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NEW PARAFERMION, SU(2) COSET AND $N = 2$ SUPERCONFORMAL FIELD THEORIES

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In this paper we construct new parafermion theories, generalizing the Z_L parafermion theory from integer L to rational L. These non-unitary parafermion theories (which are also defined as $SL(2,\mathbb{R})_L/U(1)$) have some novel features: an infinite number of currents with negative conformal dimensions for most (if not all) of them. We construct the string functions of these new parafermion theories. Generalizing Felder's BRST cohomology approach we construct from the string functions the branching functions of the $SU(2)_L \times SU(2)_K/SU(2)_{K+L}$ coset theories, where both K, L are rational. New $N = 2$ superconformal field theories and topological field theories are also constructed. Their characters are obtained in terms of the new string functions.

1. Introduction and summary

Conformal field theories (CFTs) have been studied for their possible applications to string compactifications, two-dimensional critical phenomena, integrable systems and matrix models. Except for string theory, unitarity of CFTs is in general not necessary. For example, in integrable field theories, the physical states are neither primary fields nor their descendents of the corresponding CFTs; thus, unitarity of the CFT is not necessary for the unitarity of integrable systems. In fact, non-unitary CFTs may have more applications than unitary CFTs. In this paper, we construct new rational CFTs that are generically non-unitary.

The classification of rational CFTs is far from complete. The best known example is the BPZ series $M_{p,p'}$ with $c = 1 - 6(p-p')^2/pp'$ [1]. It is useful to identify $M_{p,p'}$ with coset CFTs constructed from SU(2) Wess–Zumino–Witten (WZW) models [2]. Consider the coset

$$\mathrm{COSET}[K, L] = \frac{SU(2)_K \otimes SU(2)_L}{SU(2)_{K+L}}. \tag{1.1}$$

Goddard, Kent and Olive (GKO) [3] showed that $M_{p,p'} = \mathrm{COSET}[1, L]$ with

$L+2=p/(p'-p)$ where p, p' are coprime. When L is integer ($p'-p=1$), they correspond to the unitary series with $c<1$ [4]. Generalization of this construction to K, L both positive integers have been studied in the literature. It was pointed out by several groups [5] that the above COSET[K, L] has an extended Virasoro symmetry. We can first consider the coset $\mathrm{SU(2)}_K/\mathrm{U(1)}$ which is simply the Z_K parafermion (PF) theory constructed by Fateev and Zamolodchikov [6, 7]. Then the coset theory COSET[K, L] can be constructed by introducing a background charge for a free boson using the Feigin–Fuchs (FF) construction [8]

$$\mathrm{COSET}[K, L] = \left[\frac{\mathrm{SU}(2)_K}{\mathrm{U}(1)}\right] \oplus \mathrm{FF\,boson}[c<1]. \tag{1.2}$$

Here the coset theory is analyzed under the extended chiral Virasoro algebra, the $\{\mathscr{G}^{(K)}, T\}$ algebra, where a new current $\mathscr{G}^{(K)}$ (for $K \ngeq 2$) is added to the Virasoro generator $T(z)$. The $\mathscr{G}^{(K)}$ current is constructed from the Z_K PF theory and the FF boson [5, 9–11]. Its dimension (or spin) is $(K+4)/(K+2)$. The central charge of the boson $c=c(K, L)$ is chosen to match that of the coset theory. The $K=1,2$ cases correspond to the conformal and superconformal unitary series, respectively. For $K=2$, $\mathscr{G}^{(K)}$ is the supercurrent with spin 3/2. For higher K, the spin of $\mathscr{G}^{(K)}$ is fractional.

As noticed by several authors [12–18], it is possible to generalize the above GKO construction by considering rational levels L. In particular, $\mathrm{SU(2)}_L$, L rational (or more appropriately $\mathrm{SL}(2,\mathbb{R})_L$) has been analyzed by Kač and Wakimoto [13]. While the extension from integer L to rational L is straightforward, a new issue arises. The COSET[K, L] clearly has the duality property

$$\mathrm{COSET}[K, L] = \mathrm{COSET}[L, K]. \tag{1.3}$$

This implies that the COSET[K, L] can be analyzed under the $\{\mathscr{G}^{(K)}, T\}$ symmetry or under the $\{\mathscr{G}^{(L)}, T\}$. For example, the tricritical Ising model = COSET[1, 2] has a hidden superconformal algebra which can be explained by the duality COSET[1, 2] = COSET[2, 1]. Since the coset theory for rational L exists, the duality (1.3) implies that there is a $\{\mathscr{G}^{(L)}, T\}$ chiral symmetry even for rational L. To find this symmetry is our initial motivation.

In this paper, we generalize the Z_L PF algebra from integer level L to rational level L with $L=t/u$ where t, u are coprime integers satisfying $t+2u \geqslant 2$, $u>0$. Our construction exhausts all possible Z_L PF algebra with admissible representation and central charge less than 2. (Z_L PF theory with $u<0$, whose central charge is larger than 2, has been analyzed by Lykken. See fig. 1.) This Z_L PF theory is quite novel in that there are an infinite number of parafermion currents, most of them, if not all, having negative conformal dimensions. We also derive their characters, so-called string functions; again there are an infinite number of them. From the Z_L PF theory, we can construct the corresponding $\mathscr{G}^{(L)}$ current and define the $\{\mathscr{G}^{(L)}, T\}$ chiral symmetry algebra. Next we generalize Felder's

BRST cohomology analysis [19–21] to construct the characters (and the branching functions) for the COSET$[L, K]$, where both L and K are rational, using the Z_L PF string functions constructed earlier. These CFTs are new, in either the CFT or the GKO approach. The central charges are

$$c = \frac{3L}{L+2} + \frac{3K}{K+2} - \frac{3(K+L)}{K+L+2}, \tag{1.4}$$

where $L = t/u$, the same as defined before, and $K + 2 \equiv p/q$ with p, q coprime. Classification of modular invariant partition functions is also derived.

As a check of the above analysis, we can consider the special cases COSET$[L, 1]$, L rational, which correspond to the BPZ series, i.e. COSET$[1, L]$ and whose branching functions are well known [8]. The construction of branching functions for COSET$[L, 1]$ relies on the BRST cohomology of the $\{\mathscr{G}^{(L)}, T\}$ symmetry algebra and the new Z_L PF string functions. We illustrate this with the example COSET$[1/2, 1]$ in sect. 6. From eq. (1.4), it is clear that there are many coset models with $c < 1$ but not in the BPZ series. An example COSET$[1/2, 1/2]$ with $c = 1/5$ is given in sect. 6.

Using the new Z_L PF theory, new $N = 2$ superconformal theories can be easily constructed. The bosonization of the new PF theories is also discussed; however, their BRST cohomology properties remain to be clarified.

This paper is organized as follows: In sect. 2, we study new Z_L PF theory as a coset $SL(2)_L/U(1)$. A brief review of $SL(2)_L$ with rational L is included. The associativity of PF algebra determines the central charge and the conformal dimensions of the PF currents. The primary fields are introduced as well. The string functions and their symmetry properties are given. In sect. 3, we consider the bosonization of the parafermion theory.

In sect. 4 we apply our new PF algebra to construct $N = 2$ superconformal algebra (SCA). It is straightforward to construct $N = 2$ SCA from the generalized PF algebra using the relation

$$(N = 2)\,\mathrm{SCA} = \left[\frac{\mathrm{SL}(2)_L}{\mathrm{U}(1)}\right] \oplus \mathrm{boson}[R], \tag{1.5}$$

with a free boson compactified on an appropriate radius R. The new $N = 2$ superconformal field theories (SCFT) have central charge $c = 3t/(t + 2u)$. t and u have the same restrictions as before. These theories have an infinite number of highest weight states but a finite number, $u(t + 2u - 1)$, of chiral primary fields. From these $N = 2$ SCFT we can construct new topological field theories by twisting the energy–momentum tensor. In the resulting theories, the central charge $c = 0$ and the number of physical observables is $u(t + 2u - 1)$.

In sect. 5 we apply Z_L PF algebra to understand COSET$[L, K]$ theories. Here we construct the $\{\mathscr{G}^{(L)}, T\}$ chiral symmetry algebra and their representations based on the Z_L PF and the FF boson. The branching functions and characters of COSET$[L, K]$ can be derived by using the Z_L PF string functions given in sect. 2 and the BRST cohomology method. Their symmetry properties are also given. For the special case of $K = 1$, we reproduce the familiar characters of the BPZ series. This consistency check confirms the logical coherence of our approach based on the new PF algebra. The modular invariant partition functions for the coset theories can be easily constructed from the known results on SL(2) WZW theory with rational level. In sect. 6 we present some examples which illustrate the construction. In appendices, some technical details are explained.

2. Z_L parafermion theory from $SL(2)_L/U(1)$

In this section, we construct the generalized Z_L PF theory as a coset CFT $SL(2)_L/U(1)$ with rational level L. After deriving the generating (PF) currents and their OPEs for the Z_L PF theory, we identify the primary fields from those of $SL(2)_L/U(1)$ theory. The string functions (or the characters) for these primary fields are derived from those of $SL(2)_L$ theory first given by Kač and Wakimoto [13].

2.1. $A_1^{(1)}$ CURRENT ALGEBRA

We start with the $SL(2)_L$ WZW theory, whose chiral algebra is a direct sum of Virasoro and $A_1^{(1)}$ Kač–Moody (KM) algebras. These affine algebras can be expressed compactly in terms of the operator product expansion (OPE) of holomorphic currents

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T + O(1),$$

$$J^+(z)J^-(w) = \frac{L}{(z-w)^2} + \frac{1}{(z-w)}J^3(w) + O(1),$$

$$J^3(z)J^{\pm}(w) = \frac{\pm 2}{(z-w)}J^{\pm}(w) + O(1),$$

$$J^3(z)J^3(w) = \frac{2L}{(z-w)^2} + O(1),$$

$$T(z)J^a(w) = \frac{1}{(z-w)^2}J^a(w) + \frac{1}{(z-w)}\partial_w J^a + O(1), \quad a = \pm, 3 \quad (2.1)$$

where the energy-momentum tensor $T(z)$ is defined in terms of the currents by

$$T(z) = \frac{1}{L+2} \lim_{w \to z} \left[J^3(w) J^3(z) + \tfrac{1}{2} J^+(w) J^-(z) + \tfrac{1}{2} J^-(w) J^+(z) - \frac{3L/2}{(z-w)^2} \right], \tag{2.2}$$

with the central charge $c = 3L/(L+2)$, where L is the level of $A_1^{(1)}$ affine algebra.

Kač and Wakimoto [13] have worked out the highest weight representations, called "admissible" representations, whose characters form a finite-dimensional representation of modular group SL(2, $\mathbb{Z}$). We give a brief review of these results. Let $L = t/u$ where t and u are coprime integers ($u > 0$). Whenever $u = 1$, we will reproduce the well-known integer level results. The "admissible" representations of the $A_1^{(1)}$ KM algebra can be expressed in terms of two integers n and k for $2u + t - 2 \geqslant 0$

$$\lambda_{k,n} = [L - n + k(L+2)] \Lambda_0 + [n - k(L+2)] \Lambda_1,$$

$$0 \leqslant n \leqslant 2u + t - 2, \qquad 0 \leqslant k \leqslant u - 1, \tag{2.3}$$

where Λ_0 and Λ_1 are the affine and finite fundamental weights, respectively. The coefficient of Λ_1, denoted by l, can be represented by two integers (n, k)

$$l \equiv (n, k) = n - k(L+2), \tag{2.4}$$

which is in general a fractional number. The dimension of this spin-l representation with respect to the global $A_1^{(1)}$ algebra is infinite. This is quite different from the integer-level case in which the dimension is $l + 1$. Since each highest weight state of the affine algebra corresponds to a primary field of the SL(2)$_L$ WZW theory with the $A_1^{(1)}$ current algebra, the primary fields can be represented by $\Psi^{l,\bar{l}}(z, \bar{z})$. In terms of modes L_n, J_n^a, defined by

$$L_n = \oint \mathrm{d}z z^{n+1} T(z), \qquad J_n^a = \oint \mathrm{d}z z^n J^a(z), \tag{2.5}$$

the primary fields satisfy

$$L_n \Psi^{l,\bar{l}} = 0, \qquad J_n^a \Psi^{l,\bar{l}} = 0, \qquad \text{for } n > 0,$$

$$J_0^+ \Psi^{l,\bar{l}} = 0, \qquad L_0 \Psi^{l,\bar{l}} = \Delta \Psi^{l,\bar{l}}, \qquad J_0^3 \Psi^{l,\bar{l}} = l \Psi^{l,\bar{l}}, \tag{2.6}$$

where the conformal weight Δ is

$$\Delta(\Psi^l) = \frac{l(l+2)}{4(L+2)}\,. \tag{2.7}$$

The anti-holomorphic sector satisfies similar relations.

For the highest weight $l=(n,k)$, let

$$a=u^2(L+2)\,,\qquad b_{\pm}=u(\pm(n+1)-k(L+2))\,. \tag{2.8}$$

Then the character

$$\chi_l(\tau,z) \equiv \mathrm{Tr}_l \exp\{2\pi i\tau[L_0 - zJ_0^3 - \tfrac{1}{24}c]\} \tag{2.9}$$

is expressed in terms of theta functions

$$\Theta_{n,m}(\tau,z) = \sum_{j\in \mathbf{Z}+n/2m} \exp\{2\pi i m\tau(j^2-2jz)\} \tag{2.10}$$

as follows:

$$\chi_l(\tau,z) = \frac{\Theta_{b_+,a}(\tau,z/u) - \Theta_{b_-,a}(\tau,z/u)}{\Theta_{1,2}(\tau,z) - \Theta_{-1,2}(\tau,z)}\,. \tag{2.11}$$

For the case of $k \neq 0$, it is obvious that the character is not well defined for $z=0$ because the denominator becomes zero. If $k=0$, the numerator also becomes zero because of $b_+ = -b_-$ and the final character formula with $z=0$ can be well defined. Due to Θ-function identities, it is straightforward to show the symmetry of the character

$$\chi_l(\tau,z) = -\chi_{-l-2}(\tau,-z)\,. \tag{2.12}$$

If we denote $l=(n,k)$, the highest weight state $-l-2$ corresponds to $(t+2u-2-n, u-k)$. The symmetry (2.12) holds only for $k\neq 0$.

For the case of integer L ($u=1$), the character (2.11) is reduced to that of Kač and Peterson [27] for the integer level [23–25]. The character (2.11) has also been derived from the BRST cohomology framework [20]. We considered only holomorphic chiral algebras based on $\mathrm{SL}(2)_L$ KM algebra. The combination of this with the anti-holomorphic sector is determined by the modular invariance conditions of the partition function

$$Z = \sum_{l,l'} N_{l,l'}\chi_l(\tau,z)\bar{\chi}_{l'}(\tau,z)\,. \tag{2.13}$$

The non-negative integer matrix $N_{l,l'}$ should satisfy the modular invariance condition

$$SN = NS, \qquad TN = NT, \tag{2.14}$$

where two matrices S and T are defined by

$$\chi_l(-1/\tau) = \sum_{l'} S_{l,l'}\chi_{l'}(\tau), \qquad \chi_l(\tau+1) = \sum_{l'} T_{l,l'}\chi_{l'}(\tau). \tag{2.15}$$

Eq. (2.14) can determine the matrix N completely for the $A_1^{(1)}$ with integer level [26]. This celebrated ADE classification for $SU(2)_K$ (K integer) partition functions has been extended to rational level [15, 16]. This extended result will be used to find the modular invariant partition functions of SU(2) coset theories.

2.2. Z_L PARAFERMION CURRENT ALGEBRA

We construct the generalized PF theory as a coset $SL(2)_L/U(1)$ theory. We start by introducing the Z_L PF currents $\psi_1(z), \psi_1^+(z)$, which are defined by the relation

$$\begin{aligned}
J^+(z) &= \sqrt{L}\,\psi_1(z) :\exp\left[i\phi(z)/\sqrt{L}\right]:,\\
J^-(z) &= \sqrt{L}\,\psi_1^+(z) :\exp\left[-i\phi(z)/\sqrt{L}\right]:,\\
J^3(z) &= i\sqrt{L}\,\partial_z\phi(z),
\end{aligned} \tag{2.16}$$

where : : denotes the usual normal ordering for the bosonic field ϕ, which is normalized by the relation $\langle\phi(z)\phi(w)\rangle = -2\ln(z-w)$. Since we demand the conformal weight of the J^a to be 1, we can determine those of $\psi_1(z), \psi_1^+(z)$ to be

$$\Delta(\psi_1) = \Delta(\psi_1^+) = 1 - \frac{1}{L}. \tag{2.17}$$

Note that $\psi_1^+(z)$ need not be the hermitian conjugate of $\psi_1(z)$ because we are considering SL(2) for which $J^-(z) \neq (J^+(z))^\dagger$. Using eqs. (2.1) and (2.2), one can derive

$$\psi_1(z)\psi_1^+(w) = (z-w)^{-2\Delta_1}\left[I + (2\Delta_1/c_\psi)(z-w)^2 T_\psi(w) + O\left((z-w)^3\right)\right] \tag{2.18}$$

where $T_\psi = T - T_\phi = T + \frac{1}{4}(\partial\phi)^2$. This implies the Z_L PF theory is the coset $SL(2)_L/U(1)$ and the central charge is $c_\psi = 2(L-1)/(L+2)$.

