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OPE in planar QCD from integrability

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Abstract: We consider the operator product expansion of local gauge-invariant single-trace operators composed of self-dual components of the field strength tensor in planar QCD. Using the integrability of the 1-loop dilatation operator, we obtain a determinant expression for certain tree-level structure constants.

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1 Introduction

1.1 SYM$_4$ and integrable spin chains

The problem of computing the conformal dimensions of local, gauge-invariant single-trace composite operators in planar $\mathcal{N}=4$ supersymmetric Yang-Mills theory in (3+1)-dimensions, SYM$_4$, is integrable [1–3]. At 1-loop level, the mixing matrix Γ maps to the Hamiltonian $\mathcal{H}_{\mathrm{PSU}(2,2|4)}$ of an integrable PSU$(2,2|4)$-symmetric spin chain with nearest-neighbor interactions and periodic boundary conditions, such that *1.* The eigenstates $\{\mathcal{O}\}$ of Γ are in one-to-one correspondence with the eigenstates $\{|\mathcal{O}\rangle\}$ of $\mathcal{H}_{\mathrm{PSU}(2,2|4)}$, and *2.* The eigenvalues $\{\gamma\}$ of Γ, which are the anomalous dimensions of $\{\mathcal{O}\}$, are equal to the eigenvalues $\{\mathcal{E}\}$ of $\mathcal{H}_{\mathrm{PSU}(2,2|4)}$.[1] Since the eigenstates and eigenvalues of $\mathcal{H}_{\mathrm{PSU}(2,2|4)}$ can be computed using Bethe ansatz methods, the problem is integrable.[2] For a recent review, see [4] and references therein.

1.2 SYM$_4$ SU(2)-doublets and spin-$\frac{1}{2}$ chains

SYM$_4$ contains a vector gauge field, four chiral and four anti-chiral spin-$\frac{1}{2}$ fermions, and six real scalars that can be expressed as three complex scalars $\{X, Y, Z\}$ and their charge-conjugates $\{\bar{X}, \bar{Y}, \bar{Z}\}$. Any two complex scalars that are not charge conjugates, such as $\{X, Z\}$ or $\{X, \bar{Z}\}$, mix only amongst themselves to form an SU(2)-doublet and an SU(2)-invariant scalar subsector of SYM$_4$. In the planar limit at 1-loop level, the single-trace operators $\{\mathcal{O}\}$, that are composed of a single SU(2) doublet, and that are eigenstates of Γ, map to eigenstates $\{|\mathcal{O}\rangle\}$ of the Hamiltonian $\mathcal{H}_{\frac{1}{2}}$ of a periodic XXX spin-$\frac{1}{2}$ chain.

1.3 QCD SU(2)-triplets and spin-1 chains

In [5], Ferretti, Heise and Zarembo noted that, at 1-loop level, operators composed of self-dual components $\{f_+, f_0, f_-\}$ of the QCD field strength tensor mix only among themselves to form an SU(2)-triplet. Using that observation, as well as the fact that QCD with no matter fields is conformally invariant (the beta function vanishes) in the planar limit at 1-loop level, they showed that local single-trace operators $\{\mathcal{O}\}$ that are eigenstates of Γ correspond to eigenstates of the Hamiltonian $\mathcal{H}_1$ of an integrable XXX spin-1 [6] chain.[3] As in the spin-$\frac{1}{2}$ case, the spin-1 chain eigenstates and eigenvalues can be computed using Bethe ansatz methods [7–10].

[1] In this note, $\mathcal{O}$ is a local gauge-invariant single-trace composite operator, in SYM$_4$ or in QCD depending on context, that is an eigenstate of the mixing matrix Γ, with anomalous dimension γ. For brevity, we will refer to $\mathcal{O}$ from now on simply as *"a single-trace operator"*. $|\mathcal{O}\rangle$ is the corresponding eigenstate of the integrable spin chain Hamiltonian $\mathcal{H}$, whose eigenvalue $\mathcal{E} = \gamma$. The notation $\{\mathcal{O}\}$ stands for sets of single-trace operators, *etc.*

[2] The situation at higher loops is more complicated: Spin chains with nearest-neighbor interaction are replaced with spin chains with long range interactions, the algebraic Bethe ansatz is replaced with an asymptotic Bethe ansatz, and finite-size effects must be accounted for. In this note, we restrict our attention to 1-loop level and nearest-neighbor interacting spin chains.

[3] All spin chains mentioned in this note will be integrable (their R-matrices satisfy Yang-Baxter equations), of XXX type (their R-matrices are parametrized by rational functions in the rapidity variables), and satisfy periodic boundary conditions, hence we need not repeat this from now on.

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1.4 SYM$_4$ structure constants

Following [11–14], Escobedo, Gromov, Sever and Vieira [15] used the connection to spin-$\frac{1}{2}$ chains to obtain a sum expression for the structure constants of 3-point functions of single-trace operators $\{\mathcal{O}\}$ in SU(2) scalar subsectors of SYM$_4$. They noted that the three operators $\mathcal{O}_i$, of lengths L_i, $i \in \{1, 2, 3\}$, could be chosen to be non-BPS (their conformal dimensions are unprotected by supersymmetry) and non-extremal ($L_i < L_j + L_k$, for any choice of distinct i, j and k).

