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An HHL 3-point correlation function in the η-deformed $AdS_5 \times S^5$

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ABSTRACT

We derive the 3-point correlation function between two giant magnons heavy string states and the light dilaton operator with zero momentum in the η-deformed $AdS_5 \times S^5$ valid for any J_1 and η in the semiclassical limit. We show that this result satisfies a consistency relation between the 3-point correlation function and the conformal dimension of the giant magnon. We also provide a leading finite J_1 correction explicitly.

1. Introduction

The AdS/CFT duality [1] between string theories on curved space–times with anti-de Sitter subspaces and conformal field theories in different dimensions has been actively investigated in the last years. A lot of impressive progress has been made in this field of research based mainly on the integrability structures discovered on both sides of the correspondence (for recent review on the AdS/CFT duality, see [2]). For the most studied case of the $\mathcal{N} = 4$ super Yang–Mills theory, the anomalous dimensions of gauge-invariant single-trace operators match non-perturbatively with the string energies in the curved $AdS_5 \times S^5$ background. Integrability provides tools to solve the finite-volume spectral problem exactly.

After these successes, one direction of interesting development is to generalize the duality to larger theories which include the original AdS/CFT as a special case and the other is to go beyond the spectral problem by computing general correlation functions, in particular, the three-point functions, or the structure constants.

An interesting development for the former direction is to study the string theory on the η-deformed $AdS_5 \times S^5$ background [3]. The bosonic part of the superstring sigma model Lagrangian on this η-deformed background and perturbative worldsheet S-matrix were obtained in [4]. The TBA for spectrum and explicit dispersion relation for giant magnon [5] have been derived in [6]. Finite-size effect on the giant magnon spectrum has been computed in [7]. For three-point correlation functions, quite a lot of interesting results on both strong and weak coupling regions were accumulated although non-perturbative results are much more difficult than the spectral problem.

In this letter, we compute the three-point correlation function of two giant magnon heavy operators with finite-size J_1 and a single dilaton light operator of the string theory with the η-deformed $AdS_5 \times S^5$ background [3]. Then, we show that this result is consistent with the dispersion relation of the finite-size giant magnon solution obtained in [7] using Mathematica code.

The paper is organized as follows. In Section 2, we derive the exact semiclassical structure constant valid for any J_1 and η and prove its consistency. In Section 3, we expand it for the case of $J_1 \gg T$ (T is the tension of string) and obtain explicit expression. A brief conclusion is in Section 4 and a short Mathematica code for the consistency condition is provided in Appendix A.

2. Exact semiclassical structure constant

According to [8], the three-point functions of two "heavy" operators and a "light" operator can be approximated by a supergravity vertex operator evaluated at the "heavy" classical string configuration:

$$\langle V_H(x_1) V_H(x_2) V_L(x_3) \rangle = V_L(x_3)_{\text{classical}}.$$

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For $|x_1| = |x_2| = 1$, $x_3 = 0$, the correlation function reduces to

$$\langle V_H(x_1) V_H(x_2) V_L(0)\rangle = \frac{C_{123}}{|x_1 - x_2|^{2\Delta_H}}.$$

Then, the normalized structure constant

$$\mathcal{C}_3 = \frac{C_{123}}{C_{12}}$$

can be found from

$$\mathcal{C}_3 = c_\Delta V_L(0)_{\text{classical}}, \tag{2.1}$$

where c_Δ is the normalized constant of the "light" vertex operator. Actually, we are going to compute the normalized structure constant (2.1). For the case under consideration, the "light" state is represented by the dilaton with zero momentum.

According to [9], $\mathcal{C}_3$ for the infinite-size giant magnons and dilaton with zero momentum in the undeformed $AdS_5 \times S^5$ is given by

$$\mathcal{C}_3 = c_\Delta^d \int_{-\infty}^{+\infty} \frac{d\tau_e}{\cosh^4(\kappa\tau_e)} \int_{-\infty}^{+\infty} d\sigma \left(\kappa^2 + \partial X_K \bar{\partial} X_K\right) = \frac{4c_\Delta^d}{3\kappa} \int_{-\infty}^{+\infty} d\sigma \left(\kappa^2 + \partial X_K \bar{\partial} X_K\right), \tag{2.2}$$

where $t = \kappa\tau_e$ is the Euclidean AdS time and the term $\partial X_K \bar{\partial} X_K$ is proportional to the string Lagrangian on S^2 computed on the giant magnon solution living in the $R_t \times S^2$ subspace.