In addition to ψ_1, ψ_1^+, the Z_L PF algebra needs a set of additional currents, defined by the following OPE

$$\psi_j(z)\psi_{j'}(w) = c_{j,j'}(z-w)^{\Delta_{j+j'}-\Delta_j-\Delta_{j'}}\left[\psi_{j+j'}(w) + O(z-w)\right],$$

$$\psi_j^+(z)\psi_{j'}^+(w) = c_{j,j'}^+(z-w)^{\Delta_{j+j'}^+-\Delta_j^+-\Delta_{j'}^+}\left[\psi_{j+j'}^+(w) + O(z-w)\right], \quad (2.19)$$

where Δ_j and Δ_j^+ denote the conformal weights of $\psi_j(z)$ and $\psi_j^+(z)$, respectively. The number of the currents is in general infinite. The constants $c_{j,j'}, c_{j,j'}^+$ and the conformal weights Δ_j, Δ_j^+ are to be determined by the associativity of the Z_L PF algebra.

Let us consider the holomorphic correlation function of ψ_1 and ψ_1^+ fields in terms of $J^{\pm}$ and the bosonic field

$$\begin{aligned}
&\langle J^+(z_1)J^+(z_2)\dots J^+(z_n)J^-(w_1)J^-(w_2)\dots J^-(w_n)\rangle \\
&\quad = \left\langle \exp\left[i\phi(z_1)/\sqrt{L}\right]\dots\exp\left[i\phi(z_n)/\sqrt{L}\right]\exp\left[-i\phi(w_1)/\sqrt{L}\right]\dots \right. \\
&\qquad \left. \times\dots\times\exp\left[-i\phi(w_n)/\sqrt{L}\right]\right\rangle\langle\psi_1(z_1)\psi_1(z_2)\dots \\
&\qquad \psi_1(z_n)\psi_1^+(w_1)\psi_1^+(w_2)\dots\psi_1^+(w_n)\rangle .
\end{aligned} \quad (2.20)$$

The correlation function of SL(2) currents $\langle J^+(z_1)\dots J^-(w_n)\rangle$ can be computed from the Ward identities derived from eq. (2.1) and that of the bosonic fields. Using this information, one can write down the Ward identities for the correlation function of ψ_1 and ψ_1^+ as follows:

$$\begin{aligned}
&\langle\psi_1(z_1)\psi_1(z_2)\dots\psi_1(z_n)\psi_1^+(w_1)\psi_1^+(w_2)\dots\psi_1^+(w_n)\rangle \\
&\quad = \prod_{i=2}^{n}(z_1-z_i)^{-2/L}\prod_{j=1}^{n}(z_1-w_j)^{2/L} \\
&\qquad \times\sum_{k=1}^{n}\left\{\frac{1}{(z_1-w_k)^2} + \frac{2/L}{(z_1-w_k)}\left[\sum_{l=1}^{n}\frac{1}{w_k-z_l} - \sum_{\substack{m=1\\ m\neq k}}^{n}\frac{1}{w_k-w_m}\right]\right\} \\
&\qquad \times\prod_{q=2}^{n}(z_q-w_k)^{2/L}\prod_{p=1}^{k-1}(w_p-w_k)^{-2/L}\prod_{r=k+1}^{n}(w_k-w_r)^{-2/L} \\
&\qquad \times\langle\psi_1(z_2)\dots\psi_1(z_n)\psi_1^+(w_1)\dots\psi_1^+(w_{k-1})\psi_1^+(w_{k+1})\dots\psi_1^+(w_n)\rangle . \quad (2.21)
\end{aligned}$$

These Ward identities can be used to find all PF current correlation functions recursively.

Using the above formulae, we can fix the conformal weights and the OPEs (2.19) completely. The results are

$$\Delta(\psi_j) = \Delta(\psi_j^+) = \frac{j(L-j)}{L}, \qquad c = \frac{2(L-1)}{L+2}, \tag{2.22}$$

and the extra OPE is

$$\psi_j(z)\psi_j^+(w) = (z-w)^{-2\Delta_j}\left[1 + (2\Delta_j/c)(z-w)^2 T_\psi(w) + \mathrm{O}\left((z-w)^3\right)\right]. \tag{2.23}$$

For an illustration, we compute the four-point correlation function to derive some of the above results in appendix A. This completes the proof that the Z_L PF theory, defined by eqs. (2.18), (2.19), (2.22) and (2.23), is equivalent to the coset CFT $SL(2)_L/U(1)$ and forms an associative current algebra. For integer L, $\Delta(\psi_L) = 0$ and we can decouple all the negative dimensional currents from the theory by imposing the periodic condition $\psi_j^+ = \psi_{L-j}$. This is not possible for rational L. Hence the Z_L (L rational) PF theory contains an infinite number of negative dimensional currents. This new aspect is perfectly consistent in non-unitary CFT.

2.3. PRIMARY FIELDS OF Z_L PF ALGEBRA

The primary fields of the Z_L PF theory can be constructed from those of $SL(2)_L$ theory. From the primary fields of $SL(2)_L$, $\Psi^{l,\bar{l}}$ in eq. (2.6), we define the Virasoro primary fields

$$\Psi^{l,\bar{l}}_{m,\bar{m}} = (J_0^-)^N (\bar{J}_0^-)^{\bar{N}} \Psi^{l,\bar{l}}, \qquad l - m = 2N, \qquad \bar{l} - \bar{m} = 2\bar{N}, \tag{2.24}$$

which have the same conformal weight Δ_l with J_0^3 quantum number m. Since l is a fractional number, so is m. As already explained, the range of m is not bounded from below if the level L is a rational number. This means $SL(2)_L$ CFT has an infinite number of Virasoro primary fields. If we consider the $SL(2)_L$ chiral algebra, only a finite number of them survive as primary fields, $\Psi^{l,\bar{l}}$.

To express $\Psi^{l,\bar{l}}_{m,\bar{m}}(z,\bar{z})$ in terms of the primary fields of Z_L PF theory and U(1) boson theory, let us consider the primary fields of the boson theory. Since $J^3 \sim \partial\phi$, the U(1) primary fields must be

$$:\exp\left[im\phi(z)/2\sqrt{L} + i\bar{m}\bar{\phi}(\bar{z})/2\sqrt{L}\right]:, \tag{2.25}$$

for $\Psi^{l,\bar{l}}_{m,\bar{m}}$ to have quantum number m. If we denote the Virasoro primary fields of

Z_L PF theory by $\Phi^{l,\bar{l}}_{m,\bar{m}}$,

$$\Psi^{l,\bar{l}}_{m,\bar{m}}(z,\bar{z}) = \Phi^{l,\bar{l}}_{m,\bar{m}} :\exp\left[im\phi(z)/2\sqrt{L} + i\bar{m}\bar{\phi}(\bar{z})/2\sqrt{L}\right]:. \tag{2.26}$$

The conformal weight of $\Phi^{l,\bar{l}}_{m,\bar{m}}$ is, using $T_\psi = T - T_\phi$,

$$\Delta\left(\Phi^{l,\bar{l}}_{m,\bar{m}}\right) = \frac{l(l+2)}{4(L+2)} - \frac{m^2}{4L}, \qquad \text{for } m \leqslant l. \tag{2.27}$$

Furthermore, if we define the modes of ψ_1 and ψ_1^+ by

$$\psi_1(z)\Phi^{l,\bar{l}}_{l,\bar{l}}(0,0) = \sum_{k=-\infty}^{\infty} z^{-l/L+k-1} A_{l/L-k}\Phi^{l,\bar{l}}_{l,\bar{l}}(0,0),$$

$$\psi_1^+(z)\Phi^{l,\bar{l}}_{l,\bar{l}}(0,0) = \sum_{k=-\infty}^{\infty} z^{l/L+k-1} A^+_{-l/L-k}\Phi^{l,\bar{l}}_{l,\bar{l}}(0,0), \tag{2.28}$$

from eq. (2.6), the parafermion primary fields $\Phi^{l,\bar{l}}_{l,\bar{l}}$ satisfy

$$A_{l/L+n}\Phi^{l,\bar{l}}_{l,\bar{l}} = A^+_{-l/L+n+1}\Phi^{l,\bar{l}}_{l,\bar{l}} = 0, \qquad \text{for } n \geqslant 0. \tag{2.29}$$

All the results on the holomorphic sector holds equally for the anti-holomorphic one with similar expressions.

2.4. Z_L PF STRING FUNCTION

Let us derive the characters of the Z_L PF theory, which we refer to as string functions. Since the character is defined in the (anti-) holomorphic Hilbert space independently, we will consider only the holomorphic sector of the primary fields. The Hilbert space $\mathscr{H}_c$ of $SL(2)_L$ can be decomposed as follows:

$$\mathscr{H}_c = \bigoplus_m \mathscr{H}_{l,m},$$

$$\mathscr{H}_{l,m} = \left\{J^{a_1}_{-n_1}J^{a_2}_{-n_2}\dots J^{a_r}_{-n_r}|l,m\rangle\right\}, \tag{2.30}$$

with $n_i > 0$ and $a = \pm$ or 3. The highest weight state is defined by $|l,m\rangle = \Psi^l_m(0)|0\rangle$. Since the primary fields, $\Psi^l_m(z)$ and their descendents can be expressed by those of Z_L PF and the bosonic theory using eq. (2.16) and (2.26), we can decompose $\mathscr{H}_{l,m}$ into the direct product of $\mathscr{H}^{\mathrm{PF}}_{l,m}$ and $\mathscr{H}^{\mathrm{b}}_m$:

$$\mathscr{H}_{l,m} = \mathscr{H}^{\mathrm{PF}}_{l,m} \otimes \mathscr{H}^{\mathrm{b}}_m. \tag{2.31}$$

To see this, let us first express a state in $\mathscr{H}_{l,m}$

$$
\begin{aligned}
|\text{state}\rangle = & \oint \mathrm{d}z_1\, z_1^{-n_1} \oint \mathrm{d}z_2\, z_2^{-n_2} \dots \oint \mathrm{d}z_r\, z_r^{-n_r} \\
& \times \left[\psi_1^+(z_{t+1}) \dots \psi_1^+(z_s) \psi_1(z_{s+1}) \dots \psi_1(z_r) \Phi_m^l(0) \right] \\
& \times \left[\partial\phi(z_1) \dots \partial\phi(z_t) :\exp \frac{-i\phi(z_{t+1})}{\sqrt{L}}: \dots :\exp \frac{-i\phi(z_s)}{\sqrt{L}}: \right. \\
& \left. \times :\exp \frac{i\phi(z_{s+1})}{\sqrt{L}}: \dots :\exp \frac{i\phi(z_r)}{\sqrt{L}}: :\exp \frac{im}{2\sqrt{L}} \phi(0): \right] |0\rangle, \qquad (2.32)
\end{aligned}
$$

then one notices that the factor in the first square bracket generates the PF Hilbert space while the second one gives that of the boson. The operators $\partial\phi$ generate descendents of a bosonic state and $:\exp \pm i\phi/\sqrt{L}:$'s make the contour integrals well defined and change the quantum number m of bosons, $m \to m + 2(r + t - 2s)$. The null states in $\mathscr{H}_{l,m}$ move into $\mathscr{H}_{l,m}^{\mathrm{PF}}$ because $\mathscr{H}_m^{\mathrm{b}}$ does not have any.

Using eq. (2.9), the $\mathrm{SL}(2)_L$ character can be written as ($l - m \in 2\mathbb{Z}$)

$$
\begin{aligned}
\chi_l(\tau, z) &= \sum_{m=-\infty}^{l} \mathrm{Tr}_{\mathscr{H}_{l,m}} \mathrm{e}^{2\pi i\tau[L_0 - zJ_0^3 - c/24]} \\
&= \sum_{m=-\infty}^{\infty} \mathrm{Tr}_{\mathscr{H}_{l,m}^{\mathrm{PF}}} \mathrm{e}^{2\pi i\tau[L_0^{\psi} - c_{\psi}/24]} \times \mathrm{Tr}_{\mathscr{H}_m^{\mathrm{b}}} \mathrm{e}^{2\pi i\tau[L_0^{\phi} - zJ_0^3 - 1/24]}. \qquad (2.33)
\end{aligned}
$$

Notice the change of upper bound in the second sum, which originates from the fact that the m quantum number changes due to the action of the $:\exp \pm i\phi/\sqrt{L}:$'s in both the positive and the negative directions. Let us denote the PF character or string function by ηC_m^l, after summing up the bosonic part, we get

$$
\chi_l(\tau, z) = \sum_{\substack{m=l+2N \\ N \in \mathbb{Z}}} \eta C_m^l(\tau) \times \frac{q^{(m^2/4L - zm)}}{\eta(\tau)}, \qquad (2.34)
$$

where $\eta(\tau) = \mathrm{e}^{2\pi i\tau/24} \prod_{n=1}^{\infty}(1 - q^n)$ with $q = \exp(2\pi i\tau)$. Notice that eq. (2.34) can be viewed as the $\mathrm{SL}(2)_L$ character decomposition into that of U(1) theory and that of the PF theory. Therefore, the ηC_m^l become characters of the coset $\mathrm{SL}(2)_L/\mathrm{U}(1)$ CFT.

To compute ηC^l_m from eqs. (2.11) and (2.34), we will compare the z dependence of both sides. First, to express eq. (2.11) in power series of q^z, we use the identity [24]

$$[\Theta_{1,2}(\tau,z)-\Theta_{-1,2}(\tau,z)]^{-1}=\frac{1}{\eta^3}\sum_{p=-\infty}^{\infty}\sum_{r=0}^{\infty}(-1)^r q^{\frac{1}{2}r^2-(p-\frac{1}{2})(r+2z)}, \quad (2.35)$$

to obtain the $SL(2)_L$ character for $l=(n,k)$

$$\chi_l(\tau,z)=\chi_+(\tau,z)-\chi_-(\tau,z),$$

$$\chi_\pm(\tau,z)=\frac{1}{\eta^3}\sum_{j,p=-\infty}^{\infty}\sum_{r=0}^{\infty}(-1)^r q^{\Delta_\pm - zm_\pm},$$

$$\text{with}\quad \Delta_\pm = aj^2+b_\pm j+\left(b_\pm^2/4a\right)+\tfrac{1}{2}r^2-r\left(p-\tfrac{1}{2}\right)$$

$$m_\pm = 2p-1+2(t+2u)j\pm(n+1)-(k/u)(t+2u), \quad (2.36)$$

where a and $b_\pm$ were defined in eq. (2.8). Comparing z-dependence of eqs. (2.34) and (2.36), one can equate $m=m_\pm$ and obtain string function C^l_m as follows:

$$\eta^3 C^l_m(\tau)=\sum_{\substack{r=0\\ p=-\infty}}^{\infty}\sum_{\sigma=\pm}\sigma(-1)^r$$

$$\times\exp\left[2\pi i\tau\left\{a\left[\left(p+\frac{r}{2u}\right)+\frac{b_\sigma}{2a}\right]^2-L\left[\frac{r}{2}+\frac{m}{2L}\right]^2\right\}\right]. \quad (2.37)$$

To get a more familiar form, we can reduce eq. (2.37) using the identity

$$\sum_{r=0}^{2p-1}(-1)^r q^{r^2/2-r(p-1/2)}=0. \quad (2.38)$$

After some algebraic operations (shown in appendix B), we find

$$\eta^3 C^l_m(\tau)$$

$$=\left(\sum_{r,p\geqslant 0}-\sum_{r,p<0}\right)(-1)^r\exp\left[2\pi i\tau\left\{a\left[\left(p+\frac{r}{2u}\right)+\frac{b_+}{2a}\right]^2-L\left[\frac{r}{2}+\frac{m}{2L}\right]^2\right\}\right]$$

$$-\left(\sum_{r,p\geqslant 0}-\sum_{r,p<0}\right)(-1)^r\exp\left[2\pi i\tau\left\{a\left[\left(p+\frac{r}{2u}\right)+\frac{b_-}{2a}\right]^2-L\left[\frac{r}{2}+\frac{m}{2L}\right]^2\right\}\right]$$

(2.39)

2.5. PROPERTIES OF THE STRING FUNCTIONS

The conformal highest weight of Φ^l_m can be read off directly from string function C^l_m, eq. (2.39). However, in practice, the cancellations taking place in the summations make this less obvious than expected. Instead, we will first present the symmetries of the string functions and then derive the highest weights using the symmetries.