In [16], the sum expression of Escobedo et al. was evaluated in determinant form. This was made possible by the fact that, when expressed in spin chain terms, the essential factor in the sum expression can be identified with (a special case of) the scalar product of an eigenstate of $\mathcal{H}_{\frac{1}{2}}$ and a generic state (not an eigenstate of $\mathcal{H}_{\frac{1}{2}}$).

1.5 QCD structure constants

In this note, we extend the results of [15, 16], from SYM$_4$ and spin-$\frac{1}{2}$ chains to QCD and spin-1 chains, to gain information about QCD operator product expansions, OPE's, of the operators $\{\mathcal{O}\}$ of Ferretti et al..[4]

We show that *1.* In the general case where all three operators $\mathcal{O}_i$, $i \in \{1, 2, 3\}$ are non-BPS-like (all three states map to eigenstates of $\mathcal{H}_1$ that are not spin-chain reference states), the tree-level structure constants can be expressed in a sum form that is similar to, but even less restricted than that of Escobedo et al.[5] *2.* In the special case where one operator, e.g. $\mathcal{O}_3$, is BPS-like (it maps to a spin-chain reference state), the tree-level structure constants can be expressed in a determinant form that is similar to that in [16].

In other words, to express the tree-level structure constants in determinant form, (at least) one of the three operators must be BPS-like. In the following subsection, we outline why this is the case. More details are given in section 4.

1.6 SYM$_4$ structure constants that can be evaluated as determinants

The SYM$_4$ structure constants studied in [15, 16] involve four types of scalars, $\{X, Z, \bar{X}, \bar{Z}\}$. The only non-vanishing Wick contractions (2-point functions) are those between charge-conjugate pairs, that is $\langle X\bar{X}\rangle$, $\langle \bar{X}X\rangle$, $\langle Z\bar{Z}\rangle$, or $\langle \bar{Z}Z\rangle$. Each operator $\mathcal{O}_i$, $i \in \{1, 2, 3\}$, consists of two types of non-conjugate scalars, that is $\{X, Z\}$, $\{X, \bar{Z}\}$, $\{\bar{X}, Z\}$, and $\{\bar{X}, \bar{Z}\}$.

If $\mathcal{O}_1$ is $\{X, Z\}$-type (a composite operator of scalars of type $\{X, Z\}$), and $\mathcal{O}_2$ is $\{\bar{X}, \bar{Z}\}$-type, there are non-zero Wick contractions of both types, $\langle X\bar{X}\rangle$ and $\langle Z\bar{Z}\rangle$, between $\mathcal{O}_1$ and $\mathcal{O}_2$. Now consider $\mathcal{O}_3$. There is no way to choose the scalar content of $\mathcal{O}_3$ such that *1.* It has non-zero Wick contractions of both types with $\mathcal{O}_1$, *2.* It has non-zero Wick contractions of both types with $\mathcal{O}_2$, and *3.* The 3-point function is non-extremal, which

[4] Our results are subject to the same restrictions as in [5], and are valid only in the planar limit ($N_c \to \infty$ and $g \to 0$, with $\lambda = g^2 N_c$ constant) and at one-loop level, so that the beta function vanishes, and the theory is conformally invariant.

[5] The sum form of Escobedo et al. involves a summation over all partitions of one set of rapidity variables. The sum form that we obtain in the general case of three non-BPS-like operators involves summations over all partitions of three sets of rapidity variables with constraints between them.

requires that $\mathcal{O}_3$ has non-zero Wick contractions with both $\mathcal{O}_1$ and $\mathcal{O}_2$. The only way to have a non-extremal 3-point function is to choose $\mathcal{O}_3$ to be $\{\bar{X}, Z\}$-type or $\{X, \bar{Z}\}$-type. Either way, the Wick contractions between $\mathcal{O}_1$ and $\mathcal{O}_3$ will be of one type only, and the Wick contractions between $\mathcal{O}_2$ and $\mathcal{O}_3$ will also be of one type only, different from that between $\mathcal{O}_1$ and $\mathcal{O}_2$. These constraints simplify the structure constant and allow one to evaluate the sum form of Escobedo et al. in determinant form.

1.7 QCD structure constants that can be evaluated as determinants

The QCD structure constants studied in this note involve three types of "scalars", $\{f_+, f_0, f_-\}$. The non-vanishing Wick contractions are those between spin-conjugate pairs, that is $\langle f_+ f_- \rangle$, $\langle f_- f_+ \rangle$, and $\langle f_0 f_0 \rangle$.

Since the action of the Bethe creation operators on the spin-1 reference states generates all three scalars, each operator $\mathcal{O}_i$, $i \in \{1, 2, 3\}$, will consist of all three scalars. Consequently, there are no constraints on the Wick contractions, and the 3-point function of non-BPS operators is more complicated than in the SYM_4 case.[6] This 3-point function between three non-BPS-like operators can be expressed in sum form, as we will explain in the sequel, but that sum form will be more complicated than that in [15], and less useful.

The aim of this note is to identify the structure constants that can be evaluated in single determinant form using currently available methods of integrability.[7] Our result is that, in QCD and the spin-1 case, determinant expressions for the structure constants require that one operator is BPS-like. In other words, that it maps to a spin chain reference state.

1.8 Outline of contents

In section 2, we review the construction of the single-trace composite operators from the self-dual components of the field strength tensor, the 1-loop mixing matrix, operator product expansions, and the *"tailoring"* approach of Escobedo et al. to the structure constants. In section 3, we recall the algebraic Bethe ansatz solution for the eigenstates and eigenvalues of the mixing matrix. In section 4, we present our results for the structure constants in terms of solutions of the Bethe equations. Section 5 contains a brief discussion. In appendix A, we recall the coordinate Bethe ansatz and the $\mathcal{F}$-conjugation of [15]. In appendix B, we present the scalar products that appear in the expression for the structure constants.