Since here we are interested in *finite-size* giant magnons, we have to replace

$$\int_{-\infty}^{+\infty} d\sigma \to \int_{-L}^{+L} d\sigma = 2 \int_{\theta_{min}}^{\theta_{max}} \frac{d\theta}{\theta'},$$

where L gives the size of the giant magnon and θ is the non-isometric angle on the two-sphere [11].

Going to the η-deformed $AdS_5 \times S^5$ case, we have to compute the term $\partial X_K \bar{\partial} X_K$ for this background, which is proportional to the string Lagrangian on S^2_η for *finite-size* giant magnons:

$$L_{S^2_\eta} = -\frac{T}{2} \partial X_K \bar{\partial} X_K,$$

where $X_K = (\phi_1, \theta)$ are the isometric and non-isometric string coordinates on S^2_η correspondingly.

Working in conformal gauge and applying the ansatz

$$\phi_1(\tau,\sigma) = \tau + F_1(\xi), \quad \theta(\tau,\sigma) = \theta(\xi), \quad \xi = \alpha\sigma + \beta\tau, \quad \alpha, \beta - \text{constants},$$

one finds

$$L_{S^2_\eta} = -\frac{T}{2}\left\{(\alpha^2 - \beta^2)\frac{\theta'^2}{1 + \tilde{\eta}^2(1-\chi)} + (1-\chi)\left[(\alpha^2 - \beta^2)(F_1')^2 - 2\beta F_1' - 1\right]\right\}, \tag{2.3}$$

where $\tilde{\eta}$ is related to the deformation parameter η according to [4]

$$\tilde{\eta} = \frac{2\eta}{1-\eta^2}, \tag{2.4}$$

and a new variable χ is defined by

$$\chi = \cos^2\theta.$$

The prime here and below is a derivative $d/d\xi$. The string tension T for the η deformed case is related to the coupling constant g by

$$T = g\sqrt{1 + \tilde{\eta}^2}. \tag{2.5}$$

The first integrals of the equations of motion F_1' and θ' can be written as

$$F_1' = \frac{\beta}{\alpha^2 - \beta^2}\left(-\frac{\kappa^2}{1-\chi} + 1\right), \tag{2.6}$$

$$\theta'^2 = \frac{1 + \tilde{\eta}^2(1-\chi)}{(\alpha^2 - \beta^2)^2}\left[(\alpha^2 + \beta^2)\kappa^2 - \frac{\beta^2\kappa^4}{1-\chi} - \alpha^2(1-\chi)\right]. \tag{2.7}$$

Inserting (2.6), (2.7) into (2.3), we obtain:

$$L_{S^2_\eta} = -\frac{T}{2}\frac{\beta^2\kappa^2 + \alpha^2(\kappa^2 - 2(1-\chi))}{\alpha^2 - \beta^2}. \tag{2.8}$$

Now we introduce the new parameters

$$v=-\frac{\beta}{\alpha}, \quad W=\kappa^2,$$

which leads to

$$L_{S^2_\eta}=-\frac{T}{2}\frac{(1+v^2)W-2(1-\chi)}{1-v^2}. \tag{2.9}$$

Therefore, for the case at hand, the normalized structure constant takes the form

$$C_3^{\tilde{\eta}}=\frac{8c_\Delta^d}{3\sqrt{W}}\int_{\chi_m}^{\chi_p}\frac{d\chi}{\chi'}\left[W+\frac{(1+v^2)W-2(1-\chi)}{1-v^2}\right], \tag{2.10}$$

where

$$\chi_m=\chi_{min}, \quad \chi_p=\chi_{max}.$$

One can rewrite Eq. (2.7) as

$$\chi'=\frac{2\tilde{\eta}}{1-v^2}\sqrt{(\chi_\eta-\chi)(\chi_p-\chi)(\chi-\chi_m)\chi}, \tag{2.11}$$

where [7]

$$\chi_m=1-W, \quad \chi_p=1-v^2W, \quad \chi_\eta=1+\frac{1}{\tilde{\eta}^2}. \tag{2.12}$$

Using this, we can express all the results in terms of χ_p, χ_m by eliminating v, W.