The string function $C^l_m(\tau)$, has the following symmetries: (See appendix B.)

$$C^l_m = C^{L-l}_{L-m},$$
$$C^l_m = C^l_{-m} \quad (\text{for} \quad k=0),$$
$$C^l_m = C^l_{2L-m} \quad (\text{for} \quad k=u-1), \tag{2.40}$$

where $l = n - k(L+2)$. The third symmetry follows from the first two:

$$C^{l(n,k=u-1)}_m = C^{L-l}_{L-m} = C^{L-l}_{m-L} = C^{l(n,k=u-1)}_{2L-m}. \tag{2.41}$$

From eq. (2.26) and eq. (2.27) we have

$$\hat{h}^l_m \equiv \Delta(\Phi^l_m) = \frac{l(l+2)}{4(L+2)} - \frac{m^2}{4L} \quad (\text{for} \quad m \leqslant l). \tag{2.42}$$

From the symmetries (2.40) we can derive the highest weight of C^l_m for $m \geqslant l$. Their values are different according to the three cases $k=0$, $k=u-1$, and the rest. In summary they are

$$\begin{aligned}
k \neq 0, u-1, \quad & h^l_m = \hat{h}^l_m && (m \leqslant l) \\
& h^l_m = \hat{h}^l_m + (m-l)/2 && (l \leqslant m), \\
k = 0, \quad & h^l_m = \hat{h}^l_m && (-l \leqslant m \leqslant l) \\
& h^l_m = \hat{h}^l_m + (m-l)/2 && (l \leqslant m) \\
& h^l_m = \hat{h}^l_m + (-m-l)/2 && (m \leqslant -l), \\
k = u-1, \quad & h^l_m = \hat{h}^l_m && (m \leqslant l) \\
& h^l_m = \hat{h}^l_m + (m-l)/2 && (l \leqslant m \leqslant l + (t+2u-2) - n) \\
& h^l_m = \hat{h}^l_m + (m-l) - (t+2u-2-n) && \\
& && (l + (t+2u-2) - n \leqslant m).
\end{aligned} \tag{2.43}$$

From the requirement that $C^l_m = C^{L-l}_{L-m}$ we obtain the second line in the above three cases. For the $k=0$ case, the additional requirement $C^l_m = C^l_{-m}$ gives the case $m \leqslant -l$. For the $k = u-1$ case, the last two lines are simply from the symmetry $C^l_m = C^{L-l}_{L-m}$ and the values of h^l_m in the $k=0$ case.

In the $u=1$ case (integral level L), the above symmetries imply the periodicity: $C^l_m = C^l_{m+2L}$. The proof is:

$$C^l_m = C^l_{-m} = C^{L-l}_{L+m} = C^{L-l}_{-L-m} = C^l_{m+2L}, \tag{2.44}$$

and the highest weights become

$$\begin{aligned} h^l_m &= \hat{h}^l_m && (-l \leqslant m \leqslant l) \\ h^l_m &= \hat{h}^l_m + (m-l)/2 && (l \leqslant m \leqslant 2L-l). \end{aligned} \tag{2.45}$$

Due to the periodicity we can also rewrite the character $\chi_l(\tau, z)$ (2.34) in a more familiar form. If we divide the sum over m into two parts

$$\sum_{m=-\infty}^{\infty} \Rightarrow \sum_{s=-\infty}^{\infty} \sum_{m=-l}^{2L-l-1} \qquad m \Rightarrow m + 2sL, \tag{2.46}$$

the summation over s (i.e. summation over the winding modes) can be applied only to bosonic characters (due to eq. (2.44), which gives

$$\sum_{s=-\infty}^{\infty} q^{(n^2/(4L)-zn)}(n = m+2sL) \equiv \Theta_{m,L}(\tau, z), \tag{2.47}$$

and eq. (2.34) becomes

$$\chi_l(\tau, z) = \sum_{m=-l}^{2L-l-1} C^l_m(\tau) \Theta_{m,L}(\tau, z). \tag{2.48}$$

Hence we recover the string functions for integral L [7,27].

3. Bosonization in the generalized PF theory

In this section we introduce the bosonization of the Z_L PF theory we derived in sect. 2. This bosonization is a straightforward generalization of the integral L case [23–25,28–31]. Next we present the string functions in a particular basis of the bosonization formalism, which clearly suggests an underlying BRST cohomology. However, such a BRST cohomology analysis is beyond the scope of this paper.

3.1. THE GENERAL FORMALISM

The standard strategy is to start with three bosons to bosonize the SL(2) theory: two of them can have background charges but the third cannot. The coset $SL(2)_L/U(1)$ corresponds to stripping the third boson of the SL(2) theory, which is compactified on radius $\sqrt{1/L}$. This procedure allows us to use the first two bosons to construct the generalized PF theory, requiring that the resulting PF theory have the OPEs obtained in sect. 2, eq. (2.18). This procedure will be shown to constrain our general formalism.

Let us start with

$$\langle \varphi_i(z)\varphi_i(w)\rangle = -2\varepsilon_i \ln(z-w),$$
$$T(z) = -\tfrac{1}{4}(\partial\boldsymbol{\varphi})^2 + i\boldsymbol{\alpha}_0\cdot\partial^2\boldsymbol{\varphi}, \tag{3.1}$$

where $\varepsilon_i = +1\ (-1)$ if the boson is space-like (time-like) and $\boldsymbol{\alpha}_0 = (\alpha_{0,1}, \alpha_{0,2}, 0)$. In the above inner product we use the diagonal metric, $\eta_{ii} = \varepsilon_i$. The central charge associated with the PF energy–momentum tensor T_ψ is

$$c = 2 - 24\boldsymbol{\alpha}_0^2 = 2 - 24\left(\varepsilon_1\alpha_{0,1}^2 + \varepsilon_2\alpha_{0,2}^2\right)$$
$$= \frac{2(L-1)}{L+2}. \tag{3.2}$$

We take the PF current to be of the following form:

$$\psi_1 = i\boldsymbol{A}\cdot\partial\boldsymbol{\varphi}\exp[i\boldsymbol{C}\cdot\boldsymbol{\varphi}],$$
$$\psi_1^+ = i\boldsymbol{B}\cdot\partial\boldsymbol{\varphi}\exp[-i\boldsymbol{C}\cdot\boldsymbol{\varphi}], \tag{3.3}$$

where the two-component constant vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are to be determined.

It is straightforward to calculate the OPE of $\psi_1(z)\psi_1^+(w)$ and $T_\psi(z)\psi_1(w)$ in terms of the bosons. We can then compare the results with the following OPE to get the constraints on $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and Δ_1

$$\psi_1(z)\psi_1^+(w) = (z-w)^{-2\Delta_1}\left[1 + (2\Delta_1/c_\psi)(z-w)^2 T_\psi(w) + O\left((z-w)^3\right)\right],$$
$$T_\psi(z)\psi_1(w) = (z-w)^{-2}\left[\Delta_1\psi_1(w) + (z-w)\,\partial\psi_1 + O\left((z-w)^2\right)\right]. \tag{3.4}$$

A straightforward calculation gives

$$\Delta_1 = 1 - \frac{1}{L},$$

$$\boldsymbol{A} = \beta\big(-\varepsilon_2(\delta\alpha_{0,1} - \alpha_{0,2}), \varepsilon_1(\delta\alpha_{0,2} - \alpha_{0,1})\big),$$

$$\boldsymbol{B} = \frac{L+2}{\beta L}\left(\frac{2\varepsilon_1\alpha_{0,2}^2 - \varepsilon_1\varepsilon_2/4}{\delta\alpha_{0,1} - \alpha_{0,2}}, \frac{-2\varepsilon_2\alpha_{0,1}^2 + \varepsilon_1\varepsilon_2/4}{\delta\alpha_{0,2} - \alpha_{0,1}}\right),$$

$$\boldsymbol{C} = \pm\sqrt{\frac{L+2}{-L\varepsilon_1\varepsilon_2}}\,(\alpha_{0,2}, \alpha_{0,1}), \tag{3.5}$$

where δ is defined as $\delta = \pm\sqrt{(L+2)/(-\varepsilon_1\varepsilon_2 L)}$ and β is an arbitrary constant. The solution, $\Delta_1 = 1 - 1/L$, implies that the third boson has to be compactified on a circle of radius $R = 1/\sqrt{L}$. Now if $\alpha_{0,1}\alpha_{0,2} = 0$, there are no further constraints. However, if $\alpha_{0,1}\alpha_{0,2} \neq 0$, then we must also demand

$$\varepsilon_1\varepsilon_2 = -1 \quad (\text{if} \quad \alpha_{0,1}\alpha_{0,2} \neq 0). \tag{3.6}$$

In other words, if we choose the background charges to be non-zero for both bosons, we must have one time-like and one space-like boson.

To be specific, it is convenient to choose the background charge to be $\boldsymbol{\alpha}_0 = (\frac{1}{2}/\sqrt{L+2}, 0)$ and the metric to be $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ for the $L > 0$ case. In this basis, the parafermion currents become [29–31]

$$\psi_1(z) = \frac{i}{2}\left[\partial\varphi_2 - \sqrt{\frac{L+2}{L}}\,\partial\varphi_1\right] :\exp\left(\frac{i\varphi_2}{\sqrt{L}}\right):,$$

$$\psi_1^+(z) = \frac{-i}{2}\left[\partial\varphi_2 + \sqrt{\frac{L+2}{L}}\,\partial\varphi_1\right] :\exp\left(\frac{-i\varphi_2}{\sqrt{L}}\right):. \tag{3.7}$$

Using the bosonization of parafermion currents, we can easily derive the conformal dimensions of ψ_l. From $\psi_1\psi_1 \sim \psi_2$, $\psi_1\psi_j \sim \psi_{j+1}$ we obtain $\Delta_j = j - (j^2/L)$, where $j = 1, 2, \ldots, \infty$.

3.2. BOSONIZATION OF PRIMARY FIELDS

In the basis we specified in the last subsection, the Virasoro primary fields are defined as:

$$\Phi_m^l = V_m^l \equiv \exp\left(\frac{-il}{2\sqrt{L+2}}\varphi_1\right)\exp\left(\frac{im}{2\sqrt{L}}\varphi_2\right), \tag{3.8}$$

which has conformal dimension

$$h_m^l = \frac{l(l+2)}{4(L+2)} - \frac{m^2}{4L} \quad (m \leqslant l). \tag{3.9}$$

We can also calculate the OPE of the ψ_1, ψ_1^+ currents with these highest weight states

$$\begin{aligned}
\psi_1(z)V_m^l(w) = (z-w)^{-m/L-1}\big[(l-m) \\
&+ (z-w)\left(\sqrt{(2/L)}\,(l-m+L)i\partial\varphi_2(w) - \sqrt{2(L+2)}\,i\partial\varphi_1(w)\right) \\
&+ \mathrm{O}\left((z-w)^2\right)\big]V_{m+2}^l(w), \\
\psi_1^+(z)V_m^l(w) = (z-w)^{m/L-1}\big[(l+m) \\
&- (z-w)\left(\sqrt{(2/L)}\,(l+m+L)i\partial\varphi_2(w) + \sqrt{2(L+2)}\,i\partial\varphi_1(w)\right) \\
&+ \mathrm{O}\left((z-w)^2\right)\big]V_{m-2}^l(w).
\end{aligned} \tag{3.10}$$

It is not hard to see that when $m = l$, V_l^l satisfies eq. (2.29) and becomes the parafermionic primary field.

Here we would like to make some comments on the choice of ε_i. If we have $L < 0$ we have to flip the sign of ε_2 such that we have two space-like bosons. And the one we remove from SL(2) becomes a time-like boson. Another special case is discussed in refs. [22, 32] regarding the SU(1, 1) conformal theory in which there are the restrictions that $L < 0$ and $u < 0, t \geqslant 0$. In our notation it simply means they have two time-like bosons, $\varepsilon_1 = \varepsilon_2 = -1$, and the removed third boson is also time-like. It is clear to see the above comment from fig. 1. In this figure we use the first two bosons to construct the parafermion, and the third one is needed to recover the SL(2) CFT. Our construction covers region A, B and C. Lykken's PF theory covers region C and D.

There are three possible choices for screening currents [30, 31]

$$S_1 \equiv V_L^{L+2}, \qquad S_2 \equiv V_{-L}^{L+2}, \qquad S_3 \equiv i\partial\varphi_2 V_0^{-2}.$$

and screening charges

$$Q_i \equiv \oint \mathrm{d}z\, S_i(z). \tag{3.11}$$

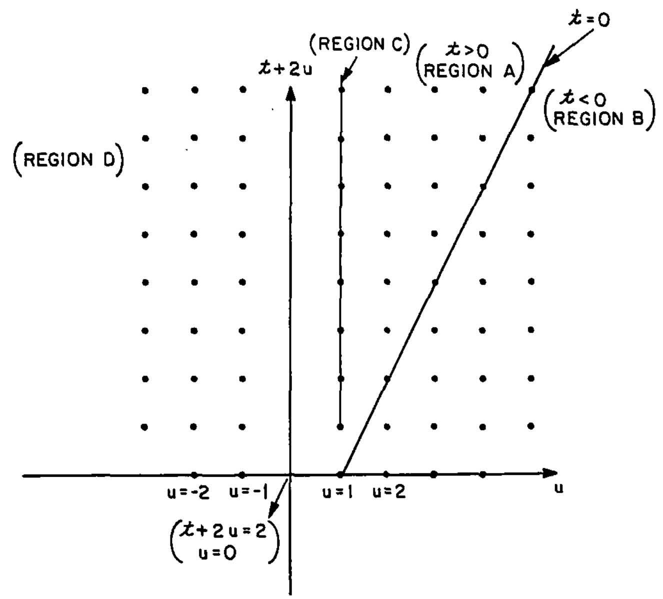

Fig. 1. This figure describes different regions of (t, u) values specifying different PF theories: Region A, B, C are considered in this paper. Region C ($t=K, u=1$) is the original Z_K PF constructed by Zamolodchikov and Fateev [6]. Region D was first considered by Lykken [22].

Using the screening currents, the string function (2.39), can also be written as (see also appendix B, eq. (B.3)):

$$\eta C^l_m(\tau) q^{-1/(4(L+2))} = \frac{1}{\eta^2}\Big(\sum_{r,p\geqslant 0} - \sum_{r,p<0}\Big)(-1)^r q^{h_1(r,p)}$$
$$-\frac{1}{\eta^2}\Big(\sum_{r\geqslant 0, p>0} - \sum_{r<0, p\leqslant 0}\Big)(-1)^r q^{h_2(r,p)}, \quad (3.12)$$

where we use the notation $h_1(r, p), h_2(r, p)$ to denote the conformal dimension of the vertex operator V^a_b. They are defined as (see also eq. (3.9))

$$h_1(r,p) = h\big(V^{l+(r+2pu)(L+2)}_{m+rL}\big) = h\big(S_1^{r+pu} S_2^{pu} V^l_m\big),$$
$$h_2(r,p) = h\big(V^{l+(2k-r-2pu)(L+2)}_{m+rL}\big) = h\big(S_1^{k-pu} S_2^{k-pu-r} V^l_m\big). \quad (3.13)$$

In order to understand this alternating sum and subtraction procedure, we plot each term on the two-dimensional lattice spanned by vectors S_1 and S_2. We show these results in fig. 2 and fig. 3. Each lattice point denotes a two bosons' Fock space generated by two bosonic creation operators acting on the highest weight

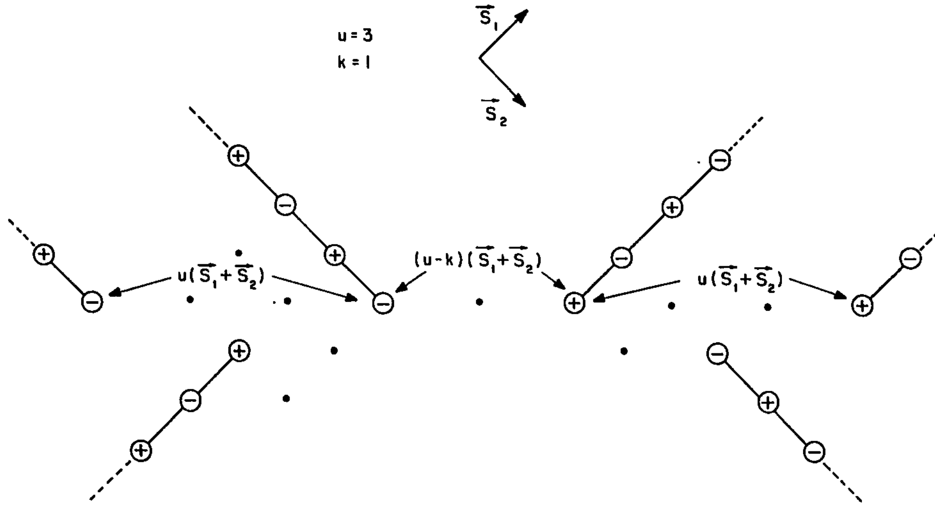

Fig. 2. A typical example of the adding and subtracting procedure in the string function C^l_m. We choose $u = 3$, $k = 1$ in this figure. S_1 and S_2 vectors are the movements of the screening charges Q_1 and Q_2 (defined in sect. 3). Black dots are the points which can be reached by S_1 and S_2 starting from the center, V^l_m. Each circle represents a bosonic Fock space associated with the highest weight state V^a_b, whose highest weight is h^a_b, defined in eq. (3.9), and character is equal to $\exp(2\pi i\tau h^a_b)/\eta^2$. The plus or the minus sign inside each circle indicates whether this Fock space has to be added or subtracted. Then the string function C^l_m is equal to the sum of all circles.

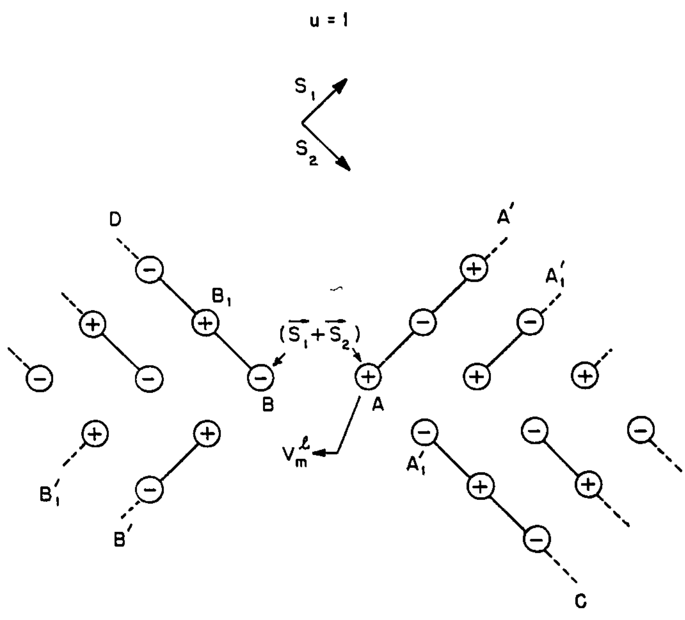

Fig. 3. This picture compares our formulae with those of Distler and Qiu [23]. The line connecting C, A, B, D is their BRST operator complex. For each circle on this line there is another line (e.g. $AA', A_1A'_1, \ldots, BB', B_1B'_1, \ldots$), which is their ghost zero mode subtraction.

state, $|V_b^a(0)\rangle$. The $\pm$ indicates whether we need to add or subtract the corresponding Fock space. So the string function becomes a sum over the traces of these Fock spaces represented by the lattice points.