[6] In particular, while integrable spin-1 chains are related to integrable spin-$\frac{1}{2}$ chains by fusion, there is no way that one can use fusion to obtain a 3-point function of non-BPS-like operators in the spin-1 case from the corresponding spin-$\frac{1}{2}$ result.

[7] What we have in mind is Slavnov's determinant expression for the scalar product of an eigenstate of the Hamiltonian and a generic state. This determinant expression is unique. It is conceivable that determinant expressions for more general scalar products, that will allow us to evaluate more general structure constants, will eventually be found, but this is obviously beyond the scope of this work.

2 Composite operators, operator product expansions and structure constants

2.1 Self-dual field-strength components as an SU(2)-triplet

Following [5], we decompose the QCD Yang-Mills field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu\,, A_\nu]$ into self-dual, $f_{\alpha\beta}$, and anti-self-dual, $\bar{f}_{\dot\alpha\dot\beta}$, components,

$$F_{\mu\nu} = \sigma_{\mu\nu}{}^{\alpha\beta} f_{\alpha\beta} + \bar{\sigma}_{\mu\nu}{}^{\dot\alpha\dot\beta} \bar{f}_{\dot\alpha\dot\beta}\,, \tag{2.1}$$

where

$$\sigma_{\mu\nu} = \frac{i}{4}\sigma_2\left(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu\right),\ \ \bar\sigma_{\mu\nu} = -\frac{i}{4}\left(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu\right)\sigma_2,\ \ \sigma_\mu = (1,\vec\sigma),\ \ \bar\sigma_\mu = (1,-\vec\sigma)\,. \tag{2.2}$$

We further define

$$f_A = \left(\sigma_2\sigma_A\right)^{\alpha\beta} f_{\alpha\beta}\,, \qquad \bar{f}_{\dot A} = \left(\sigma_{\dot A}\sigma_2\right)^{\dot\alpha\dot\beta} \bar{f}_{\dot\alpha\dot\beta}\,, \tag{2.3}$$

where $A, \dot A = 1, 2, 3$. The 2-point function of the field strength tensor has the structure

$$\langle F_{\mu\nu}{}^a{}_b(x) F_{\rho\sigma}{}^c{}_d(0)\rangle = \phi(x)\left(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\right)\delta^a_d\delta^c_b\,, \tag{2.4}$$

where $a, b, c, d = 1, \ldots, N_c$ are color indices, and $\phi(x)$ is a scalar function. Hence,

$$\langle f_A{}^a{}_b(x) f_B{}^c{}_d(0)\rangle = \phi(x)\delta_{AB}\delta^a_d\delta^c_b\,, \qquad \langle f_A{}^a{}_b(x) \bar{f}_{\dot B}{}^c{}_d(0)\rangle = 0\,. \tag{2.5}$$

Following [17], we write

$$f_+ = f_{11} = \frac{1}{2}\left(f_2 + i f_1\right),\ \ f_0 = \frac{1}{\sqrt2}\left(f_{12} + f_{21}\right) = -\frac{i}{\sqrt2} f_3,\ \ f_- = f_{22} = \frac{1}{2}\left(f_2 - i f_1\right). \tag{2.6}$$

From equation (2.5), $\langle f_\pm(x) f_\pm(0)\rangle = \langle f_\pm(x) f_0(0)\rangle = 0$, and the only nonzero Wick contractions (2-point functions) are $\langle f_\pm f_\mp\rangle$ and $\langle f_0 f_0\rangle$. $\{f_+, f_0, f_-\}$ is an SU(2) triplet, and transforms in the spin-1 representation of SU(2).

2.2 Single-trace operators from the self-dual components

We focus on the single-trace operators of length L that are composed of self-dual components only

$$\mathcal{O}(x) = \mathrm{tr}\left(f_{A_1}(x)\cdots f_{A_L}(x)\right)\,. \tag{2.7}$$

Following [5], at 1-loop level, in the planar limit, these operators mix only among themselves, as in equation (2.5), and their mixing matrix is given by

$$\Gamma = \frac{\lambda}{48\pi^2}\sum_{l=1}^{L}\left(7 + 3\vec{S}_l\cdot\vec{S}_{l+1} - 3(\vec{S}_l\cdot\vec{S}_{l+1})^2\right)\,, \tag{2.8}$$

where $\lambda = g^2 N_c$, and $\vec{S}_l$ are SU(2) spin-1 generators,

$$\left(S^j f\right)_A = -i\epsilon_{jAB} f_B\,. \tag{2.9}$$

Note that $\{f_+, f_0, f_-\}$ are eigenstates of S^3 with eigenvalues $\{+1, 0, -1\}$, respectively. Since Γ commutes with $\vec{S}^2$ (where $\vec{S} = \sum_{l=1}^{L} \vec{S}_l$ is the total spin), and S^3, all three operators can be diagonalized simultaneously. An eigenstate of Γ is an operator of definite conformal dimension $\Delta = 2L + \gamma$, where γ is the corresponding eigenvalue.