Replacing (2.11) in (2.10) and using (2.12), we can express $C_3^{\tilde{\eta}}$ by

$$C_3^{\tilde{\eta}}=\frac{8c_\Delta^d}{3\tilde{\eta}\sqrt{1-\chi_m}}\int_{\chi_m}^{\chi_p}\sqrt{\frac{\chi-\chi_m}{(\chi_\eta-\chi)(\chi_p-\chi)\chi}}\,d\chi. \tag{2.13}$$

The integral can be easily expressed by $\mathbf{K}$ and $\mathbf{\Pi}$, the complete elliptic integrals of the first and the third kind, respectively, are as follows:

$$C_3^{\tilde{\eta}}=\frac{16c_\Delta^d}{3\tilde{\eta}}\frac{\chi_m}{\sqrt{\chi_p(1-\chi_m)(\chi_\eta-\chi_m)}}\left[\mathbf{\Pi}\left(1-\frac{\chi_m}{\chi_p},1-\epsilon\right)-\mathbf{K}(1-\epsilon)\right], \tag{2.14}$$

where we introduced a short notation ϵ by

$$\epsilon=\frac{\chi_m(\chi_\eta-\chi_p)}{\chi_p(\chi_\eta-\chi_m)}. \tag{2.15}$$

Eq. (2.14) is the main result of this paper, which is an *exact* semiclassical result for the normalized structure constant $C_3^{\tilde{\eta}}$ valid for any value of $\tilde{\eta}$ and J_1. Here, χ_p and χ_m are determined by the angular momentum J_1 and world-sheet momentum p from the following equations[1]:

$$J_1=\frac{2T}{\tilde{\eta}}\frac{1}{\sqrt{\chi_p(\chi_\eta-\chi_m)}}\left[\chi_p\mathbf{K}(1-\epsilon)-\chi_m\mathbf{\Pi}\left(1-\frac{\chi_m}{\chi_p},1-\epsilon\right)\right], \tag{2.16}$$

$$p=\frac{2\chi_m}{\tilde{\eta}}\sqrt{\frac{1-\chi_p}{\chi_p(1-\chi_m)(\chi_\eta-\chi_m)}}\left[\mathbf{K}(1-\epsilon)-\mathbf{\Pi}\left(\frac{\chi_p-\chi_m}{\chi_p(1-\chi_m)},1-\epsilon\right)\right]. \tag{2.17}$$

The world-sheet energy of the giant magnon is given by

$$E=\frac{2T}{\tilde{\eta}}\frac{\chi_p-\chi_m}{\sqrt{\chi_p(1-\chi_m)(\chi_\eta-\chi_m)}}\mathbf{K}(1-\epsilon). \tag{2.18}$$

One of nontrivial checks is that the g derivative of $\Delta=E-J_1$ should be proportional to the normalized structure constant $C_3^{\tilde{\eta}}$ since the g derivative of the two-point function inserts the dilaton (Lagrangian) operator into the two-point function of the heavy operators [10]. This can be expressed by

$$C_3^{\tilde{\eta}}=\frac{8c_\Delta^d}{3\sqrt{1+\tilde{\eta}^2}}\frac{\partial\Delta}{\partial g}. \tag{2.19}$$

[1] We express J_1 and p in terms of different but equivalent combinations of elliptic functions compared to Eqs. (3.23) and (3.25) in [7].

To check that Eqs. (2.14), (2.16), and (2.18) satisfy Eq. (2.19), we use the fact that

$$\frac{\partial J_1}{\partial g} = \frac{\partial p}{\partial g} = 0 \tag{2.20}$$

as noticed in [11] for the case of undeformed giant magnon. From these, we can obtain the expressions for $\partial\chi_p/\partial g$ and $\partial\chi_m/\partial g$ which can be inserted into $\partial\Delta/\partial g$. The η-deformed case involves much more complicated expressions which can be dealt with Mathematica. In Appendix A, we provide our Mathematica code which confirms that the structure constant $C_3^{\tilde\eta}$ in Eq. (2.14) do satisfy the consistency condition (2.19) exactly.