When $u = 1$, the above picture was explained by Distler and Qiu [23] in terms of ghost fields. In fig. 3 the line connecting points C, A, B, D is the movement in their BRST cohomology. The lines AA′, $A_1A'_1, \ldots$ and BB′, $B_1B'_1, \ldots$ are the ghost zero-mode subtraction. Fig. 2 shows an example of the rational L case, where a definite pattern emerges. However, a complete BRST cohomology analysis is beyond the scope of this paper.

4. New $N = 2$ superconformal field theories

In this section we formulate new $N = 2$ superconformal field theories (SCFT) in terms of the Z_L PF theory and a boson with an appropriate radius of compactification. We then use this formalism to derive the $N = 2$ superconformal (SC) characters based on the string function we derived in sect. 2. Also we introduce the chiral primary fields and the twisted $N = 2$ SCFT in subsect. 4.3.

4.1. $N = 2$ SUPERCONFORMAL ALGEBRA

The superconformal field theory is defined by the energy–momentum tensor, $T(z)$, two supercurrents, $G^{\pm}$, and the U(1) current $J(z)$. (Later we use $N = 2$ SC currents to denote these four currents.) They satisfy the following OPE:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w) + O(1),$$

$$T(z)J(w) = \frac{1}{(z-w)^2}J(w) + \frac{1}{(z-w)}\partial J(w) + O(1),$$

$$T(z)G^{\pm}(w) = \frac{3/2}{(z-w)^2}G^{\pm}(w) + \frac{1}{(z-w)}\partial G^{\pm}(w) + O(1),$$

$$J(z)J(w) = \frac{c/3}{(z-w)^2} + O(1),$$

$$J(z)G^{\pm}(w) = \frac{\pm 1}{(z-w)}G^{\pm}(w) + O(1),$$

$$G^{+}(z)G^{-}(w) = \frac{2c/3}{(z-w)^3} + \frac{2}{(z-w)^2}J(w) + \frac{1}{(z-w)}\left[\partial J(w) + 2T(w)\right] + O(1), \qquad (4.1)$$

where c is equal to $3L/(L+2)$. Corresponding to these currents are the generators of superconformal algebra, which are defined by: $T(z)=\sum_{n=-\infty}^{\infty} L_n z^{-n-2}$, $G^{\pm}(z)=\sum_{n=-\infty}^{\infty} G_{n\pm a}^{\pm} z^{-(n\pm a)-3/2}$ and $J(z)=\sum_{n=-\infty}^{\infty} J_n z^{-n-1}$, where a is a free real parameter which dictates the branch cuts in $G(z)^{\pm}$. Shifting a by an integer yield isomorphic algebras. Therefore we restrict a within the range $0 \leqslant a < 1$. It is the Neveu–Schwarz (NS) sector when $a=1/2$ and the Ramond sector when $a=0$.

The highest weight state (assuming Ψ is a primary field) $|\Psi\rangle$ is defined by

$$G_{\mu}^{\pm}|\Psi\rangle = 0,$$

$$L_{\mu}|\Psi\rangle = J_{\mu}|\Psi\rangle = 0, \qquad \mu > 0. \tag{4.2}$$

This completes our review of SCFT.

4.2. THE CHARACTER FORMULAE OF $N=2$ SCFT

The relation between the $N=2$ superconformal minimal model and the Z_L PF theory (L integer) is well known [33]. We now generalize to rational L and construct new non-unitary $N=2$ SCFT, with central charge $c=3L/(L+2)$. We shall write down the representation for the currents, the highest weight states and the characters associated with these highest weight states.

The $N=2$ SC currents are written in terms of ψ_1, ψ_1^+ and φ

$$T(z) = -\tfrac{1}{4} :\partial\varphi\partial\varphi(z): + T_{\psi}(z),$$

$$J(z) = i\sqrt{\frac{L}{2(L+2)}}\,\partial\varphi(z),$$

$$G^{+}(z) = \sqrt{\frac{2L}{L+2}}\,\psi_1(z)\exp\left[i\sqrt{\frac{L+2}{2L}}\,\varphi(z)\right],$$

$$G^{-}(z) = \sqrt{\frac{2L}{L+2}}\,\psi_1^{+}(z)\exp\left[-i\sqrt{\frac{L+2}{2L}}\,\varphi(z)\right]. \tag{4.3}$$

They satisfy eq. (4.1) if we use eq. (2.18) for the OPE between ψ_1, ψ_1^+ and choose the radius of boson's compactification to be $R=\sqrt{2L/(L+2)}$ such that $G^+(z), G^-(w)$ have dimension $3/2$. The primary fields in $N=2$ SCFT can also be represented by the generalized PF and a boson, of the form $\Phi_k^l \exp(i\alpha_k^l \varphi)$ which are already primary with respect to the energy momentum tensor. The only thing to be checked is their OPE with $G^{\pm}(z)$. Using the OPE in eq. (3.9), we find, for $l \neq k$ case

$$G^{+}(z)\Phi_k^l(w)\exp\left[i\alpha_k^l\varphi(w)\right] = (z-w)^{-k/L-1}(z-w)^{2\alpha_k^l\sqrt{(L+2)/(2L)}} \times\Phi_{k+2}^l\exp\left[i\left(\alpha_k^l+\sqrt{(L+2)/2L}\right)\varphi(w)\right]+\dots,$$

$$G^{-}(z)\Phi_k^l(w)\exp\left[i\alpha_k^l\varphi(w)\right] = (z-w)^{k/L-1}(z-w)^{-2\alpha_k^l\sqrt{(L+2)/(2L)}} \times\Phi_{k-2}^l\exp\left[i\left(\alpha_k^l-\sqrt{(L+2)/2L}\right)\varphi(w)\right]+\dots. \tag{4.4}$$

So the highest weight condition is equivalent to the following inequalities:

$$-k/L-1+2\alpha_k^l\sqrt{(L+2)/2L}+a+1/2\geqslant 0,$$
$$k/L-1-2\alpha_k^l\sqrt{(L+2)/2L}-a+3/2\geqslant 0. \tag{4.5}$$

This implies

$$\alpha_k^l=\frac{k+L\left(\frac{1}{2}\operatorname{sign}(0)-a\right)}{\sqrt{2L(L+2)}}. \tag{4.6}$$

The $\operatorname{sign}(0)=\pm 1$ indicates the degeneracy of the ground states ($P^{\pm}$) in the Ramond sector when $a=0$. $\operatorname{sign}(0)=1$ when $a\neq 0$. We use the notation N_k^l to denote the highest weight state $\Phi_k^l\exp[i\alpha_k^l\varphi]$. Its corresponding U(1) charge, q, and conformal dimension, h_k^l, are

$$q=\frac{k+L\left(\frac{1}{2}\operatorname{sign}(0)-a\right)}{L+2},$$

$$h_k^l=\frac{l(l+2)}{4(L+2)}-\frac{k^2}{4L}+\frac{\left(k+L\left[\frac{1}{2}\operatorname{sign}(0)-a\right]\right)^2}{2L(L+2)}, \tag{4.7}$$

which can be easily obtained from eq. (4.3). When $l=\pm k$, α_k^l will have one more solution in addition to (4.6). But that one is ruled out by the requirement of integral U(1) charge.

Now we can use our generalized PF as a machinery to calculate the $N=2$ SC character. First we can see that any state in the highest weight representation (HWR) can be reached by the negative modings of the $N=2$ SC currents. They can be written as:

$$|\text{state}\rangle = G^{+}_{-n_1}\dots G^{+}_{-n_{j_1}} G^{-}_{-m_1}\dots G^{-}_{-m_{j_2}} L_{-n'_1}\dots L_{-n'_{j_3}} J_{-m'_1}\dots J_{-m'_{j_4}}|N^l_k(0)\rangle\,. \quad (4.8)$$

From the OPE's between the $N=2$ SC currents, eq. (4.1), and the parafermionic representation of the $N=2$ currents, we can rewrite the above states in terms of

$$|\text{state}\rangle = (\psi_1)_{-n'_1}\dots(\psi_1)_{-n'_{j_1}}(\psi_1^+)_{-m'_1}\dots(\psi_1^+)_{-m'_{j_2}}|\Phi^l_k(0)\rangle$$

$$\otimes(\partial\varphi)_{-\tilde{m}'_1}\dots(\partial\varphi)_{-\tilde{m}'_{j_4}}\Big|\exp\Big[i\Big(\alpha^l_k + (j_1-j_2)\sqrt{(L+2)/2L}\Big)\varphi(0)\Big]\Big\rangle. \quad (4.9)$$

When we take the trace over those states in the HWR we can decompose them into the following factors and sum over

$$\chi^l_k = \sum_{n=-\infty}^{\infty} \eta C^l_{k+2n}(\tau)\frac{1}{\eta}\exp\left[\frac{2\pi i\tau}{4}\Big((k+L[\text{sign}(0)/2-a])\sqrt{2/L(L+2)} + 2n\sqrt{(L+2)/(2L)}\Big)^2\right]$$

$$= \sum_{n=-\infty}^{\infty} C^l_{k+2n}(\tau)\exp\left[\frac{2\pi i\tau}{2L(L+2)}(k+L[\text{sign}(0)/2-a]+n(L+2))^2\right]. \quad (4.10)$$

We shall emphasize here the specific features in our construction. For $u\neq 1$, we have an infinite number of N^l_k. This is because $k=l+2m$ and $m=0,\pm1,\pm2,\dots,\pm\infty$. In subsect. 4.3 we will introduce new conditions to restrict the primary fields. The resulting fields are called chiral primary fields.

If L is integer we have the symmetry $C^l_{k+2tL}=C^l_k$. We then perform the change of variables $k+2n\equiv m+2tL$ and the summation in n can be decomposed into the summation in m and t. So the character becomes

$$\chi^l_k = \sum_{m=-l}^{2L-l-1}\sum_{t=-\infty}^{\infty} C^l_{m+2tL}(\tau)$$

$$\times\exp\left[\frac{2\pi i\tau}{2L(L+2)}\left((m+2tL)\left(\frac{L+2}{2}\right)-\frac{L}{2}(k-\text{sign}(0)/2+2a)\right)^2\right]$$

$$= \sum_{m=-l}^{2L-l-1} C^l_m(\tau)\Theta_{m(L+2)/2-L(k-\text{sign}(0)+2a)/2,\,L(L+2)/2}(\tau)\,, \quad (4.11)$$

where the Θ-function is defined in eq. (2.10). This result is in agreement with the early results of $N=2$ superconformal minimal model obtained by Qiu, Ravanini and Yang [34,35].

4.3. THE NEW TWISTED $N=2$ SCFT

In this section we first define the chiral primary fields in our $N=2$ SCFT and then obtain the new topological field theories (TFTs) by twisting our new $N=2$ SCFT. In this subsection we only consider the NS sector for simplicity and follow closely the work done by Eguchi and Yang [36].

The chiral primary states, $|\phi_{\rm cp}(0)\rangle$, are defined by:

$$G^{+}_{-1/2}|\phi_{\rm cp}(0)\rangle = G^{-}_{1/2}|\phi_{\rm cp}(0)\rangle = 0. \tag{4.12}$$

We can assume that $\phi_{\rm cp}$ has similar form as found in subsect. 4.2. We can then repeat the calculation, eq. (4.4), in terms of the generalized PF and the boson. The only difference from the previous subsection is due to $G^{+}_{-1/2}|\phi_{\rm cp}(0)\rangle = 0$, which results into

$$-k/L - 1 + 2\alpha^{l}_{k}\sqrt{(L+2)/2L} \geqslant 0. \tag{4.13}$$

Combining this with the second one of eq. (4.5), we found there is no solution for α^{l}_{k} if $l \neq k$. When $l=k$, the OPE between $G^{\pm}(z)$ and $\Phi^{l}_{l}(w)\exp[i\alpha^{l}_{l}\varphi(w)]$ becomes

$$\begin{aligned} G^{+}(z)\Phi^{l}_{l}(w)\exp\left[i\alpha^{l}_{l}\varphi(w)\right] &= (z-w)^{-l/L}(z-w)^{2\alpha^{l}_{l}\sqrt{(L+2)/(2L)}} \\ &\quad\times\Phi^{l}_{l+2}\exp\left[i\left(\alpha^{l}_{l}+\sqrt{(L+2)/2L}\right)\varphi(w)\right]+\dots, \\ G^{-}(z)\Phi^{l}_{l}(w)\exp\left[i\alpha^{l}_{l}\varphi(w)\right] &= (z-w)^{l/L-1}(z-w)^{-2\alpha^{l}_{l}\sqrt{(L+2)/(2L)}} \\ &\quad\times\Phi^{l}_{l-2}\exp\left[i\left(\alpha^{l}_{l}-\sqrt{(L+2)/2L}\right)\varphi(w)\right]+\dots. \end{aligned} \tag{4.14}$$

So in order to satisfy eq. (4.12), we must have

$$-l/L + 2\alpha^{l}_{l}\sqrt{(L+2)/2L} \geqslant 0 \quad \text{and} \quad l/L - 2\alpha^{l}_{l}\sqrt{(L+2)/2L} \geqslant 0. \tag{4.15}$$

We then obtain α^{l}_{l} and the chiral primary field, $\Phi^{l}_{\rm cp}$ as

$$\begin{aligned} \Phi^{l}_{\rm cp} &\equiv \Phi^{l}_{l}\exp\left(i\alpha^{l}_{l}\phi\right) \\ \alpha^{l}_{l} &= l/\sqrt{2L(L+2)}. \end{aligned} \tag{4.16}$$

The number of chiral primary fields are finite and are determined by the number of spin fields. It is easy to calculate the U(1) charge, q_l, and the highest weight, h_l, of this chiral primary field

$$q_l = l/(L+2) \quad \text{and} \quad h_l = l/2(L+2)\,. \tag{4.17}$$

They satisfy the property, $h_l = q_l/2$, which can also be derived purely from the $N=2$ SCFT [37].

Now we can try to obtain the corresponding topological field theory (TFT) by twisting this $N=2$ SCFT. The basic idea is to add the term $\partial J/2$ to the energy–momentum tensor, $T_{N=2}$. We use the $\tilde{T}$ to denote the twisted energy–momentum tensor:

$$\tilde{T} = T_{N=2} + \tfrac{1}{2}\partial J\,. \tag{4.18}$$

This new energy–momentum tensor has zero central charge as calculated from the OPEs (4.1). It also changes the conformal dimension of $G^{\pm}$ from 3/2 to 1 and 2 respectively.

It is now straightforward to identify the nilpotent operator, Q_{BRST} in the TFT and the ancestor of the energy–momentum tensor, $\tilde{T}$, under Q_{BRST}'s action. We can see from the last equation in (4.1) that if we identify $Q_{\mathrm{BRST}} \equiv \oint \mathrm{d}z\, G^+$, we will have

$$\{Q_{\mathrm{BRST}}, G^-(w)\} = 2\tilde{T}(w)\,. \tag{4.19}$$

From the property of the TFT, $G^-(w)$ is just the ancestor of the $\tilde{T}(w)$.

The chiral primary fields in our theory are the physical observables in the TFT

$$Q_{\mathrm{BRST}}|\Phi^l_{\mathrm{cp}}(0)\rangle = \oint \mathrm{d}z\, G^+ |\Phi^l_{\mathrm{cp}}(0)\rangle = 0\,. \tag{4.20}$$

The last equality is entirely due to eq. (4.12). To complete the story we also have to calculate the correlation function of these observables. It must be independent of the positions where these observables are located.

5. The new coset theories

In this section we construct new SU(2) COSET$[K, L]$ theory where both K, L are rational (see eq. (1.1)). To do so, we first introduce the fractional supersymmetry current $\mathscr{G}^{(L)}$ which survives after we change the compactification radius of the third boson in SL(2) CFT and turn on the background charge associated with that boson. The primary fields in the coset theory, defined with respect to $\{\mathscr{G}^{(L)}, T\}$, can be obtained through the requirement of having non-trivial BRST cohomology, an approach forwarded by Felder [19]. The character formulae for these primary fields, and the closely related branching functions can be derived through this BRST cohomology. The analysis follows that of ref. [21]. The symmetry properties of the branching functions are also derived. A non-trivial check on the construction is provided when we specialize to the $K=1$ case and compare the results with the BPZ series. The last subsection is devoted to modular invariant partition functions of the new coset theories.