2.3 Operator product expansion of single-trace operators

Following [15, 16], we normalize the operators of definite conformal dimension according to

$$\langle \mathcal{O}_i(x_i)\, \bar{\mathcal{O}}_j(x_j)\rangle \sim (\mathcal{N}_i \mathcal{N}_j)^{\frac{1}{2}} \frac{\delta_{ij}}{|x_{ij}|^{\Delta_i+\Delta_j}} \tag{2.10}$$

for $x_{ij} \equiv x_i - x_j \to 0$, where $\mathcal{N}_i$ will be specified below in equation (B.10). The OPE of a pair of these operators $\mathcal{O}_1(x)$ and $\mathcal{O}_3(x)$ is given by

$$\mathcal{O}_1(x_1)\, \mathcal{O}_3(x_3) \sim \sum_{\mathcal{O}_2} \left(\frac{\mathcal{N}_1 \mathcal{N}_3}{\mathcal{N}_2}\right)^{\frac{1}{2}} \frac{C_{132}}{|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}\, \mathcal{O}_2(x) + \ldots\,, \qquad x = \frac{1}{2}(x_1 + x_3)\,, \tag{2.11}$$

for $x_{13} \to 0$, where the ellipsis denotes subleading corrections involving conformal descendants of $\mathcal{O}_2$ [18]. The structure constants C_{132} have a perturbative expansion in λ,

$$N_c\, C_{132} = c^{(0)}_{132} + \lambda c^{(1)}_{132} + \ldots\,, \tag{2.12}$$

In this note, we focus on the leading (tree-level) contribution $c^{(0)}_{132}$.

2.4 *"Tailoring"* the structure constants

Following [15], we construct $c^{(0)}_{132}$ in four steps.

Step 1. We map the length-L_i single-trace operator $\mathcal{O}_i$ to an eigenstate $|\mathcal{O}_i\rangle$ of a length-L_i periodic spin-1 chain Hamiltonian $\mathcal{H}_1$.

Step 2. We *"split"* the spin chains into left and right subchains of lengths[8]

$$L_{i,l} = \frac{1}{2}\left(L_i + L_j - L_k\right)\,, \quad L_{i,r} = \frac{1}{2}\left(L_i + L_k - L_j\right)\,, \tag{2.13}$$

respectively, with (i, j, k) in cyclic order. We perform a corresponding split of the states,

$$|\mathcal{O}_i\rangle = \sum_a |\mathcal{O}_{i_a}\rangle_l \otimes |\mathcal{O}_{i_a}\rangle_r\,, \tag{2.14}$$

where, roughly speaking, the sum is over all possible ways of distributing the component fields into the left and right subchains. (A more precise definition of this splitting, as well as a more accurate version of equation (2.14), will be given below after introducing the Bethe ansatz.) Note that $|\mathcal{O}_{i_a}\rangle_l$ and $|\mathcal{O}_{i_a}\rangle_r$ are states of subchains with lengths $L_{i,l}$ and $L_{i,r}$, respectively.

[8]We restrict the discussion to the *"non-extremal"* case where all $L_{i,l}, L_{i,r} > 0$, for which there is no mixing with double-trace operators [15].

Step 3. We *"flip"* or $\mathcal{F}$-conjugate the right kets into right bras

$$|\mathcal{O}_i\rangle = \sum_a |\mathcal{O}_{i_a}\rangle_l \otimes |\mathcal{O}_{i_a}\rangle_r \to \sum_a |\mathcal{O}_{i_a}\rangle_l \otimes {}_r\langle\mathcal{O}_{i_a}| \,. \tag{2.15}$$

Given a pair of elementary fields A and B that are associated with the kets $|\Psi_i\rangle_r$ and $|\Psi_{i+1}\rangle_l$, respectively, the flipped state ${}_r\langle\Psi_i|$ is defined such that

$$\langle AB\rangle \sim {}_r\langle\Psi_i|\Psi_{i+1}\rangle_l \,. \tag{2.16}$$

In view of the fact that the only non-zero 2-point functions are between f_+ and f_-, and between two f_0 fields, the prescription (2.16) implies that

$$|f_\pm\rangle_r \to {}_r\langle f_\mp|, \qquad |f_0\rangle_r \to {}_r\langle f_0| \,. \tag{2.17}$$

Our convention is that $\langle f_\pm|f_\pm\rangle = 1$, $\langle f_0|f_0\rangle = 1$, while all other 2-point functions are zero.

Step 4. We construct the structure constants by taking scalar products of bra and ket states [15, 16], to obtain

$$c^{(0)}_{132} = \mathcal{N}_{132} \sum_{a,b,c} {}_r\langle\mathcal{O}_{2_b}|\mathcal{O}_{1_a}\rangle_l \; {}_r\langle\mathcal{O}_{1_a}|\mathcal{O}_{3_c}\rangle_l \; {}_r\langle\mathcal{O}_{3_c}|\mathcal{O}_{2_b}\rangle_l \,, \tag{2.18}$$

where

$$\mathcal{N}_{132} = \left(\frac{L_1 L_2 L_3}{\langle\mathcal{O}_1|\mathcal{O}_1\rangle\langle\mathcal{O}_2|\mathcal{O}_2\rangle\langle\mathcal{O}_3|\mathcal{O}_3\rangle}\right)^{\frac{1}{2}} \,. \tag{2.19}$$

This is represented graphically in figure 1. In order to further evaluate the expression (2.18) for the structure constants, it is necessary to have a more explicit construction of the states with definite conformal dimensions. To this end, we now turn to the Bethe ansatz.

