilon.\end{aligned}$$

The explicit solutions for the coefficients above are given by

$$\begin{aligned}v_0 &= \frac{\sin(p)}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}, \tilde{v}_0=\cos(p/2), \nu_0=\frac{\mathcal{J}_2}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}},\\ v_1 &= \frac{1}{4\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)}\left\{\frac{1}{\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)^{3/2}}\left[\cos(\Phi)\sin(p)\sin^2(p/2)\right.\right.\\ &\quad\times\left(\mathcal{J}_2^4(\log(256)-4)-16(\log(16)-1)\sin^4(p/2)+8\mathcal{J}_2^2\log(2)\sin^2(p)\right.\end{aligned}$$

$$
\begin{aligned}
&+\ 4\sin^2(p/2)\left(\mathcal{J}_2^2(\log(16)-3)+(\log(16)-1)\sin^2(p)\right)\big]-\frac{1}{\mathcal{J}_2^2+4\sin^2(p/2)}\\
&\times\ \big[2\Lambda\mathcal{J}_2\left(4\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)(\log(4)-1)\sin^2(p/2)+\sin^2(p)\left(\mathcal{J}_2^2(1-\log(16))\right.\right.\\
&\left.\left.+\sin^2(p)-\ \log(16)\cos(p)(1-\cos(p))\right)\right)\sin(\Phi)\big]\big\}\,,\\
v_2 &= -\frac{\sin^2(p/2)}{8\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)^2\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)}\left\{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\,\big[4\mathcal{J}_2^4+6\mathcal{J}_2^2-10\right.\\
&+\ \left(15-4\mathcal{J}_2^2\right)\cos(p)-2\left(3+\mathcal{J}_2^2\right)\cos(2p)+\cos(3p)\big]\sin(p)\cos(\Phi)\\
&+2\Lambda\mathcal{J}_2\left[\left(4\mathcal{J}_2^4+17\mathcal{J}_2^2+34\right)\cos(p)-4\left(2+\mathcal{J}_2^2\right)\cos(2p)\right.\\
&\left.\left.-\ \left(2+\mathcal{J}_2^2\right)\cos(3p)+\cos(4p)-12\mathcal{J}_2^2-25\right]\sin(\Phi)\right\},\\
\tilde{v}_1 &= \frac{\sin(p/2)}{8\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)}\left\{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\right.\\
&\times\ \left[\left(4\left(\mathcal{J}_2^2-2\right)\log(2)\cos(\Phi)-\mathcal{J}_2^2-2\right)\sin(p)+(1+\log(16)\cos(\Phi))\right.\\
&\left.\times\ \left(\sin^2(p)+\sin(2p)\right)\right]+8\Lambda\mathcal{J}_2\left(4\cos(p)+\cos(2p)-2\mathcal{J}_2^2-5\right)\log(2)\sin^2(p/2)\sin(\Phi)\big\}\,,\\
\tilde{v}_2 &= \frac{\sin(p/2)}{16\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)}\left[\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\right.\\
&\times\ \left(3-2\mathcal{J}_2^2-4\cos(p)+\cos(2p)\right)\sin(p)\cos(\Phi)\\
&\left.-4\ \Lambda\mathcal{J}_2\left(4\cos(p)+\cos(2p)-2\mathcal{J}_2^2+-5\right)\sin^2(p/2)\sin(\Phi)\right],\\
\omega_1 &= \frac{2\sin^4(p/2)}{\mathcal{J}_2^2+4\sin^4(p/2)}\left(\sin^2(p/2)\cos(\Phi)+\frac{\Lambda\mathcal{J}_2\sin(p)\sin(\Phi)}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}\right),\\
\nu_1 &= \frac{\sin^2(p/2)}{2\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)}\left\{\frac{1}{\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)^{3/2}}\left[\mathcal{J}_2\cos(\Phi)\right.\right.\\
&\times\ \left(9-2\cos(p)\left(6+\mathcal{J}_2^2-8\log(2)\right)-20\log(2)-4\mathcal{J}_2^2\left(\log(4)-1\right)\right.\\
&\left.+\ \cos(2p)\left(\log(16)+3\right)\right)\sin^2(p/2)\big]+\frac{\Lambda\sin(p)\sin(\Phi)}{\mathcal{J}_2^2+4\sin^2(p/2)}\left[6\log(4)+2\log(4)\cos(2p)\right.\\
&\left.\left.-\ \mathcal{J}_2^2\left(\log(256)-2\right)-2\cos(p)\left(\mathcal{J}_2^2+\log(256)\right)\right]\right\},\\
\nu_2 &= \frac{1}{4\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)^{3/2}\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)}\left\{2\mathcal{J}_2\sin^4(p/2)\cos(\Phi)\right.\\
&\times\ \left(5+2\mathcal{J}_2^2-4\cos(p)-\cos(2p)\right)+\frac{4\Lambda\sin^3(p/2)\cos(p/2)\sin(\Phi)}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}\\
&\left.\times\ \left[2\mathcal{J}_2^4+\mathcal{J}_2^2-10+15\cos(p)-\left(\mathcal{J}_2^2+6\right)\cos(2p)+\cos(3p)\right]\right\},\\
A_{21} &= -\Lambda\sin^2(p/2)\tan(p/2)\sin(\Phi),\\
&\chi_{n1} = -\sin^2(p/2)\sin^2(\Phi/2).
\end{aligned}
$$