5.1. FRACTIONAL SUPERSYMMETRY CURRENT

In addition to a direct product of $A_1^{(1)}$ KM and Virasoro algebra, the $SL(2)_L$ theory can also be considered as representations of a non-local algebra, which includes a fractional supersymmetry current, denoted by $\mathscr{G}^{(L)}(z)$, and the stress–energy tensor $T(z)$. The $\mathscr{G}^{(L)}(z)$ current has conformal dimension $\Delta_L = (L+4)/(L+2)$ and is expressed in the following form [5,9–11]:

$$\mathscr{G}^{(L)}(z) = \sum_{a,b} q_{ab} J^a_{-1} \Phi^b(z), \tag{5.1}$$

where J^a_{-1} are the generators of the $A_1^{(1)}$ KM current as defined in eq. (2.5). $\Phi^a(z)$ is the chiral primary field in the generalized adjoint representation of $SL(2)_L$. This current exists whenever $t+2u \geqslant 4$. (See eqs. (2.3) and (2.4) for clarity.) For $L=2$, $\mathscr{G}^{(L)}(z)$ is just the usual $N=1$ supercurrent. Together with the stress–energy tensor $T(z)$, it generates an extended Virasoro algebra.

In sect. 2 we use the Z_L PF and an U(1) boson compactified on the radius of $\sqrt{1/L}$ as a representation of the $SL(2)_L$ theory. Here we can also bosonize the fractional supercurrent by using the Z_L PF. Following the similar calculation done in ref. [11], we can derive the following form for $\mathscr{G}^{(L)}(z)$ current:

$$\mathscr{G}^{(L)}(z) = \sqrt{\frac{L(L+4)}{4c(L+2)}} \left\{ \epsilon_1 \partial\varphi(z) + \frac{2i\sqrt{L}}{L+4} \left[\hat{\epsilon}_1(z) - \hat{\epsilon}_1^+(z) \right] \right\}, \tag{5.2}$$

where $c = 3L/(L+2)$ is the central charge of $SL(2)_L$, and

$$\begin{aligned} \epsilon_1(z) &\equiv \tfrac{1}{2}\left(A^+_{-2/L} \Phi^2_2(z) + A_{-2/L} \Phi^{2^+}_2(z) \right), \qquad \hat{\epsilon}_1(z) \equiv A^+_{-2/L-1} \Phi^2_2(z), \\ \hat{\epsilon}_1^+(z) &\equiv A_{-2/L-1} \Phi^{2^+}_2(z) = \tfrac{1}{2} L A_{-2/L-1} A^+_{2/L} \epsilon_1(z). \end{aligned} \tag{5.3}$$

The overall normalization is chosen such that $\mathscr{G}^{(L)}(z)$ has the following OPE with itself and with the energy–momentum tensor:

$$\begin{aligned} \mathscr{G}^{(L)}(z)\mathscr{G}^{(L)}(w) &= (z-w)^{-2\Delta_L}\left\{1 + (2\Delta_L/c)(z-w)^2 T(w)\right\} \\ &\quad + \lambda_L(c)(z-w)^{-\Delta_L}\left\{\mathscr{G}^{(L)}(w) + \tfrac{1}{2}(z-w)\partial\mathscr{G}^{(L)}(w)\right\} + \text{reg.}, \\ T(z)\mathscr{G}^{(L)}(w) &= \frac{\Delta_L}{(z-w)^2}\mathscr{G}^{(L)}(w) + \frac{\partial\mathscr{G}^{(L)}(w)}{(z-w)}, \end{aligned} \tag{5.4}$$

where the structure constant $\lambda_L(c)$ is determined by associativity and closure of the algebra.

What we have here are a free U(1) boson compactified on radius $\sqrt{1/L}$ and the Z_L PF. As is well known [10], we can change the compactification radius to obtain a deformed fractional CFT. Changing the compactification radius does not change the energy–momentum tensor,

$$T(z) = T_\varphi(z) + T_\psi = -\tfrac{1}{4}:\partial\varphi\partial\varphi: + T_\psi\,. \tag{5.5}$$

The modular invariant partition function can be obtained by combining the partition function of the parafermion and that of the U(1) boson with twisted boundary conditions [10]. Here we are only concerned with the diagonal modular invariant partition function and denote it as $Z(R)$ for compactification radius R. We demand the whole theory should satisfy the duality symmetry, $R/R_* \to R_*/R$ with $R_* = \sqrt{1/L}$. That is,

$$Z(R) = Z(R_*^2/R)\,,$$

$$\mathscr{G}_R^{(L)}(z) = \sqrt{\frac{L(L+4)}{4c(L+2)}}\left\{\epsilon_1\partial\varphi(z) + \frac{1}{R^*}\left(\frac{R}{R^*} + \frac{R^*}{R}\right)\frac{i}{L+4}\left[\hat{\epsilon}_1(z) - \hat{\epsilon}_1^+(z)\right]\right\}. \tag{5.6}$$

It is only at the self-dual radius that we have the $SL(2)_L$ symmetry. Once we move away from this radius, we lose the symmetry. Instead, the fractional supersymmetry current, $\mathscr{G}_R^{(L)}(z)$, and the energy–momentum tensor, $T(z)$, remain symmetry currents.

5.2. NEW COSET THEORIES

The coset theories we are going to construct are COSET$[L, K]$, where L and K are fractional numbers. Let $L = t/u$, the same as before, and K is defined by $K + 2 = p/q$, where p and q are coprime integers. For later convenience, we also define $p' \equiv p + Lq$. The central charge of this new coset theory can be derived from the usual GKO construction. It gives

$$c = \frac{3L}{L+2} + \frac{3K}{K+2} - \frac{3(K+L)}{K+L+2}$$

$$= \frac{2(L-1)}{L+2} + 1 - 24\alpha_0^2\,, \tag{5.7}$$

where $\alpha_0^2 \equiv (p-p')^2/(4Lpp') = tq^2/[4p(pu+tq)]$. From eq. (5.7), it is clear that the new coset theories can be constructed from the Z_L PF and a boson with some background charge α_0. It is usually known as the generalized Feigin–Fuchs construction. In other words we construct the Z_L PF by removing from the SL(2) theory an U(1) free boson compactified at certain radius $\sqrt{1/L}$. Now we can move to the coset theories just by putting the boson back at different radius and turning on the background charge associated with the boson. To have a consistent deformed CFT [5, 10, 11], we must choose the radius to be $R = \sqrt{p/(Lp')}$.

The PF part of the energy–momentum tensor does not change when we turn on the background charge:

$$T(z) = T_{\psi}(z) + T_{\varphi} = T_{\psi}(z) - \tfrac{1}{4} :\partial\varphi\partial\varphi: + i\alpha_0 \partial^2\varphi. \tag{5.8}$$

But the appearance of the background charge changes the primary status of the $\mathcal{G}_R^{(L)}(z)$. So we have to add a term $\partial\epsilon_1$ with coefficient proportional to α_0, such that $\mathcal{G}^{(L)}(z)$ remains primary with respect to $T(z)$

$$\mathcal{G}^{(L)}(z) = \sqrt{\frac{L(L+4)}{4c(L+2)}} \times \left\{ [\epsilon_1 \partial\phi(z) - i\alpha_0(L+2)\partial\epsilon_1(z)] + \frac{iL(\alpha_+ - \alpha_-)}{L+4}[\hat{\epsilon}_1 - \hat{\epsilon}_1^+] \right\}, \tag{5.9}$$

where we have used

$$\alpha_+ = R/R^{*2} = \sqrt{\frac{p'}{Lp}} = \sqrt{\frac{pu+tq}{pt}},$$

$$\alpha_- = -R = -\sqrt{\frac{p}{Lp'}} = -\sqrt{\frac{pu^2}{t(pu+tq)}}. \tag{5.10}$$

Even though we lose the $SL(2)_L$ symmetry, the off-diagonal currents, $J^{\pm}(z)$, can be modified such that their dimensions remain equal to one. They are now the screening currents

$$S^+(z) = \psi_1(z) :\exp(i\alpha_+\phi(z)):,$$

$$S^-(z) = \psi_1^+(z) :\exp(i\alpha_-\phi(z)):. \tag{5.11}$$

It can be verified easily that the OPE of $S^{\pm}(z)$ with $\mathcal{G}^{(L)}(z)$ has the following simple form:

$$S^{\pm}(z)\mathcal{G}^{(L)}(w) \sim \frac{W(w)}{(z-w)^2} + \text{reg.}, \tag{5.12}$$

where $W(w)$ is some operator, and the residue of the single pole piece vanishes. We define the screening charge as the contour integral of $S^{\pm}(z)$,

$$Q^{+} \equiv \oint \mathrm{d}z\, \psi_1(z) :\exp[i\alpha_{+}\phi(z)]:,$$

$$Q^{-} \equiv \oint \mathrm{d}z\, \psi_1^{+}(z) :\exp[i\alpha_{-}\phi(z)]:. \tag{5.13}$$

Since the screening currents, $S^{\pm}(z)$, are dimension-one operators and their OPE with $\mathcal{G}^{(L)}(z)$ is given by eq. (5.12), we can conclude that $Q^{\pm}$ commute with $T(z)$ and $\mathcal{G}^{(L)}(z)$

$$[Q^{\pm}, T(z)] = 0, \qquad [Q^{\pm}, \mathcal{G}^{(L)}(z)] = 0. \tag{5.14}$$

These screening charges will be used later as the basic ingredients of the BRST operator.

In sect. 2 we discussed the relation between the primary fields with respect to $\mathrm{SL}(2)_L$ chiral algebra and those with respect to Z_L parafermionic algebra (eq. (2.26)). The relation is

$$\Psi_l^l(z) = \Phi_l^l :\exp[il\phi(z)/2\sqrt{L}]:. \tag{5.15}$$

When we turn on the background charge, only the bosonic part of eq. (5.15) has to be modified and the parafermionic part remains intact. According to the requirement that they are primary with respect to the $\mathcal{G}^{(L)}(z)$ and the $T(z)$ currents, we modify them into

$$V_{j,j'}^{l}(z) = \Phi_l^l :\exp[i\beta_{j,j'}\phi(z)]:,$$

$$\text{where} \quad \beta_{j,j'} = \tfrac{1}{2}(1-j)\alpha_{+} + \tfrac{1}{2}(1-j')\alpha_{-}, \tag{5.16}$$

and j, j' are arbitrary integers satisfying the condition:

$$|n-2k| = |j-j' \pmod{2[t+2u-2]}|,$$

$$1 \leqslant j < p, \qquad 1 \leqslant j' < \tilde{p}', \tag{5.17}$$

where $\tilde{p}' \equiv (pu+tq)/D$ and $D \equiv \text{g.c.d.}\{pu+tq, uq\}$. We derive these conditions in appendix C. The second equation in (5.17) comes from demanding a non-trivial 0th cohomology class which we will discuss in subsect. 5.3. According to the appearance of the background charge, the conformal dimension of the primary field $V^l_{j,j'}(z)$ becomes

$$h^l_{j,j'} = \Delta(\Phi^l_l) + \beta^2_{j,j'} - 2\alpha_0\beta_{j,j'}$$
$$= \frac{l(l+2)}{4(L+2)} - \frac{l^2}{4L} + \frac{(jp'-j'p)^2-(p-p')^2}{4Lpp'}. \qquad (5.18)$$

5.3. CHARACTER IN THE COSET THEORY

In this subsection we calculate the characters of the primary fields with respect to $\{\mathscr{G}^{(L)}, T\}$ by using the BRST symmetry. We follow Felder's approach [19,21]. Here the reducible Verma module is defined by applying the negative modings of the extended Virasoro algebra

$$\text{Reducible Verma Module} \equiv \left\{ \sum_{q,l=1}^{\infty} \prod_{i,k=1}^{q,l} \mathscr{G}^{(L)}_{-n_1} \cdots \mathscr{G}^{(L)}_{-n_i} T_{-m_1} \cdots T_{-m_k} \left| V^l_{j,j'}(0) \right\rangle \right\}. \qquad (5.19)$$

From the parafermion content of the current $G^{(L)}$ in eq. (5.9), it is natural to conclude that the reducible Verma module is embedded in

$$\tilde{\mathscr{H}}^l_{j,j'} \equiv \bigoplus_{\substack{r=0 \\ n-r=0 (\text{mod}\, 2)}}^{t+2u-2} \mathscr{H}^{l'(r),l}_{j,j'} = \bigoplus_{\substack{r=0 \\ n-r=0 (\text{mod}\, 2)}}^{t+2u-2} \mathscr{H}^{\text{PF}}_{l'(r),l} \otimes \mathscr{H}^{\text{b}}_{j,j'}, \qquad (5.20)$$

where $l \equiv n - k(L+2)$ and $l'(r) \equiv r - k(L+2)$. The Fock space $\mathscr{H}^{\text{PF}}_{l'(r),l}$ has been defined in eq. (2.31) and the other one is defined as

$$\mathscr{H}^{\text{b}}_{j,j'} \equiv \left\{ \sum_{i=0}^{\infty} \prod_{j=0}^{i} a_{-n_1} a_{-n_2} \cdots a_{-n_j} \left| \exp[i\beta_{j,j'}\varphi(0)] \right\rangle \right\}, \qquad (5.21)$$

where a_{-n_i} are the creation operators of the boson.

The BRST operator we need is defined as

$$Q^{(n)}_{\text{BRST}} \equiv \prod_{i=1}^{n} \oint_{\mathscr{C}} \mathrm{d}z_i\, \psi_1(z_i) :\exp[i\alpha_+\varphi(z_i)]:. \qquad (5.22)$$

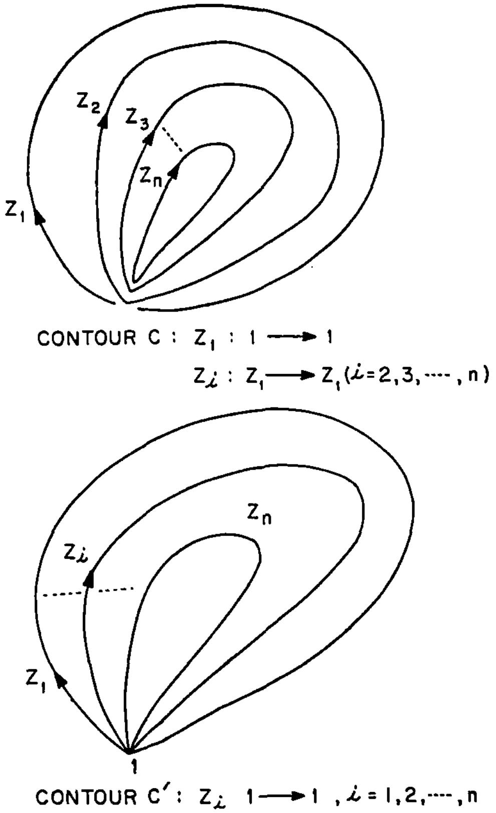

Fig. 4. The contour $\mathscr{C}$ is defined as follows: z_1 goes from 1 to 1. z_i goes from z_1 to z_1, for $i = 2, 3, \dots, n$. This is an ordered contour which means that the imaginary part of inner circle is smaller than the outer ones. We use this contour to define the BRST operator. The contour $\mathscr{C}'$ is defined as follows: z_i goes from 1 to 1, for $i = 1, 2, \dots, n$. The difference between contours $\mathscr{C}$ and $\mathscr{C}'$ is a phase factor discussed in appendix C.

The contour $\mathscr{C}$ is chosen in the same way as Felder did: z_1 goes from 1 to 1 as a closed circle. z_i goes from z_1 to z_1, for $i = 2, 3, \dots, n$. This is described in fig. 4. All the following arguments and definitions can be applied to α_- and $S^-(z)$ as well. So we shall concentrate on $S^+(z)$. In appendix C we also discuss the other contour choice and show the nilpotency property of the BRST operator

$$Q_{\mathrm{BRST}}^{(p-n)} Q_{\mathrm{BRST}}^{(n)} = 0 . \tag{5.23}$$

The power of the screening charges we need to construct the BRST operator depends on the Fock space on which $Q_{\mathrm{BRST}}^{(n)}$ acts. The criterion is based on the closure of the outer contour, z_1. It is shown in appendix C that the well defined BRST operator on the Fock space $\tilde{\mathscr{H}}_{j,j'}^{l}$ is $Q_{\mathrm{BRST}}^{(j)}$.

From the property of ψ_1 when acting on the Virasoro primary fields, $\Phi^l_m(0)$ and the $j\alpha_+$ charge carried by the BRST operator, we see that the BRST operator has the following action:

$$Q^{(j)}_{\mathrm{BRST}}: \mathscr{H}^{l,m}_{j,j'} \to \mathscr{H}^{l,m+2j}_{-j,j'} .$$

So the BRST complex can be described as

$$\dots \to \tilde{\mathscr{H}}^{l-2(p-j)}_{-j+2p,j'} \xrightarrow{Q^{(p-j)}_{\mathrm{BRST}}} \tilde{\mathscr{H}}^{l}_{j,j'} \xrightarrow{Q^{(j)}_{\mathrm{BRST}}} \tilde{\mathscr{H}}^{l+2j}_{-j,j'} \to \dots . \tag{5.24}$$

Now the irreducible Verma module defined with respect to the negative modings of the $\mathscr{G}^{(L)}$ and $T(z)$ currents is identified with the 0th cohomology class

$$\text{Irred. Verma}\left(\Phi^l_l \exp\left[i\beta_{j,j'}\varphi(0)\right]|0\rangle\right) = \left(\text{Ker}\, Q^{(j)}_{\mathrm{BRST}}/\text{Im}\, Q^{(p-j)}_{\mathrm{BRST}}\right)\Big|_{\text{on}} \tilde{\mathscr{H}}^{l}_{j,j'} \tag{5.25}$$

and the other cohomology classes are empty. In appendix C we also show the restriction on j, j' such that we have non-trivial 0th cohomology class by examining the non-vanishing two-point functions. These restrictions are

$$1 \leqslant j < p \quad \text{and} \quad 1 \leqslant j' < \tilde{p}', \tag{5.26}$$

where $\tilde{p}'$ is defined by $\tilde{p}' \equiv (pu + tq)/D$ and $D \equiv \text{g.c.d.}\{pu + tq, uq\}$.