3 Algebraic Bethe ansatz

3.1 Diagonalizing the Hamiltonian $\mathcal{H}_1$

The 1-loop QCD mixing matrix Γ (2.8) is identical to the Hamiltonian $\mathcal{H}_1$ of an anti-ferromagnetic spin-1 chain, with periodic boundary conditions, that is integrable [6], and therefore can be diagonalized using the algebraic Bethe ansatz [7–10]. The basic strategy to diagonalize $\mathcal{H}_1$ is to diagonalize a transfer matrix $t^{(\frac{1}{2})}(u)$ that is constructed from a monodromy matrix with a 2-dimensional (that is, spin-$\frac{1}{2}$) auxiliary space. Although $t^{(\frac{1}{2})}(u)$ does not generate $\mathcal{H}_1$ (2.8), it is related by the fusion procedure to another transfer matrix $t^{(1)}(u)$ that is constructed from a monodromy matrix with a 3-dimensional (that is, spin-1) auxiliary space and that contains $\mathcal{H}_1$ [8–10]. By diagonalizing $t^{(\frac{1}{2})}(u)$, we diagonalize $t^{(1)}(u)$, $\mathcal{H}_1$ and Γ, all in one go.

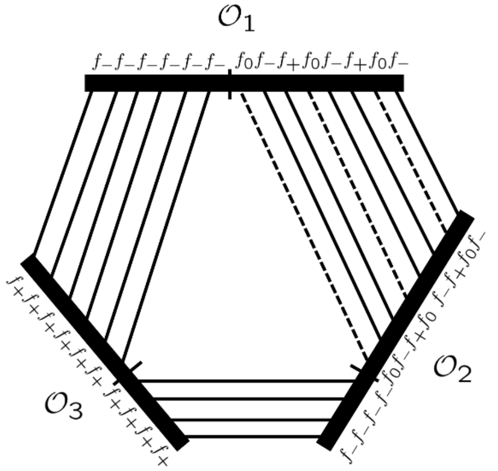

Figure 1. A configuration of 3-point functions with contractions among the self-dual Yang-Mills fields (a solid line is $\langle f_+ f_- \rangle$ and a dotted line is $\langle f_0 f_0 \rangle$). $\mathcal{O}_3$ is chosen to consist of f_+ fields only, so it maps to a spin-chain reference state. This will be the case that can be evaluated in determinant form.

3.2 The *R*- and the monodromy matrices

The transfer matrix $t^{(\frac{1}{2})}(u)$ can be constructed using the 6×6 R-matrix

$$R^{(\frac{1}{2},1)}(u,v) = \frac{1}{(u-v-\eta)} \left(\begin{array}{ccc|ccc} u-v+\eta & & & & & \\ & u-v & & \sqrt{2}\eta & & \\ & & u-v-\eta & & \sqrt{2}\eta & \\ \hline & \sqrt{2}\eta & & u-v-\eta & & \\ & & \sqrt{2}\eta & & u-v & \\ & & & & & u-v+\eta \end{array} \right), \tag{3.1}$$

where eventually we shall set $\eta = i$. The matrix elements that are zero are left empty. We regard $R^{(\frac{1}{2},1)}(u,v)$ as an operator acting on $\mathcal{C}^2 \otimes \mathcal{C}^3$. This R-matrix can be obtained by fusion [7, 8] from $R^{(\frac{1}{2},\frac{1}{2})}(u,v) = u - v + \eta\mathcal{P}$, where $\mathcal{P}$ is the permutation matrix on $\mathcal{C}^2 \otimes \mathcal{C}^2$, together with a *"gauge"* transformation that makes the matrix symmetric.

The (inhomogeneous) monodromy matrix is constructed from the R-matrix as[9]

$$T_0^{(\frac{1}{2})}[u;\{z\}_L] = R_{01}^{(\frac{1}{2},1)}(u,z_1) \dots R_{0L}^{(\frac{1}{2},1)}(u,z_L)\,, \tag{3.2}$$

where we have introduced the inhomogeneities $\{z\}_L = \{z_1,\dots,z_L\}$ for later convenience. The auxiliary space (labeled 0) is 2-dimensional, while each of the quantum spaces (labeled $1,\dots,L$) are 3-dimensional. By tracing over the auxiliary space, we arrive at the

[9]In the sequel, we use different brackets to indicate the type of enclosed arguments. We write $f(x,y)$ when neither x nor y is a set of variables, $f\{x,y\}$ when both x and y are sets of variables, and $f[x,\{y\}]$ when x is not a set of variables, but y is.

(inhomogeneous) transfer matrix

$$t^{(\frac{1}{2})}[u;\{z\}_L] = \mathrm{tr}_0\, T_0^{(\frac{1}{2})}[u;\{z\}_L]\,. \tag{3.3}$$

It has the commutativity property

$$\left[t^{(\frac{1}{2})}[u;\{z\}_L]\,, t^{(\frac{1}{2})}[v;\{z\}_L]\right] = 0\,, \tag{3.4}$$

by virtue of the fact that the R-matrix obeys the Yang-Baxter equation.