In the limit $\tilde\eta \to 0$ with $\tilde\eta^2\chi_\eta \to 1$, Eq. (2.14) becomes

$$C_3^0 = \frac{16c_\Delta^d}{3}\sqrt{\frac{\chi_p}{1-\chi_m}}\left[\mathbf{E}(1-\epsilon) - \epsilon\mathbf{K}(1-\epsilon)\right], \quad \epsilon = \frac{\chi_m}{\chi_p} \tag{2.21}$$

where we used the identity $(1-a)\mathbf{\Pi}(a,a) = \mathbf{E}(a)$. This is the structure constant of the undeformed theory derived in [11].

3. Leading finite-size effect on $C_3^{\tilde\eta}$

It is straightforward to compute a leading finite-size effect on $C_3^{\tilde\eta}$ for $J_1 \gg g$ by the limit $\epsilon \to 0$ in (2.14).

First we expand the parameters χ_p, W and v for small ϵ as follows:

$$\begin{aligned}
\chi_p &= \chi_{p0} + (\chi_{p1} + \chi_{p2}\log\epsilon)\epsilon, \\
W &= 1 + W_1\epsilon, \\
v &= v_0 + (v_1 + v_2\log\epsilon)\epsilon.
\end{aligned} \tag{3.1}$$

Inserting into Eq. (2.14), we obtain

$$\begin{aligned}
C_3^{\tilde\eta} \approx \frac{16c_\Delta^d}{3\tilde\eta^2\sqrt{\left(1+\frac{1}{\tilde\eta^2}\right)\chi_{p0}}}\Bigg\{&\sqrt{(1+\tilde\eta^2)\chi_{p0}}\,\mathrm{arctanh}\frac{\tilde\eta\sqrt{\chi_{p0}}}{\sqrt{1+\tilde\eta^2}} \\
&- \Bigg[\frac{W_1}{2}\sqrt{(1+\tilde\eta^2)\chi_{p0}}\,\mathrm{arctanh}\frac{\tilde\eta\sqrt{\chi_{p0}}}{\sqrt{1+\tilde\eta^2}} + \frac{\tilde\eta}{4\left(1+\tilde\eta^2(1-\chi_{p0})\right)} \\
&\times\left((1+\tilde\eta^2)(\chi_{p0}-2\chi_{p1}) - 4\left((1+\tilde\eta^2)\chi_{p0} + 2W_1\left(1+\tilde\eta^2(1-\chi_{p0})\right)\right)\log 2\right)\Bigg]\epsilon \\
&- \frac{\tilde\eta}{4\left(1+\tilde\eta^2(1-\chi_{p0})\right)}\left(\left((1+\tilde\eta^2)(\chi_{p0}-2\chi_{p2}) + 2W_1\left(1+\tilde\eta^2(1-\chi_{p0})\right)\right)\right)\epsilon\log\epsilon\Bigg\}.
\end{aligned} \tag{3.2}$$

In view of Eqs. (2.12) and (2.15), we can express all the auxiliary parameters in terms of v (or its coefficients v_0, v_1, and v_2):

$$\chi_{p0} = 1 - v_0^2, \quad \chi_{p1} = 1 - v_0^2 - 2v_0v_1 - \frac{(1-v_0^2)^2}{1+\tilde\eta^2v_0^2}, \quad \chi_{p2} = -2v_0v_2, \quad W_1 = -\frac{(1+\tilde\eta^2)(1-v_0^2)}{1+\tilde\eta^2v_0^2}. \tag{3.3}$$

This leads to

$$\begin{aligned}
C_3^{\tilde\eta} \approx \frac{16c_\Delta^d}{3\tilde\eta}\Bigg\{&\mathrm{arctanh}\frac{\tilde\eta\sqrt{1-v_0^2}}{\sqrt{1+\tilde\eta^2}} + \frac{1}{4\sqrt{(1+\tilde\eta^2)(1-v_0^2)}\left(1+\tilde\eta^2v_0^2\right)^2} \\
&\times\Bigg[(1+\tilde\eta^2)\left((1-v_0^2)\left(1+\tilde\eta^2v_0^2\right)\left(2\sqrt{(1+\tilde\eta^2)\,((1-v_0^2)}\mathrm{arctanh}\frac{\tilde\eta\sqrt{1-v_0^2}}{\sqrt{1+\tilde\eta^2}} - \tilde\eta\log 16\right)\right. \\
&\left.- \tilde\eta\left(1 - v_0(3v_0 - 2v_0^3 - 4v_1 + v_0(1-v_0^2-4v_0v_1)\tilde\eta^2)\right)\right)\Bigg]\epsilon \\
&+ \frac{\tilde\eta(1+\tilde\eta^2)(1-v_0^2-4v_0v_2)}{4\sqrt{(1+\tilde\eta^2)(1-v_0^2)}(1+\tilde\eta^2v_0^2)}\epsilon\log\epsilon\Bigg\}.
\end{aligned} \tag{3.4}$$