Now that we have the BRST nilpotence property and have identified the irreducible Verma module as the 0th cohomology class, we can calculate the character, $\chi^l_{j,j'}$, which is the trace over the irreducible Verma module with the highest weight state $|V^l_{j,j'}(0)\rangle$

$$\begin{aligned}\chi^l_{j,j'} = &\sum_{s=-\infty}^{\infty} \mathrm{Tr}\left(q^{L_0 - c/24}\right) \quad \left(\text{over } \tilde{\mathscr{H}}^{l-2sp}_{j+2sp,j'}\right)\\ &- \sum_{s=-\infty}^{\infty} \mathrm{Tr}\left(q^{L_0 - c/24}\right) \quad \left(\text{over } \tilde{\mathscr{H}}^{l+2j-2sp}_{-j+2sp,j'}\right).\end{aligned} \tag{5.27}$$

The trace over the Fock space $\tilde{\mathscr{H}}^{l-2sp}_{j+2sp,j'}$ is equal to

$$\begin{aligned}\mathrm{Tr}\left(q^{L_0 - c/24}\right) \quad \left(\text{over } \tilde{\mathscr{H}}^{l-2sp}_{j+2sp,j'}\right) = &\sum_{\substack{r=0\\ n-r=0 (\mathrm{mod}\, 2)}}^{t+2u-2} \mathrm{Tr}\left(q^{L^{\mathrm{PF}}_0 - c_{\mathrm{p}}/24}\right) \quad \left(\text{over } \mathscr{H}^{\mathrm{PF}}_{l'(r), l-2sp}\right)\\ &\otimes \mathrm{Tr}\left(q^{L^{\mathrm{b}}_0 - c_{\mathrm{b}}/24}\right) \quad \left(\text{over } \mathscr{H}^{\mathrm{b}}_{j+2sp,j'}\right)\\ = &\left(\sum_{\substack{r=0\\ n-r=0 (\mathrm{mod}\, 2)}}^{t+2u-2} C^{l'(r)}_{l-2sp}(\tau)\eta(\tau)\right) \otimes \left(\frac{1}{\eta(\tau)} q^{\beta_{j,j'(s)}}\right).\end{aligned}$$

(5.28)

So eq. (5.27) can be simplified as

$$\chi^l_{j,j'} = \sum_{s=-\infty}^{\infty} \left\{ \sum_{\substack{r=0 \\ n-r=0 (\mathrm{mod}\, 2)}}^{t+2u-2} C^{l'(r)}_{m_1(s)}(\tau) q^{\beta_{j,j'}(s)} - \sum_{\substack{r=0 \\ n-r=0 (\mathrm{mod}\, 2)}}^{t+2u-2} C^{l'(r)}_{m_2(s)}(\tau) q^{\beta_{-j,j'}(s)} \right\}, \tag{5.29}$$

where $C^{l'(r)}_{m_i}(\tau)$ is the string function we defined in eq. (2.39), and $\beta_{j,j'}(s)$ is defined as

$$\beta_{j,j'}(s) = \frac{[2spp' + p'j - pj']^2 - (p-p')^2}{4Lpp'},$$

$$\beta_{-j,j'}(s) = \frac{[2spp' - p'j - pj']^2 - (p-p')^2}{4Lpp'}, \tag{5.30}$$

and

$$m_1(s) \equiv l - 2sp, \qquad m_2(s) \equiv l + 2j - 2sp. \tag{5.31}$$

Before we close this subsection, we would like to answer the question why we need to specify the admissible representation by restricting the level to satisfying $t + 2u \geqslant 2$ from the point of view of constructing the coset theories. First we argue that if $t + 2u \leqslant -2$ we can flip the signs of t and u without changing the level. So this range of t and u is excluded because of the redundancy. Now if $t + 2u = 0$ or $t + 2u = 1$ we can see from eq. (5.17) that the coset theory $SU(2)_1 \otimes SU(2)_L / SU(2)_{1+L}$ is empty because $p = t + 2u$ and $1 \leqslant j \leqslant p - 1$. So the existence of the $SU(2)_L$ WZW theory is a contradiction. Thus we obtain the condition $t + 2u \geqslant 2$, which is the same as the one required from having admissible representations.

5.4. THE DERIVATION OF THE BRANCHING FUNCTION

In this subsection we first define the branching function and derive their symmetry properties. Next we use the BRST cohomology method developed earlier to calculate some of the branching functions. We can then obtain the rest of the branching functions via the symmetry properties. Also we should point out here the checkpoint of the string functions derived in sect. 2. We can obtain the same branching functions in the coset construction in two ways: (i) using the Z_L PF and the corresponding BRST complex, (ii) derived from its dual theory, namely, by using the Z_K PF and the corresponding BRST complex [21]. The two approaches give the same results, as should be the case.

The branching function $B^{l_1}_{l_2,l_3}(\tau)$ is defined by

$$\chi^{(L)}_{l_1}(\tau,z)\chi^{(K)}_{l_2}(\tau,z) = \sum_{l_3} B^{l_1}_{l_2,l_3}(\tau)\chi^{(K+L)}_{l_3}(\tau,z)\,, \tag{5.32}$$

where K and L are the same as we have defined before, and $\chi^{(L_i)}_{l_i}(\tau,z)$ is the character of the admissible representation for level L_i, which is defined by Kač and Wakimoto [13]. The spins l_i are defined by $l_i = n_i - k_i(L_i+2) \equiv (n_i, k_i)$, where $L_i = L$, K, and $(L+K)$, respectively. The summation in l_3 has to be restricted due to the requirement that z's coefficient in the SU(2) characters must be matched: From eq. (2.34), this condition leads to $m_3 = m_1 + m_2$. Since $l_i - m_i =$ even integer, l_3 should satisfy the following selection rule:

$$l_3 - l_2 - l_1 = \text{even integer}. \tag{5.33}$$

We can rewrite eq. (5.33) into two parts

$$\begin{aligned} j &= (k_3 - k_1)L + (k_3 - k_2)K = \text{integer}, \\ n_3 &= n_1 + n_2 + j \,(\text{mod}\,2)\,. \end{aligned} \tag{5.34}$$

The first equation in (5.34) turns out to be a strong condition on k_3; for given values of k_1 and k_2, there is at most one solution of k_3. Therefore, the summation over l_3 becomes the summation over all possible n_3's.

The branching function has the following symmetries:

$$\begin{aligned} &\text{I.} \quad B^{l_1}_{l_2,l_3}(\tau) = -B^{-l_1-2}_{-l_2-2,\,-l_3-2}(\tau) \quad (\text{for all } k_i \neq 0)\,, \\ &\text{II.} \quad B^{l_1}_{l_2,l_3}(\tau) = B^{L-l_1}_{K-l_2,\,K+L-l_3}(\tau)\,, \\ &\text{III.} \quad B^{l_1}_{l_2,l_3}(\tau) = B^{l_2}_{l_1,l_3}(\tau)\,. \end{aligned} \tag{5.35}$$

The first one is derived directly from the symmetry of the character, eq. (2.12). The second one is confirmed by comparing the conformal dimensions

$$\Delta\left(B^{l_1}_{l_2,l_3}\right) = \Delta\left(B^{L-l_1}_{K-l_2,\,K+L-l_3}\right) (\text{mod integer})\,. \tag{5.36}$$

Since each primary field with a given conformal dimension forms an irreducible representation of chiral algebras underlying in a given CFT, eq. (5.36) means the two primary fields $\Phi^{l_1}_{l_2,l_3}$ and $\Phi^{L-l_1}_{K-l_2,\,K+L-l_3}$ should be identified. The third symmetry in (5.35) is the duality symmetry, which is obvious from eq. (5.32). In terms of the components of l_i, (n_i, k_i), eq. (5.35) means the following identifications of the branching functions:

$$\text{I.}\quad (n_i,k_i)\to(t_i+2u_i+2-n_i,u_i-k_i)\,,$$

$$\text{II.}\quad (n_i,k_i)\to(t_i+2u_i+2-n_i,u_i-1-k_i)\,. \tag{5.37}$$

Now let us discuss how to obtain explicit expressions for these branching functions. We can first use the BRST cohomology method in subsect. 5.3 to obtain only branching functions with $k_2=0$. This is because the BRST operator $Q^j_{\rm BRST}$ must be an integral power j of the screening charges and $j\equiv l_2+1$. If k_2 is non-zero j is not an integer. So the BRST cohomology method only gives the branching functions with $k_2=0$. Then we will obtain the others ($k_2\neq 0$) using the symmetries in eq. (5.35) or (5.37).

From subsect. 5.3 we know how to form a BRST complex starting from the highest weight state, $V^l_{j,j'}(z)$ which is associated with a Fock space $\mathscr{H}^l_{j,j'}$. This highest weight state is defined with respect to both currents, $\mathscr{G}^{(L)}(z)$ and $T(z)$. But we also know that $\Phi^l_m\exp(i\beta_{j,j'}\varphi)$ is a highest weight defined with respect to $T(z)$. The Fock space associated with this highest weight state is $\mathscr{H}^{l,m}_{j,j'}$. Then the relation between the branching function and the BRST cohomology complex starting from $\mathscr{H}^{l,m}_{j,j'}$ is

$$\begin{aligned}B^{l_1}_{l_2,l_3}&=\text{Irred.}\,\widetilde{\text{Verma}}\big(\Phi^{l_1}_{m_1}\exp[i\beta_{j,j'}\phi(0)]\big)\\&=\big(\text{Ker}\,Q^{(j)}_{\rm BRST}/\text{Im}\,Q^{(p-j)}_{\rm BRST}\big)\big|_{\rm on}\,\mathscr{H}^{l_1,m_1}_{j,j'}\,,\end{aligned} \tag{5.38}$$

where $\widetilde{\text{Verma}}$ defines the Verma module with respect to the energy–momentum tensor only, and

$$j=l_2+1\,,\qquad j'=l_3+1\,,\quad\text{and}\quad m_1=j'-j=l_3-l_2\,. \tag{5.39}$$

Since $k_2=0$, j is an integer so that $Q^{(j)}_{\rm BRST}$ is well defined. Here p is determined through $K+2=p/q$. The Fock space, $\mathscr{H}^{l_1,m_1}_{j,j'}$, is equal to $\mathscr{H}^{l_1,m_1}_{\rm PF}\otimes\mathscr{H}^{j,j'}_{\rm b}$, which is a subset of $\bar{\mathscr{H}}^l_{j,j'}$ without summing over the $l'(r)$ in the previous definition (5.20). For simplicity we neglect l_1's subscript in the following.

The properties of the BRST operator do not change; in particular, its action on the Fock spaces is the same as before, eq. (5.24):

$$Q^{(j)}_{\rm BRST}:\ \mathscr{H}^{l,m}_{j,j'}\to\mathscr{H}^{l,m+2j}_{-j,j'}\,,$$

$$\dots\to\mathscr{H}^{l,m-2(p-j)}_{-j+2p,j'}\xrightarrow{Q^{(p-j)}_{\rm BRST}}\mathscr{H}^{l,m}_{j,j'}\xrightarrow{Q^{(j)}_{\rm BRST}}\mathscr{H}^{l,m+2j}_{-j,j'}\to\dots\,. \tag{5.40}$$

By the same alternating sum and subtraction procedure as eqs. (5.27) and (5.29), we arrive at

$$\begin{aligned} B^{l_1}_{l_2,l_3} &= \mathrm{Tr}\left(q^{L_0-c/24}\right) \quad \left(\text{over } \widetilde{\text{Verma}}\left[\Phi^l_m \exp\left(i\beta_{j,j'}\phi(0)\right)\right]\right) \\ &= \sum_{s=-\infty}^{\infty} \mathrm{Tr}\left(q^{L_0-c/24}\right) \quad \left(\text{over } \mathscr{H}^{l,m-2sp}_{j+2sp,j'}\right) \\ &\quad - \sum_{s=-\infty}^{\infty} \mathrm{Tr}\left(q^{L_0-c/24}\right) \quad \left(\text{over } \mathscr{H}^{l,m+2j-2sp}_{-j+2sp,j'}\right) \\ &= \sum_{s=-\infty}^{\infty} C^l_{m_1(s)}(\tau) q^{\beta_{j,j'}(s)} - \sum_{s=-\infty}^{\infty} C^l_{m_2(s)}(\tau) q^{\beta_{-j,j'}(s)}, \end{aligned} \tag{5.41}$$

where $\beta_{j,j'}(s)$ and $m_i(s)$ are the same as we defined in eqs. (5.30) and (5.31).

Since eq. (5.41) covers all the branching functions with $k_2 = 0$, which we denote $B[k_2 = 0]$, we can then use symmetries I, II in eqs. (5.35) and (5.37) to find all the other branching functions $B[k_2 \neq 0]$ as follows:

$$B[k_2=0] \xrightarrow{\text{II}} B[k_2=u_2-1] \xrightarrow{\text{I}} B[k_2=1] \xrightarrow{\text{II}} B[k_2=u_2-2] \xrightarrow{\text{I}} B[k_2=2] \dots \tag{5.42}$$

Since the symmetry I holds for $k_i \neq 0$, if $u_1 < u_2$ it may be possible that k_1 becomes zero in the sequence (5.42) before we can cover all the k_2 from 0 to $u_2 - 1$. If this happens, we can just use dual symmetry III to start the sequence with $B[k_1 = 0]$. Therefore, the branching function $B[k_2 = 0]$ in eq. (5.41) and the symmetries in eq. (5.35) determine all the branching functions completely.

While most of these branching functions are new, we can derive the BPZ characters for minimal models using eq. (5.41). Considering COSET$[L, 1]$ theory, we can find the corresponding branching functions which should be identified with those of the BPZ minimal models because of duality. Since the BPZ characters are well known, this gives a good consistency check for all of our construction of new coset theories. Using the "MATHEMATICA" program, we checked these two character formulae up to the order of (q^{50}). We will show some examples in sect. 6.

The relation between the character of the chiral algebra, $\mathscr{G}^{(L)}(z)$ and $T(z)$, and the branching functions is

$$\chi^l_{j,j'} = \sum_{\substack{r=0 \\ n-r=0\,(\mathrm{mod}\,2)}}^{t+2u-2} B^{l'}_{l_2=j-1,\,l_3=j'-1}, \tag{5.43}$$

where $l = n - k(L+2)$, $l' = r - k(L+2)$ and $|n - 2k| = |j - j' (\mathrm{mod}\, 2[t + 2u - 2])|$. For some cases, the branching functions can appear in the character in multiple copies.

5.5. MODULAR INVARIANT PARTITION FUNCTIONS

The modular invariant partition functions of the COSET[L, K] theory can be constructed straightforwardly from the ADE classification of the SU(2) Kač–Moody algebra. Since the non-negative integer matrices $N_{l,l'}$ in eq. (2.13) have been constructed in ref. [16], we use this result to find partition functions of the coset model. The following combinations of holomorphic and anti-holomorphic branching functions of the COSET[L, K] theory are modular invariant:

$$Z_{\text{coset}} = \sum_{l_1, l_2, l_3} \sum_{\bar{l}_1, \bar{l}_2, \bar{l}_3} \mathscr{N}_{l_1, l_2, l_3; \bar{l}_1, \bar{l}_2, \bar{l}_3} B^{l_1}_{l_2, l_3}(\tau) \bar{B}^{\bar{l}_1}_{\bar{l}_2, \bar{l}_3}(\tau),$$

$$\mathscr{N}_{l_1, l_2, l_3; \bar{l}_1, \bar{l}_2, \bar{l}_3} = [N^{(L)} \otimes N^{(K)} \otimes N^{(L+K)}]_{l_1, l_2, l_3; \bar{l}_1, \bar{l}_2, \bar{l}_3} = N^{(L)}_{l_1, \bar{l}_1} N^{(K)}_{l_2, \bar{l}_2} N^{(K+L)}_{l_3, \bar{l}_3}. \tag{5.44}$$

To prove this, we should find how the branching functions are changed under the modular transformations. From the definition of the branching function (5.32), one can find

$$B^{l_1}_{l_2, l_3}(-1/\tau) = \sum_{l'_1, l'_2, l'_3} \mathscr{S}_{l_1, l_2, l_3; l'_1, l'_2, l'_3} B^{l'_1}_{l'_2, l'_3}(\tau),$$

$$B^{l_1}_{l_2, l_3}(\tau + 1) = \sum_{l'_1, l'_2, l'_3} \mathscr{T}_{l_1, l_2, l_3; l'_1, l'_2, l'_3} B^{l'_1}_{l'_2, l'_3}(\tau), \tag{5.45}$$

where $\mathscr{S}$ and $\mathscr{T}$ are expressed in terms of S and T of eq. (2.15)

$$\begin{aligned} \mathscr{S}_{l_1, l_2, l_3; l'_1, l'_2, l'_3} &= \left[S^{(L)} \otimes S^{(K)} \otimes \left(S^{(L+K)} \right)^{-1} \right]_{l_1, l_2, l_3; l'_1, l'_2, l'_3} \\ &= S^{(L)}_{l_1, l'_1} S^{(K)}_{l_2, l'_2} \left(S^{(L+K)} \right)^{\dagger}_{l_3, l'_3}, \\ \mathscr{T}_{l_1, l_2, l_3; l'_1, l'_2, l'_3} &= \left[T^{(L)} \otimes T^{(K)} \otimes \left(T^{(L+K)} \right)^{-1} \right]_{l_1, l_2, l_3; l'_1, l'_2, l'_3} \\ &= T^{(L)}_{l_1, l'_1} T^{(K)}_{l_2, l'_2} \left(T^{(L+K)} \right)^{-1}_{l_3, l'_3}, \end{aligned} \tag{5.46}$$

where we used the unitarity of S matrix. Using these matrices, it is easy to check the modular invariance of (5.44) using eq. (2.14):

$$\mathscr{S}\mathscr{N} = \mathscr{N}\mathscr{S}, \qquad \mathscr{T}\mathscr{N} = \mathscr{N}\mathscr{T}. \tag{5.47}$$

Since the matrices N are classified under the name of ADE, the partition functions of the COSET[L, K] theory may be denoted by $(X_{n_1}, Y_{n_2}, Z_{n_3})$ where each one of $X_{n_1}, Y_{n_2}, Z_{n_3}$ is one of A, D or E according to n_i defined by $t_i + 2u_i$ for the levels t_i/u_i. A-series is for any n_i and D-series is for n_i even. E-type exists only for $n_i = 12, 18, 30$. For example, (A, A, A) partition function represents the coset CFT with only spinless primary fields;

$$Z_{\text{coset}} = \sum_{l_1, l_2, l_3} |B^{l_1}_{l_2, l_3}|^2 .$$

This classification completes the construction of the COSET[L, K] theories which include many new rational conformal field theories.