3.3 Constructing the eigenstates

The eigenstates of this transfer matrix can be readily obtained by algebraic Bethe ansatz: we define the operators A, B, C, D by

$$T_0^{(\frac{1}{2})}[u;\{z\}_L] = \begin{pmatrix} A[u;\{z\}_L] & B[u;\{z\}_L] \\ C[u;\{z\}_L] & D[u;\{z\}_L] \end{pmatrix}\,. \tag{3.5}$$

We also introduce the reference states with all spins up or all spins down,

$$|0\rangle_\pm = |f_\pm\rangle^{\otimes L} \equiv |f_\pm^L\rangle\,. \tag{3.6}$$

These states are eigenstate of both $A[u;\{z\}_L]$ and $D[u;\{z\}_L]$,

$$\begin{aligned} A[u;\{z\}_L]|0\rangle_+ &= \left(\prod_{l=1}^{L}\frac{u-z_l+\eta}{u-z_l-\eta}\right)|0\rangle_+\,, \qquad & D[u;\{z\}_L]|0\rangle_+ &= |0\rangle_+\,, \\ A[u;\{z\}_L]|0\rangle_- &= |0\rangle_-\,, & D[u;\{z\}_L]|0\rangle_- &= \left(\prod_{l=1}^{L}\frac{u-z_l+\eta}{u-z_l-\eta}\right)|0\rangle_-\,. \end{aligned} \tag{3.7}$$

We note that

$$B[u;\{z\}_L]^\dagger = -\left(\prod_{l=1}^{L}\frac{u^*-z_l^*-\eta}{u^*-z_l^*+\eta}\right)C[u^*;\{z^*\}_L]\,, \tag{3.8}$$

where we have used $\eta = i$, and $*$ denotes complex conjugation. Choosing $|0\rangle_+$ as the reference state, one finds that the states

$$|\{u\}_N\rangle_+ = \left(\prod_{j=1}^{N} B[u_j;\{z\}_L]\right)|0\rangle_+ \tag{3.9}$$

are eigenstates of the transfer matrix $t^{(\frac{1}{2})}[u;\{z\}_L]$ provided that $\{u\}_N = \{u_1,\ldots,u_N\}$ are distinct and satisfy the spin-1 Bethe equations

$$\prod_{l=1}^{L}\frac{u_j-z_l+\eta}{u_j-z_l-\eta} = \prod_{\substack{k=1\\k\neq j}}^{N}\frac{u_j-u_k+\eta}{u_j-u_k-\eta}\,. \tag{3.10}$$

In the homogeneous limit $z_l = 0$, these states are eigenstates of $\mathcal{H}_1$ (2.8) with eigenvalues (anomalous dimensions) [5]

$$\gamma = \frac{\lambda}{48\pi^2}\left(7L - \sum_{k=1}^{N}\frac{12}{u_k^2+1}\right). \tag{3.11}$$

The conformal dimensions are therefore given by $\Delta = 2L + \gamma$. The Bethe eigenstates (3.9) are SU(2) highest-weight states, with spin

$$s = s^3 = L - N\,, \tag{3.12}$$

and therefore $N \leq L$. If we choose $|0\rangle_-$ as the reference state, then the Bethe states are given by

$$|\{u\}_N\rangle_- = \left(\prod_{j=1}^{N} C[u_j;\{z\}_L]\right)|0\rangle_-\,, \tag{3.13}$$

which are lowest-weight states, with $s = -s^3 = L - N$, so again $N \leq L$.

In order to properly define the splitting of states (2.14), we follow [15] and split the monodromy matrix (3.2),

$$T_0^{(\frac{1}{2})}[u;\{z\}_L] = T_{0,l}^{(\frac{1}{2})}[u;\{z\}_{L_l}]\, T_{0,r}^{(\frac{1}{2})}[u;\{z\}_{L_r}]\,, \tag{3.14}$$

where

$$\begin{aligned} T_{0,l}^{(\frac{1}{2})}[u;\{z\}_{L_l}] &= R_{01}^{(\frac{1}{2},1)}(u,z_1)\ldots R_{0L_l}^{(\frac{1}{2},1)}(u,z_{L_l})\,,\\ T_{0,r}^{(\frac{1}{2})}[u;\{z\}_{L_r}] &= R_{0,L_l+1}^{(\frac{1}{2},1)}(u,z_{L_l+1})\ldots R_{0L}^{(\frac{1}{2},1)}(u,z_L)\,, \end{aligned} \tag{3.15}$$

and $\{z\}_{L_l} = \{z_1,\ldots,z_{L_l}\}$, $\{z\}_{L_r} = \{z_{L_l+1},\ldots,z_L\}$. Correspondingly,

$$\begin{pmatrix} A[u;\{z\}_L] & B[u;\{z\}_L]\\ C[u;\{z\}_L] & D[u;\{z\}_L]\end{pmatrix} = \begin{pmatrix} A_l[u;\{z\}_{L_l}] & B_l[u;\{z\}_{L_l}]\\ C_l[u;\{z\}_{L_l}] & D_l[u;\{z\}_{L_l}]\end{pmatrix}\begin{pmatrix} A_r[u;\{z\}_{L_r}] & B_r[u;\{z\}_{L_r}]\\ C_r[u;\{z\}_{L_r}] & D_r[u;\{z\}_{L_r}]\end{pmatrix}. \tag{3.16}$$

In particular,

$$\begin{aligned} B[u;\{z\}_L] &= A_l[u;\{z\}_{L_l}]\, B_r[u;\{z\}_{L_r}] + B_l[u;\{z\}_{L_l}]\, D_r[u;\{z\}_{L_r}]\,,\\ C[u;\{z\}_L] &= C_l[u;\{z\}_{L_l}]\, A_r[u;\{z\}_{L_r}] + D_l[u;\{z\}_{L_l}]\, C_r[u;\{z\}_{L_r}]\,. \end{aligned} \tag{3.17}$$