Let us give some details about the derivation of A_{21} and χ_{n1}, which are zero for the undeformed case. In our third condition on p. 8

$$
\mathcal{E}-\mathcal{J}_1=\frac{2\sqrt{1-v_0^2-\nu_0^2}}{1-\nu_0^2}-\frac{(1-v_0^2-\nu_0^2)^{3/2}}{2(1-\nu_0^2)}\cos(\Phi)\epsilon
$$

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we introduced the angle Φ to describe in a simple way the change of the finite-size correction to the dispersion relation due to the γ-deformation. However, Φ is not an independent new variable. Solving the equations for our parameters, we found

$$\chi_{n1} = -\frac{1 - v_0^2 - \nu_0^2}{1 - \nu_0^2} \sin^2(\Phi/2) = -\sin^2(p/2)\sin^2(\Phi/2), \tag{A.1}$$

i.e. we use Φ instead of χ_{n1}.

There is alternative way to obtain the above relation between χ_{n1} and Φ. After expanding in ϵ, the leading order of the equation for the angle δ is given by

$$\begin{aligned}\delta = &\frac{A_{21} v_0}{\chi_{n1}\sqrt{1 - v_0^2 - \nu_0^2}} \sqrt{\frac{-\chi_{n1}}{1 - \frac{v_0^2}{1-\nu_0^2} + \chi_{n1}}} \arctan \sqrt{\frac{-\chi_{n1}}{1 - \frac{v_0^2}{1-\nu_0^2} + \chi_{n1}}} \\ &+ \frac{\nu_0 v_0}{2\sqrt{1 - v_0^2 - \nu_0^2}} \log\left(\frac{\epsilon}{16}\right). \end{aligned} \tag{A.2}$$

Let us point out that the second term in (A.2) is zero for the one-spin case, since $\nu_0 = 0$ means $J_2 = 0$. If we introduce the angle Φ as

$$\frac{\Phi}{2} = \arctan \sqrt{\frac{-\chi_{n1}}{1 - \frac{v_0^2}{1-\nu_0^2} + \chi_{n1}}},$$

this gives (A.1), and the first term in (A.2) takes the form

$$-\frac{A_{21} v_0 (1 - \nu_0^2)}{(1 - v_0^2 - \nu_0^2)^{3/2}} \Phi \csc \Phi. \tag{A.3}$$

If we impose the natural condition (A.3) to be an angle proportional to the angle Φ, this gives

$$A_{21} = -\Lambda \frac{(1 - v_0^2 - \nu_0^2)^{3/2}}{v_0(1 - \nu_0^2)} \sin(\Phi) = -\Lambda \sin^2(p/2)\tan(p/2)\sin(\Phi),$$

where Λ does not depend on Φ.

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