6. Examples

In this section we will present some examples to demonstrate our new constructions. We first show some properties of the $Z_{1/2}$ PF, which is going to be used to construct the coset theories, COSET[1/2, 1] and COSET[1/2, 1/2]. We compare the former with the BPZ minimal model, COSET[1, 1/2], as a consistent check of our whole construction. Finally we use the latter to demonstrate some special novel properties.

6.1. $Z_{1/2}$ PF

The central charge of the $Z_{1/2}$ PF is $c_{\text{PF}} = -2/5$. ψ_1 has conformal dimension -1. In general, ψ_j has conformal dimension $j(1-2j)$, where $j = 1, 2, \dots, \infty$.

The Virasoro primary fields are Φ^l_m. l takes the values $0, 1, 2, 3$ and $-5/2, -3/2, -1/2, 1/2$. m is equal to $l + 2n$ and n is an integer running from $-\infty$ to ∞. The parafermionic primary fields are Φ^l_l with highest weights,

$$h^l = \frac{l(l+2)}{10} - \frac{l^2}{2} .$$

6.2. COSET THEORY COSET[1/2, 1]

In this subsection we consider the coset COSET[1/2, 1]. This implies $p = 3$, $q = 1$, $p' = 7/2$ and $\tilde{p}' = 7$. (See eq. (5.17).) From eq. (5.10) we obtain

$$\alpha_+ = \sqrt{7/3} \quad \text{and} \quad \alpha_- = -\sqrt{12/7} .$$

The primary fields, with respect to $T(z)$ and $\mathcal{G}^{(L)}(z)$, are the first two columns ($j = 1, 2$) of table 1. The conformal dimensions are given by eq. (5.18)

$$h^l_{j,j'} = \frac{l(l+2)}{10} - \frac{l^2}{2} + \frac{(7j - 6j')^2 - 1}{84} .$$

Now we also present BPZ minimal model, $M_{p=5,\, p'=7}$ (= COSET[1, 1/2]), in the following table, where $c = 1 - 6(p - p')^2/pp' = 11/35$.

TABLE 1
The Kač table of the highest weight states $h^l_{j,j'}$ in the coset theory, COSET[1/2, 1], with central charge $c = 11/35$. The numbers in parentheses (n, k) denote the spin $l = n - k(L+2) = n - k(5/2)$. The number below is the highest weight.

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$j'=1$	(0,0) 0	(1,0) 11/20			
$j'=2$	(1,0) 3/35	(0,0) 1/28		(0,1) 1/28	
$j'=3$	(2,0) 8/35	(1,0) −3/140		(1,1) −3/140	(0,1) 3/7
$j'=4$	(3,0) 3/7	(2,0) −3/140		(2,1) −3/140	(1,1) 8/35
$j'=5$		(3,0) 1/28		(3,1) 1/28	(2,1) 3/35
$j'=6$					(3,1) 0

We can see from these two tables that only part of table 2 appears in the first two columns of table 1, which are supposed to be the primary fields in the coset theory, COSET[1/2, 1]. The rest of the table 2 become the descendents of the primary fields in this coset theory. For example*

$$\begin{aligned}
&H^{l=0}_{1,1} = h_{1,1} \xrightarrow{4/5} h_{3,1}, \qquad H^{l=0}_{2,2} = h_{1,2} \xrightarrow{4/5} h_{3,2},\\
&H^{l=1}_{2,1} = h_{2,1} \xrightarrow{6/5} h_{4,1}, \qquad H^{l=1}_{1,2} = h_{2,2} \xrightarrow{6/5} h_{4,2}, \qquad H^{l=1}_{2,3} = h_{2,3} \xrightarrow{6/5} h_{4,3},\\
&H^{l=2}_{2,4} = h_{3,4} \xrightarrow{1-4/5} h_{1,4}, \qquad H^{l=2}_{1,3} = h_{3,3} \xrightarrow{1-4/5} h_{1,3},\\
&H^{l=3}_{1,4} = h_{4,4} \xrightarrow{1-6/5} h_{2,4},
\end{aligned} \tag{6.1}$$

where the number above the arrow indicates the negative moding (mod integer) generated by the $\mathscr{G}^{(L)}$ current [11]. According to their analysis, we can write down the following sequence:

$$\dots \leftarrow H^{l-2}(\text{descendent}) \xleftarrow{1-l/(L+2)} H^l_{j,j'} \xrightarrow{(l+2)/(L+2)} H^{l+2}(\text{descendent}) \rightarrow \dots, \tag{6.2}$$

where we use "$H^{l'}$(descendent)" to denote the descendents of the coset primary fields. "$H^{l'}$(descendent)" are still primary with respect to the Virasoro algebra.

* In the following $H^l_{j,j'}$, and $h_{m,m'}$ are used to denote respectively the primary fields in the coset theory (= COSET[1/2, 1]) and those in the BPZ minimal models (= COSET[1, 1/2]).

TABLE 2
The Kač table of the highest weight states $h_{m,m'}$ in the BPZ minimal model, $M_{p=5,p'=7}$, with central charge $c = 11/35$

	$m=1$	$m=2$	$m=3$	$m=4$
$m'=1$	0	11/20	9/5	15/4
$m'=2$	1/28	3/35	117/140	16/7
$m'=3$	3/7	−3/140	8/35	33/28
$m'=4$	33/28	8/35	−3/140	3/7
$m'=5$	16/7	117/140	3/35	1/28
$m'=6$	15/4	9/5	11/20	0

The number above the arrow indicates the conformal dimension difference generated by the $\mathscr{G}^{(L)}$ current.

In addition to the above relationship between the highest weight states, there is another more restricted relation. It is that the branching functions derived from both approaches must be the same due to the obvious dual symmetry (1.3). Recall the formula, eq. (5.32),

$$\chi_{l_1}^{(1/2)}\chi_{l_2}^{(1)} = \sum_{n_3} B_{l_2,l_3}^{l_1}\chi_{l_3}^{(3/2)} = \sum_{n_3} \tilde{B}_{l_1,l_3}^{l_2}\chi_{l_3}^{(3/2)} . \tag{6.3}$$

The first one, $B_{l_2,l_3}^{l_1}$, is derived from the cohomology complex using the Z_L PF based on the highest weight state $H_{j,j'}^{l_1,m_1=l_3-l_2} = \Phi_{m_1}^{l_1}\exp(i\beta_{j,j'}\phi)$ where $j=l_2+1$ and $j'=l_3+1$. The second one, $\tilde{B}_{l_1,l_3}^{l_2}$, uses the Z_1 PF (which is trivial, hence giving just the BPZ's minimal model) based on the highest weight state $h_{m=l_1+1,m'=l_3+1}$. Notice that the cohomology method is applicable only in the case where the first subscript is an integer, which means $k_2=0$ in $B_{l_2,l_3}^{l_1}$ and $k_1=0$ in $\tilde{B}_{l_1,l_3}^{l_2}$.

In the following we want to compare the branching functions derived from these two approaches by Taylor expansion up to order of q^{40}. The calculation is based on eqs. (5.41) and (2.39) in the COSET[1/2, 1] case. We also need the symmetry of the branching function, eq. (5.35), when $k_1 \neq 0$ (the case (c) below). We only cite three typical examples in detail, which are the branching functions based on the primary field, $H_{1,1}^{l_1=0,\,m_1=0}$, its descedent (but Virasoro primary), $H_{1,1}^{l_1=2,\,m_1=0}$, and finally the $H_{4,5}^{l_1=1/2,\,m_1=1/2}$ ($k_1 \neq 0$ case).

(a) $H_{1,1}^{l_1=0,\,m_1=0}$: (We must have $B_{l_2=0,\,l_3=0}^{l_1=0} = \tilde{B}_{l_1=0,\,l_3=0}^{l_2=0}$ in this case.)

$$\begin{aligned}\eta B_{l_2=0,\,l_3=0}^{l_1=0} &= q^{1/84}\big[C_0^0 - q^2C_2^0 - q^{10}C_{-4}^0 + q^{20}C_6^0 + q^{22}C_{-6}^0 - q^{36}C_8^0 \\ &\quad - q^{60}C_{-10}^0 + q^{82}C_{12}^0 + q^{86}C_{-12}^0 - q^{112}C_{14}^0 - q^{152}C_{-16}^0 \\ &\quad + q^{186}C_{18}^0 + q^{192}C_{-16}^0 - \dots\big] \\ &= q^{1/35}\big[1 - q - q^{24} + q^{33} + q^{37} - O(q^{40})\big] .\end{aligned} \tag{6.4}$$

From the BPZ minimal model we also have

$$\eta \tilde{B}^{l_2=0}_{l_1=0,\, l_3=0} = \eta \chi_{m=1,\, m'=1} = q^{1/35}\left[1 - q - q^{24} + q^{33} + q^{37} - \mathrm{O}(q^{40})\right], \quad (6.5)$$

where $\chi_{m,m'}$ is the character with highest weight $h_{m,m'}$ which was first given by Rocha-Caridi [8]. As expected, up to the terms calculated, eq. (6.4) from COSET[1/2, 1] and eq. (6.5) from COSET[1, 1/2] agree.

(b) $H^{l_1=2,\, m_1=0}_{1,1}$: (We must have $B^{l_1=0}_{l_2=0,\, l_3=0} = \tilde{B}^{l_2=0}_{l_1=2,\, l_3=0}$ in this case.)

$$\begin{aligned}\eta B^{l_1=2}_{l_2=0,\, l_3=0} &= q^{1/84}\left[C^2_0 - q^2 C^2_2 - q^{10} C^2_{-4} + q^{20} C^2_6 + q^{22} C^2_{-6} - q^{36} C^2_8\right.\\ &\quad - q^{60} C^2_{-10} + q^{82} C^2_{12} + q^{86} C^2_{-12} - q^{112} C^2_{14} - q^{152} C^2_{-16}\\ &\quad \left. + q^{186} C^2_{18} + q^{192} C^2_{-16} - \dots\right]\\ &= q^{29/35}\left[q - q^4 - q^{13} + q^{20} + \mathrm{O}(q^{40})\right]. \end{aligned} \quad (6.6)$$

The corresponding one in the BPZ minimal model is

$$\eta \tilde{B}^{l_2=0}_{l_1=2,\, l_3=0} = \eta \chi_{m=3,\, m'=1} = q^{29/35}\left[q - q^4 - q^{13} + q^{20} + \mathrm{O}(q^{40})\right] \quad (6.7)$$

Again they agree, as expected.

(c) $H^{l_1=1/2,\, m_1=1/2}_{4,5} \cong H^{l_1=1/2,\, m_1=1/2}_{1,3/2}$: In this example we first have to shift $j=4$ to the range $1 \leqslant j < 3 = p$ because of the requirement of the BRST cohomology. The highest weight of $H^{l_1=1/2,\, m_1=1/2}_{4,5}$ is equal to that of $H^{l_1=1/2,\, m_1=1/2}_{1,3/2}$. The branching function is derived based on $H^{l_1=1/2,\, m_1=1/2}_{1,3/2}$. This is why we use the symbol $\cong$ above. In this case, $k_1 \neq 0$, so that the cohomology method can not be directly applied to the derivation of $\tilde{B}^{l_2}_{l_1, l_3}$. Therefore, we must first use the symmetry (5.35), either on $B^{l_1=1/2}_{l_2=0,\, l_3=1/2}$ (to get $B^{l_1=1/2}_{l_2=0,\, l_3=1/2} = B^{l_1=0}_{l_2=1,\, l_3=1}$) or on $\tilde{B}^{l_2=0}_{l_1=1/2,\, l_3=1/2}$ (to get $\tilde{B}^{l_2=0}_{l_1=1/2,\, l_3=1/2} = \tilde{B}^{l_2=1}_{l_1=0,\, l_3=1}$). Then we can see the duality easily.

$$\begin{aligned}\eta B^{l_1=1/2}_{l_2=0,\, l_3=1/2} &= \eta B^{l_1=0}_{l_2=1,\, l_3=1}\\ &= q^{1/21}\left[C^{1/2}_{1/2} - q^3 C^{1/2}_{5/2} - q^8 C^{1/2}_{-7/2} + q^{19} C^{1/2}_{-11/2} + q^{23} C^{1/2}_{13/2}\right.\\ &\quad - q^{40} C^{1/2}_{17/2} - q^{55} C^{1/2}_{-19/2} + q^{80} C^{1/2}_{-23/2} + q^{88} C^{1/2}_{25/2} - q^{119} C^{1/2}_{29/2}\\ &\quad \left. - q^{144} C^{1/2}_{-31/2} + q^{195} C^{1/2}_{37/2} + q^{183} C^{1/2}_{-35/2} - \dots\right]\\ &= q^{9/140}\left[1 - q^2 - q^{20} + q^{32} + q^{38} - \mathrm{O}(q^{40})\right]. \end{aligned} \quad (6.8)$$

The corresponding one in the BPZ minimal model is

$$\eta \tilde{B}^{l_2=0}_{l_1=1/2,\, l_3=1/2} = \eta \tilde{B}^{l_2=1}_{l_1=0,\, l_3=1} = \eta \chi_{m=1,\, m'=2}$$
$$= q^{9/140}\left[1 - q^2 - q^{20} + q^{32} + q^{38} - \mathrm{O}(q^{40})\right]. \tag{6.9}$$

Up to the terms calculated, this checks COSET[1, 1/2] = COSET[1/2, 1], as expected.

6.3. COSET[1/2, 1/2]

This coset model is an example of the new cosets constructed in this paper. The central charge is equal to 1/5, which is absent from the BPZ minimal model. We first have $p = 5$, $q = 2$, $p' = 6$ and $\tilde{p}' = 3$ and then

$$\alpha_+ = \sqrt{7/3} \quad \text{and} \quad \alpha_- = -\sqrt{12/7}\,.$$

The highest weight is equal to

$$h^l_{j,j'} = \frac{l(l+2)}{10} - \frac{l^2}{10} + \frac{(6j - 5j')^2 - 1}{60}$$

The Kač table is shown in table 3.

The characters are obtained using eqs. (5.41), (5.43) and (2.39).

We point out the similarity between, COSET[1/2, 1/2] and COSET[1/2, 1]. In the COSET[1/2, 1] case we can generate finitely many branching functions, i.e. Virasoro primary fields, from the coset primary fields such that they form a closed algebra, dictated by the fusion rule. These finitely many Virasoro primary fields are the same as those obtained from COSET[1, 1/2], which is a BPZ mini-mal model. In the COSET[1/2, 1/2] case we can also find a closed set with only finite number of branching functions which close under fusion rule. (Closure under fusion rule follows from the fact that they form an admissible representation under modular transformation and Verlinde's conjecture [38].) Since COSET[1/2, 1/2] with $c = 1/5$ is absent from BPZ's analysis. Our construction for $c < 1$ can be considered as a generalization of the BPZ series.

TABLE 3
The Kač table of the highest weight states $h^l_{j,j'}$ in the coset theory, COSET[1/2, 1/2] with central charge $c = 1/5$. The top line in each entry specifies the spin l

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
	$l = 0$	$l = 1$	$l = 2$	$l = 3$
$j' = 1$	0	3/5	8/5	3
	$l = 1$	$l = 0$	$l = 1$	$l = 2$
$j' = 2$	1/20	1/20	51/60	41/20

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Appendix A

FOUR-POINT CORRELATION FUNCTION

From the current–current correlation function $\langle J^+(z)J^-(w)\rangle = L/(z-w)^2$, we obtain

$$\langle \psi_1(z)\psi_1^+(w)\rangle = \frac{1}{(z-w)^{2-2/L}}\,.$$

Using the Ward identity (2.21), we can derive all correlation functions of ψ_1 and ψ_1^+. In particular, the four-point correlation function is given by

$$\begin{aligned}&\langle \psi_1(z_1)\psi_1(z_2)\psi_1^+(w_1)\psi_1^+(w_2)\rangle\\ &\quad= \left[(z_1-w_1)(z_2-w_2)\right]^{-2+2/L}\xi^{-2/L}\left[\frac{2(L-1)}{L}+\frac{2(L-1)}{L}\xi+\xi^2\right],\end{aligned} \tag{A.1}$$

where $\xi \equiv (z_1-z_2)(w_1-w_2)/(z_1-w_2)(z_2-w_1)$.