The $\mathcal{F}$-conjugation (A.13) implies that

$$\begin{aligned} B_r[u;\{z\}_{L_r}]|f_+^{L_r}\rangle_r &\to {}_r\langle f_-^{L_r}|B_r[u;\{z\}_{L_r}]\,,\\ C_r[u;\{z\}_{L_r}]|f_-^{L_r}\rangle_r &\to {}_r\langle f_+^{L_r}|C_r[u;\{z\}_{L_r}]\,. \end{aligned} \tag{3.18}$$

4 Evaluating the structure constants

4.1 3-point functions with three non-BPS-like operators in sum form

We start with the general case where all three composite operators $\mathcal{O}_i$, $i \in \{1,2,3\}$ are non-BPS-like (they are not of highest or lowest conformal dimension), so they map to Bethe eigenstates $|\mathcal{O}_i\rangle$ that are not spin-chain reference states, and that can be split into left and right parts as

$$
\begin{aligned}
|\mathcal{O}_i\rangle &= \prod_{j=1}^{N_i} \left(A_l[u_{i,j}, \{z_{L_{i,l}}\}]B_r[u_{i,j}, \{z_{L_{i,r}}\}] + D_r[u_{i,j}, \{z_{L_{i,r}}\}]B_l[u_{i,j}, \{z_{L_{i,l}}\}]\right) |f_+^{L_{i,l}}\rangle_l \otimes |f_+^{L_{i,r}}\rangle_r \\
&= \sum_{\alpha_i \cup \bar{\alpha}_i = \{u_i\}_{N_i}} H_i\{\alpha_i, \bar{\alpha}_i\} |\mathcal{O}_{i,\alpha_i}\rangle_l \otimes |\mathcal{O}_{i,\bar{\alpha}_i}\rangle_r
\end{aligned} \tag{4.1}
$$

where

$$
|\mathcal{O}_{i,\alpha_i}\rangle_l = \left(\prod_{j\in\alpha_i} B_l[u_{i,j}, \{z\}_{L_{i,l}}]\right) |f_+^{L_{i,l}}\rangle_l, \quad |\mathcal{O}_{i,\bar{\alpha}_i}\rangle_r = \left(\prod_{j\in\bar{\alpha}_i} B_r[u_{i,j}, \{z\}_{L_{i,r}}]\right) |f_+^{L_{i,r}}\rangle_r, \tag{4.2}
$$

the coefficients $H_i\{\alpha_i, \bar{\alpha}_i\}$ are computed from equation (3.7) to be

$$
H_i\{\alpha_i, \bar{\alpha}_i\} = \prod_{u_{i,j}\in\alpha_i} \prod_{z_k \in\{z\}_{L_{i,l}}} \frac{u_{i,j} - z_k + \eta}{u_{i,j} - z_k - \eta}, \tag{4.3}
$$

and $\{u_i\}_{N_i}$ satisfy the Bethe equations (3.10) with $L = L_i$. Under $\mathcal{F}$-conjugation, this operator becomes

$$
|\mathcal{O}_i\rangle \to \sum_{\alpha_i \cup \bar{\alpha}_i = \{u_i\}_{N_i}} H_i\{\alpha_i, \bar{\alpha}_i\} |\mathcal{O}_{i,\alpha_i}\rangle_l \otimes {}_r\langle \mathcal{O}_{i,\bar{\alpha}_i}|\,, \tag{4.4}
$$

where

$$
{}_r\langle \mathcal{O}_{i,\bar{\alpha}_i}| = {}_r\langle f_-^{L_{i,r}}| \left(\prod_{j\in\bar{\alpha}_i} B_r[u_{i,j}, \{z\}_{L_{i,r}}]\right). \tag{4.5}
$$

Substituting the above expressions into equation (2.18), we obtain the following sum expression for the structure constant of the 3-point function with three non-BPS-like operators

$$
c_{132} = \lim_{z_l\to 0} \mathcal{N}_{132} \sum_{\alpha_i \cup \bar{\alpha}_i = \{u_i\}_{N_i}} \left(\prod_{i=1}^{3} H_i\{\alpha_i, \bar{\alpha}_i\}\right) {}_r\langle \mathcal{O}_{2,\bar{\alpha}_2} | \mathcal{O}_{1,\alpha_1}\rangle_l \; {}_r\langle \mathcal{O}_{1,\bar{\alpha}_1} | \mathcal{O}_{3,\alpha_3}\rangle_l \; {}_r\langle \mathcal{O}_{3,\bar{\alpha}_3} | \mathcal{O}_{2,\alpha_2}\rangle_l\,, \tag{4.6}
$$

where each of the three factors of type ${}_r\langle \mathcal{O}_{i+1,\bar{\alpha}_{i+1}} | \mathcal{O}_{i,\alpha_i}\rangle_l$ in the summand is a generic scalar product as in equation (B.3), subject to the conditions in equations (B.5), (B.6).