To fix v_0, v_1, and v_2, one can use the small ϵ expansion of the angular difference

$$\Delta\phi_1 = \phi_1(\tau, L) - \phi_1(\tau, -L) \equiv p,$$

where we identified the angular difference $\Delta\phi_1$ with the magnon momentum p on the worldsheet. The result is [7]

$$v_0 = \frac{\cot\frac{p}{2}}{\sqrt{\tilde{\eta}^2 + \csc^2\frac{p}{2}}}, \tag{3.5}$$

and

$$v_1 = \frac{v_0(1-v_0^2)\left[1-\log 16 + \tilde{\eta}^2\left(2 - v_0^2(1+\log 16)\right)\right]}{4(1+\tilde{\eta}^2 v_0^2)}, \quad v_2 = \frac{1}{4}v_0(1-v_0^2). \tag{3.6}$$

By using (3.5), (3.6) in (3.4), one finds

$$C_3^{\tilde{\eta}} \approx \frac{16 c_\Delta^d}{3\tilde{\eta}}\left\{\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right) + \frac{(1+\tilde{\eta}^2)\sin^2\frac{p}{2}}{4\sqrt{\tilde{\eta}^2+\csc^2\frac{p}{2}}} \right.$$
$$\left. \times \left[\left(2\sqrt{\tilde{\eta}^2+\csc^2\frac{p}{2}}\,\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right) - \tilde{\eta}(1+\log 16)\right)\epsilon + \tilde{\eta}\epsilon\log\epsilon\right]\right\}. \tag{3.7}$$

The expansion parameter ϵ in the leading order is given by [7]

$$\epsilon = 16\exp\left[-\left(\frac{J_1}{g} + \frac{2\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde{\eta}^2\sin^2\frac{p}{2}}{(1+\tilde{\eta}^2)\sin^2\frac{p}{2}}}\right]. \tag{3.8}$$

Here we used Eq. (2.5) for the string tension T.

The final expression for the normalized structure costant is given by

$$C_3^{\tilde{\eta}} \approx \frac{16 c_\Delta^d}{3\tilde{\eta}}\left\{\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right) - 4\frac{\tilde{\eta}(1+\tilde{\eta}^2)\sin^3\frac{p}{2}}{\sqrt{1+\tilde{\eta}^2\sin^2\frac{p}{2}}}\left[1+\frac{J_1}{g}\sqrt{\frac{\tilde{\eta}^2+\csc^2\frac{p}{2}}{1+\tilde{\eta}^2}}\right]\right.$$
$$\left.\times\exp\left[-\left(\frac{J_1}{g} + \frac{2\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde{\eta}^2\sin^2\frac{p}{2}}{(1+\tilde{\eta}^2)\sin^2\frac{p}{2}}}\right]\right\}. \tag{3.9}$$

Let us point out that in the limit $\tilde{\eta}\to 0$, (3.9) reduces to

$$C_3 \approx \frac{16}{3}c_\Delta^d \sin\frac{p}{2}\left[1 - 4\sin\frac{p}{2}\left(\sin\frac{p}{2} + \frac{J_1}{g}\right)\exp\left(-\frac{J_1}{g\sin\frac{p}{2}} - 2\right)\right],$$

which reproduces the result for the undeformed case found in [11]. Another check is that this satisfies Eq. (2.19) with Δ computed in [7]

$$\Delta \equiv E - J_1 \approx 2g\sqrt{1+\tilde{\eta}^2}\left\{\frac{1}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right) - 4\frac{(1+\tilde{\eta}^2)\sin^3\frac{p}{2}}{\sqrt{1+\tilde{\eta}^2\sin^2\frac{p}{2}}}\right.$$
$$\left.\times\exp\left[-\left(\frac{J_1}{g} + \frac{2\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde{\eta}^2\sin^2\frac{p}{2}}{(1+\tilde{\eta}^2)\sin^2\frac{p}{2}}}\right]\right\}. \tag{3.10}$$