Now, using eq. (2.19)

$$\begin{aligned}\psi_1(z)\psi_1(w) &= c_{1,1}(z-w)^{\Delta_2-2\Delta_1}\psi_2(w)+\dots\,,\\ \psi_1^+(z)\psi_1^+(w) &= c_{1,1}^+(z-w)^{\Delta_2^+-2\Delta_1}\psi_2^+(w)+\dots\,,\end{aligned} \tag{A.2}$$

one can read off the most singular power of (z_1-z_2) and (w_1-w_2) to find

$$\begin{aligned}\Delta_2 = \Delta_2^+ &= 2(L-2)/L\,,\\ c_{1,1}c_{1,1}^+ &= 2(L-1)/L\,,\end{aligned} \tag{A.3}$$

and using eq. (2.23)

$$\psi_2(z)\psi_2^+(w) = (z-w)^{-2\Delta_2}\left[1+(2\Delta_2/c)(z-w)^2 T_\psi(w) + \mathrm{O}\left((z-w)^3\right)\right], \tag{A.4}$$

one can derive $c = 2(L-1)/(L+2)$. To compute other conformal weights Δ_l and structure constants $c_{j,j'}$, $c_{j,j'}^+$, we need to consider higher-point correlation functions using eq. (2.21) recursively. In general, it is easier to use the bosoization approach to calculate Δ_l.

Appendix B

THE SYMMETRIES OF THE STRING FUNCTIONS

In this appendix we will first show how to obtain eq. (2.39) and then present the detailed proof of the symmetries of the string functions.

Starting from eq. (2.36), we first separate the range of summation in p, i.e.

$$\sum_{j,\,p=-\infty}^{\infty}\sum_{r=0}^{\infty}\sum_{\sigma=\pm} = \sum_{j=-\infty}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\sigma=\pm} + \sum_{j=-\infty}^{\infty}\sum_{p=-1}^{-\infty}\sum_{r=0}^{\infty}\sum_{\sigma=\pm}. \tag{B.1}$$

We then simplify the second terms in the above equation by using eq. (2.38) to delete the first $2p$ terms in the r summation. Finally we obtain eq. (2.39) by comparing the z-dependence with that in eq. (2.34) and substituting the j value in terms of m.

The string functions have the following symmetries:

$$C_m^l = C_{L-m}^{L-l},$$
$$C_m^l = C_{-m}^l \quad (\text{for} \quad k=0), \qquad C_m^l = C_{2L-m}^l \quad (\text{for} \quad k=u-1), \tag{B.2}$$

where $l = n - k(L+2)$. To prove it we have to rewrite eq. (2.39) into slightly different forms by changing the p summation's range according to $\sigma = +1$ or $\sigma = -1$. Let us start from eq. (2.36) again, where the summation is split in two different ways,

$$\begin{aligned}
\sum_{j,\,p=-\infty}^{\infty}\sum_{r=0}^{\infty}\sum_{\sigma=\pm} &= \sum_{j=-\infty}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}(\sigma=+1) + \sum_{j=-\infty}^{\infty}\sum_{p=1}^{\infty}\sum_{r=0}^{\infty}(\sigma=-1)\\
&\quad + \sum_{j=-\infty}^{\infty}\sum_{p=-1}^{-\infty}\sum_{r=0}^{\infty}(\sigma=+1) + \sum_{j=-\infty}^{\infty}\sum_{p=0}^{-\infty}\sum_{r=0}^{\infty}(\sigma=-1)\\
&= \sum_{j=-\infty}^{\infty}\sum_{p=1}^{\infty}\sum_{r=0}^{\infty}(\sigma=+1) + \sum_{j=-\infty}^{\infty}\sum_{p=1}^{\infty}\sum_{r=0}^{\infty}(\sigma=-1)\\
&\quad + \sum_{j=-\infty}^{\infty}\sum_{p=0}^{-\infty}\sum_{r=0}^{\infty}(\sigma=+1) + \sum_{j=-\infty}^{\infty}\sum_{p=0}^{-\infty}\sum_{r=0}^{\infty}(\sigma=-1).
\end{aligned}$$

Then we repeat the same procedures described immediately below eq. (B.1) and obtain two other forms for C_m^l

$$\eta^3 C_m^l(\tau) = \Bigg(\sum_{p,\,r\geqslant 0} - \sum_{p,\,r<0}\Bigg)(-1)^r q^{g_1} - \Bigg(\sum_{p>0,\,r\geqslant 0} - \sum_{p\leqslant 0,\,r<0}\Bigg)(-1)^r q^{g_2} \tag{B.3}$$

$$\eta^3 C_m^l(\tau) = \Bigg(\sum_{p>0,\,r\geqslant 0} - \sum_{p\leqslant 0,\,r<0}\Bigg)(-1)^r q^{g_1} - \Bigg(\sum_{p>0,\,r\geqslant 0} - \sum_{p\leqslant 0,\,r<0}\Bigg)(-1)^r q^{g_2}, \tag{B.4}$$

where we define

$$g_1(n,k,m,p,r) \equiv a[(p+r/2u)+b_+/2a]^2 - L[r/2+m/(2L)]^2$$
$$= \frac{[n+1-k(L+2)+(r+2pu)(L+2)]^2}{4(L+2)} - \frac{(rL+m)^2}{4L}$$
$$g_2(n,k,m,p,r) \equiv a[(p+r/2u)+b_-/2a]^2 - L[r/2+m/(2L)]^2$$
$$= \frac{[n+1+k(L+2)-(r+2pu)(L+2)]^2}{4(L+2)} - \frac{(rL+m)^2}{4L}, \tag{B.5}$$

which corresponds to the exponents in eqs. (2.39), (B.3) and (B.4).

Now it is easy to see the first symmetry. C^{L-l}_{L-m} defines its exponents in the summation, denoted by $g_i(\tilde{n},\tilde{k},\tilde{m},p,r)$. $\tilde{n},\tilde{k}$ are defined by $L-l=\tilde{n}-\tilde{k}(L+2)$. After some algebraic manipulation, we obtain

$$g_1(\tilde{n},\tilde{k},\tilde{m},p,r) = g_1(t+2u-2-n,u-k-1,\tilde{m},r,p)$$
$$= g_1(n,k,m,-r-1,-p)$$
$$g_2(\tilde{n},\tilde{k},\tilde{m},p,r) = g_2(t+2u-2-n,u-k-1,\tilde{m},r,p)$$
$$= g_2(n,k,m,-r-1,-p+1). \tag{B.6}$$

So we finally have

$$\eta^3 C^{L-l}_{L-m}(\tau) = \left(\sum_{p,r\geqslant 0} - \sum_{p,r<0}\right)(-1)^r q^{g_1(t+2u-2-n,u-k-1,\tilde{m},r,p)}$$
$$-\left(\sum_{p>0,r\geqslant 0} - \sum_{p\leqslant 0,r<0}\right)(-1)^r q^{g_2(t+2u-2-n,u-k-1,\tilde{m},r,p)}$$
$$= \left(\sum_{p>0,r\geqslant 0} - \sum_{p\leqslant 0,r<0}\right)(-1)^r q^{g_1(n,k,m,r,p)}$$
$$-\left(\sum_{p>0,r\geqslant 0} - \sum_{p\leqslant 0,r<0}\right)(-1)^r q^{g_2(n,k,m,r,p)}$$
$$= \eta^3 C^l_m(\tau). \tag{B.7}$$

Before we prove the second symmetry we would like to emphasize here that $\chi_l(\tau, z=0)$ is well defined only when $b_+ = -b_-$, which holds only when $k=0$. Then we can simply write

$$\chi_l(\tau, z=0) = \sum_{m=-\infty}^{\infty} C^l_m(\tau) \times q^{m^2/(4L)} = \sum_{m=-\infty}^{\infty} C^l_{-m}(\tau) \times q^{m^2/(4L)}. \quad \text{(B.8)}$$

This gives us the second symmetry when $k=0$. Notice that when $k=0$, $l+m \in 2\mathbb{Z}$, both J^3 eigenstates m and $-m$ exist. The third one can be easily derived from the first and the second as we have done in sect. 2. This completes the proof of string function's symmetries.

Appendix C

BRST PROPERTIES IN THE COSET THEORIES, $SU(2)_L \otimes SU(2)_K / SU(2)_{L+K}$

In this appendix we show the relation between different choices of the contours in defining the BRST operator, and derive the condition for non-vanishing two-point function which determines the non-trivial cohomology complex.

We first define the BRST operator [19,28]

$$Q^{(n)}_{\mathrm{BRST}} \equiv \prod_{i=1}^{n} \oint_{\mathscr{C}} \mathrm{d}z_i \, \psi_1(z_i) :\exp[i\alpha_+ \phi(z_i)]:. \quad \text{(C.1)}$$

The contour $\mathscr{C}$ is chosen in the same way as Felder did: z_1 goes from 1 to 1 as a closed circle. z_i goes from z_1 to z_1, for $i = 2, 3, \dots, n$. This is described in fig. 4 and we use the symbol $\mathscr{C}$ to denote this contour.

There is also another choice of the contour. Instead of restricting z_i, for $i = 2, 3, \dots, j$, in going from z_1 to z_1, we choose z_i to be the same as z_1 contour. This contour is denoted as $\mathscr{C}'$. The difference between contours $\mathscr{C}$ and $\mathscr{C}'$ is a phase factor which comes from switching operators $S^+(z_i) = \psi_1(z) :\exp(i\alpha_+\phi(z))$: to form an ordered contour. From the OPE of $\psi_1(z_i)$ described in sect. 2 and the OPE among the bosons, we can have

$$S^+(z_i) S^+(z_j) = (z_i - z_j)^{2(\alpha_+^2 - 1/L)} :S^+(z_i) S^+(z_j):. \quad \text{(C.2)}$$

So in order to get the integral along the contour $\mathscr{C}$ from that along $\mathscr{C}'$, we have to re-order the screening operators. This re-ordering will give us

$$\prod_{i=1}^{n} \oint_{\mathscr{C}'} \mathrm{d}z_i \, \psi_1(z_i) :\exp[i\alpha_+ \phi(z_i)]:$$

$$= \frac{1 - \exp[4\pi i n(\alpha_+^2 - 1/L)]}{1 - \exp[4\pi i(\alpha_+^2 - 1/L)]} \prod_{i=1}^{n} \oint_{\mathscr{C}} \mathrm{d}z_i \, \psi_1(z_i) :\exp[i\alpha_+ \phi(z_i)]:. \quad \text{(C.3)}$$

Then from the definition of $\alpha_\pm$, eq. (5.10) and the property of p, q being coprime we can conclude that

$$\alpha_+^2 - 1/L = q/p\,,$$

$$Q_{\mathrm{BRST}}^{(p-n)} Q_{\mathrm{BRST}}^{(n)} = 0\,. \tag{C.4}$$

To have a well defined BRST operator on the Fock space, $\tilde{\mathscr{H}}^l_{j,j'}$ which we have defined in eq. (5.20), is equivalent to have a well defined operation of $Q_{\mathrm{BRST}}^{(k)}$ on $|\Phi^l_l :\exp[i\beta_{j,j'}\phi(0)]:\rangle$. It is essentially the same as demanding the closure of the outer contour z_1. The condition is

$$k = j, \quad \text{and} \quad |n - 2k| = |j - j' \,(\mathrm{mod}\, 2[t + 2u - 2])|\,. \tag{C.5}$$

The derivation is based on the operator product expansion of the boson and that of the parafermion. By the standard method, we have

$$:\exp[i\alpha_+\varphi(z_1)]:\ldots:\exp[i\alpha_+\phi(z_k)]::\exp[i\beta_{j,j'}\phi(0)]:$$

$$= \prod_{i<n}(z_i - z_n)^{2\alpha_+^2} \prod_{i=1}^{k} z_i^{2\alpha+\beta_{j,j'}} :\exp\left[i\alpha_+ \sum_{i=1}^{k} \varphi(z_i) + i\beta_{j,j'}\phi(0)\right]:\,,$$

$$\psi_1(z_1)\ldots\psi_1(z_k)\Phi^l_l(0)$$

$$= \prod_{i<n}(z_i - z_n)^{-2/L} \prod_{i=1}^{k} z_i^{-l/L-1}\left(:\Phi^l_{l+2k}(0): + \text{higher order terms}\right). \tag{C.6}$$

Then after we make the change of the variables $z_i \to z_1 u_i$, for $i = 2, 3 \ldots k$, and demand the exponent of z_1 be integer, we obtain the above condition (C.5).

We next examine the condition for the non-vanishing of the two-point function which determines whether there is a state that has non-zero overlap with $Q_{\mathrm{BRST}}^{(j)}|\Phi^l_l :\exp[i\beta_{j,j'}\phi(0)]:\rangle$. We first see the bosonic part:

$$:\exp\left[i\alpha_+ \sum_{i=1}^{j} \phi(z_i) + i\beta_{j,j'}\phi(0)\right]:|0\rangle$$

$$\in \left\{ \prod_{n_i} a_{-n_1}\ldots a_{-n_l}\middle| :\exp[i\beta_{-j,j'}\phi(0)]:\rangle \right\} \equiv \mathscr{H}^{\mathrm{b}}_{-j,j'}\,. \tag{C.7}$$

Due to the similar argument from Felder, there must exist a covector, $\chi_{\mathrm{b}}^{-j,j'}$, in $(\mathscr{H}^{\mathrm{b}}_{-j,j'})^*$, which is the dual space of $\mathscr{H}^{\mathrm{b}}_{-j,j'}$, such that

$$\begin{aligned}&\langle \chi_{\mathrm{b}}^{-j,j'} :\exp\left[i\alpha_+\phi(z_1)\right]:\ldots:\exp\left[i\alpha_+\phi(z_j)\right]::\exp\left[i\beta_{j,j'}\phi(0)\right]:|0\rangle\\ &\quad=\prod_{i<n}(z_i-z_n)^{2\alpha_+^2}\prod_{i=1}^{j} z_i^{2\alpha_+\beta_{j,j'}}.\end{aligned}\tag{C.8}$$

The parafermionic part can be obtained in the same way:

$$\langle \chi_{\mathrm{PF}}\psi_1(z_1)\ldots\psi_1(z_j)\Phi_l^l(0)|0\rangle=\prod_{i<n}(z_i-z_n)^{-2/L}\prod_{i=1}^{j} z_i^{-l/L-1}.\tag{C.9}$$

So the two-point function is proportional to

$$\begin{aligned}&\left\langle \chi_{\mathrm{PF}}\chi_{\mathrm{b}}^{-j,j'}\middle|Q_{\mathrm{BRST}}^{(j)}\Phi_l^l:\exp\left[i\beta_{j,j'}\phi(0)\right]:\right\rangle\\ &\quad=\prod_{i=2}^{j}\int \mathrm{d}u_i\, u_i^{(2\alpha_+\beta_{j,j'}-l/L-1)}(1-u_i)^{2(\alpha_+^2-1/L)}\prod_{i<n}(u_i-u_n)^{2(\alpha_+^2-1/L)}\\ &\quad=\prod_{i=2}^{j}\int \mathrm{d}u_i\, u_i^{(\alpha_+^2-1/L)(1-j)-1}(1-u_i)^{2(\alpha_+^2-1/L)}\prod_{i<n}(u_i-u_n)^{2(\alpha_+^2-1/L)}.\end{aligned}\tag{C.10}$$

By the same procedure as Felder's calculation to evaluate the above integral, we arrive at

$$\begin{aligned}&\left\langle \chi_{\mathrm{PF}}\chi_{\mathrm{b}}^{-j,j'}\middle|Q_{\mathrm{BRST}}^{(j)}\Phi_l^l:\exp\left[i\beta_{j,j'}\phi(0)\right]:\right\rangle\\ &\quad=\frac{(2\pi i)^{j-1}}{(j-1)!}\exp\left[-i\pi(\alpha_+^2-1/L)(j-1)\right]\frac{\Gamma\left[1+j(\alpha_+^2-1/L)\right]}{\Gamma\left[1+(\alpha_+^2-1/L)\right]}\\ &\qquad\times\prod_{n=1}^{j-1}\frac{\sin\left[\pi n(\alpha_+^2-1/L)\right]}{\sin\left[\pi(\alpha_+^2-1/L)\right]}.\end{aligned}\tag{C.11}$$

So the non-trivial cohomology (non-vanishing two-point function) will give us the constraint that $1\leqslant j<p$ because p,q are two coprime integers. (Remember that $\alpha_+^2-1/L=q/p$ and $\alpha_-^2-1/L=-q/p'$.) By the same procedure but with the α_- counterpart, we will obtain the constraints on j'. It is $1\leqslant j'<\tilde{p}'$, where $\tilde{p}'$ is defined by

$$\bar{p}' \equiv (pu + tq)/D, \qquad D \equiv \text{g.c.d.}\{pu + tq, uq\}. \tag{C.12}$$

These two constraints on j and j' are just eq. (5.26).

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