4. Concluding remarks

Here we obtained the *exact* semiclassical 3-point correlation function between two finite-size giant magnons "heavy" string states and the "light" dilaton operator with zero momentum in the η-deformed $AdS_5 \times S^5$. It is given in terms of the complete elliptic integrals of the first and third kind. We proved the consistency of our result by taking a derivative of the conformal dimension w.r.t. the coupling constant. We also provided the leading finite-size effect expansion of the structure constant.

It will be interesting to compute other three-point correlation functions of the η-deformed background such as HHH to which our result may be useful.

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Appendix A. The Mathematica code for the consistency check (Eq. (2.19))

$$J[g\_] := \frac{2T}{\eta\sqrt{(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}}$$
$$\left(\mathrm{xp}[g]\,\mathrm{EllipticK}\left[\frac{(\mathrm{xp}[g]-\mathrm{xm}[g])\mathrm{xn}}{(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}\right]-\mathrm{xm}[g]\,\mathrm{EllipticPi}\left[1-\frac{\mathrm{xm}[g]}{\mathrm{xp}[g]},\frac{(\mathrm{xp}[g]-\mathrm{xm}[g])\mathrm{xn}}{(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}\right]\right);$$

$$p[g\_] := \frac{2\mathrm{xm}[g]}{\eta}\sqrt{\frac{1-\mathrm{xp}[g]}{(1-\mathrm{xm}[g])(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}}$$
$$\left(\mathrm{EllipticK}\left[\frac{(\mathrm{xp}[g]-\mathrm{xm}[g])\mathrm{xn}}{(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}\right]-\mathrm{EllipticPi}\left[\frac{\mathrm{xp}[g]-\mathrm{xm}[g]}{(1-\mathrm{xm}[g])\mathrm{xp}[g]},\frac{(\mathrm{xp}[g]-\mathrm{xm}[g])\mathrm{xn}}{(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}\right]\right);$$

$$\mathrm{En}[g\_] := \frac{2T(\mathrm{xp}[g]-\mathrm{xm}[g])}{\eta\sqrt{(1-\mathrm{xm}[g])(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}}\,\mathrm{EllipticK}\left[\frac{(\mathrm{xp}[g]-\mathrm{xm}[g])\mathrm{xn}}{(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}\right];$$

$$T=\sqrt{1+\eta^2}g;$$

$$\mathrm{Eq1}=D[J[g],g]==0;$$

$$\mathrm{Eq2}=D[p[g],g]==0;$$

$$\mathrm{sol}=\mathrm{Solve}[\{\mathrm{Eq1},\mathrm{Eq2}\},\{D[\mathrm{xm}[g],g],D[\mathrm{xp}[g],g]\}];$$

$$\mathrm{xpd}=D[\mathrm{xp}[g],g]/.\mathrm{sol}[[1]];$$

$$\mathrm{xmd}=D[\mathrm{xm}[g],g]/.\mathrm{sol}[[1]];$$

$$\mathrm{threept}=\frac{8c}{3\sqrt{1+\eta^2}}\,\mathrm{FullSimplify}[D[\mathrm{En}[g]-J[g],g]/.\{D[\mathrm{xp}[g],g]\to\mathrm{xpd},D[\mathrm{xm}[g],g]\to\mathrm{xmd}\}]$$

$$\frac{16c\left(-\mathrm{EllipticK}\left[\frac{\mathrm{xn}(-\mathrm{xm}[g]+\mathrm{xp}[g])}{(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}\right]+\mathrm{EllipticPi}\left[1-\frac{\mathrm{xm}[g]}{\mathrm{xp}[g]},\frac{\mathrm{xn}(-\mathrm{xm}[g]+\mathrm{xp}[g])}{(\mathrm{xn}-\mathrm{xm}[g])\mathrm{xp}[g]}\right]\right)\mathrm{xm}[g]}{3\eta\sqrt{(-1+\mathrm{xm}[g])(-\mathrm{xn}+\mathrm{xm}[g])\mathrm{xp}[g]}}$$

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