4.2 The structure of the sum form in equation (4.6)

The non-BPS-like operator $\mathcal{O}_i$, $i \in \{1,2,3\}$, is composed of the operators $\{f_+, f_0, f_-\}$ with multiplicities $\{n_{i,+}, n_{i,0}, n_{i,-}\}$, such that $n_{i,+}+n_{i,0}+n_{i,-}=L_i$ and $2n_{i,-}+n_{i,0} = N_i$. Splitting $\mathcal{O}_i$ into a left-part $\mathcal{O}_{i,l}$ of length $L_{i,l}$, and a right-part $\mathcal{O}_{i,r}$ of length $L_{i,r}$, $L_{i,l} + L_{i,r} = L_i$, the operators $\{f_+, f_0, f_-\}$ can be on either part, such that

$$n^l_{i,+} + n^r_{i,+} = n_{i,+}, \qquad n^l_{i,0} + n^r_{i,0} = n_{i,0}, \qquad n^l_{i,-} + n^r_{i,-} = n_{i,-}, \tag{4.7}$$

where $n^l_{i,+}$ is the number of f_+-operators on the left-part of $|\mathcal{O}_i\rangle$, *etc.*

Let us consider one type of these operators, for example f_+, to be a reference state operator, in the sense that if all elementary operators in a single-trace operator $\mathcal{O}$ are of type f_+, then $\mathcal{O}$ maps to a spin-chain reference state. In that case, the other two operators, f_0 and f_-, become *"excitations"*.[10] Since the total number of elementary operators in $\mathcal{O}_i$, $i \in \{1,2,3\}$ is fixed, one can think of single-trace operators that are *not eigenstates* of the mixing matrix Γ, but whose weighted sum is a single-trace $\mathcal{O}_i$ that is an eigenstate, as labeled by the positions of the excitations in the trace.

The crucial point is that, while the lengths of the left- and right-parts are fixed once and for all,[11] the distribution of the excitations on the left and the right parts of $\mathcal{O}_i$ is not fixed. This means that the sum in equation (4.6) is over all possible distributions of excitations in $\mathcal{O}_i$, $i \in \{1,2,3\}$ over its left and right parts, subject to the conditions

$$n_{i,+} = n_{i+1,-}, \qquad n_{i,-} = n_{i+1,+}, \qquad n_{i,0} = n_{i+1,0}, \qquad i+3 \equiv i\,. \tag{4.8}$$

In spin-chain terms, the action of the Bethe operators on a reference state, that consists of one type of operators, generates excitations of both types. Thus every state $\mathcal{O}_i$, $i \in \{1,2,3\}$ that is not BPS-like will consist of all three types $\{f_+, f_0, f_-\}$, and we need to sum over all possible positions of $\{f_+, f_0, f_-\}$ in $\mathcal{O}_i$. The result is that *1.* The sum over partitions in equation (4.6) is computationally non-trivial, particularly when the number of Bethe roots involved is not small; and as mentioned above, *2.* Each of the three factors of type ${}_r\langle \mathcal{O}_{i+1,\bar{\alpha}_{i+1}} | \mathcal{O}_{i,\alpha_i}\rangle_l$ in the summand is a generic scalar product as in equation (B.3), subject to the conditions in equation (B.5). This is a complicated expression.

To reduce the complexity of the sum form in equation (4.6) and obtain a computationally tractable expression, which in our case is a determinant, we choose one of the operators to be BPS-like so that it maps to a spin-chain reference state. We will choose $\mathcal{O}_3$ to be BPS-like.

4.3 3-point functions with one BPS-like state in determinant form

Choosing $\mathcal{O}_3$ to consist of f_+-operators only, the corresponding state is

$$|\mathcal{O}_3\rangle = |f_+^{L_3}\rangle = |f_+^{L_{3,l}}\rangle_l \otimes |f_+^{L_{3,r}}\rangle_r \to |f_+^{L_{3,l}}\rangle_l \otimes {}_r\langle f_-^{L_{3,r}}|\,, \tag{4.9}$$

[10] Either f_+ or f_- can be chosen as a reference state operator, as we will see in the sequel.

[11] This follows from the fact that the lengths L_i, $i \in \{1,2,3\}$ are fixed as initial conditions, and the lengths of the left and right parts are fixed from equation (2.13).

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where $L_{3,l}$ and $L_{3,r}$ are given by (2.13). Evidently, since there is only one way to split this state, no summation is necessary. We write the algebraic Bethe state for operator $\mathcal{O}_1$ as in (4.1) with $i = 1$, and we define the corresponding parameters $u_j \equiv u_{1,j}$, which satisfy the Bethe equations (3.10) with $L = L_1$. Under $\mathcal{F}$-conjugation, this state becomes (4.4) with $i = 1$.

However, we write the state corresponding to the operator $\mathcal{O}_2$ instead as

$$|\mathcal{O}_2\rangle \to \sum_{\beta\cup\bar\beta=\{v\}_{N_2}} H_2\{\beta,\bar\beta\}|\mathcal{O}_{2,\beta}\rangle_l \otimes {}_r\langle\mathcal{O}_{2,\bar\beta}|\,, \tag{4.10}$$

with

$$|\mathcal{O}_{2,\beta}\rangle_l = \left(\prod_{j\in\beta} C_l[v_j;\{z\}_{L_{2,l}}]\right)|f_-^{L_{2,l}}\rangle_l\,, \qquad {}_r\langle\mathcal{O}_{2,\bar\beta}| = {}_r\langle f_+^{L_{2,r}}|\left(\prod_{j\in\bar\beta} C_r[v_j;\{z\}_{L_{2,r}}]\right)\,, \tag{4.11}$$

where $\{v\}_{N_2}$ satisfy the Bethe equations (3.10) with $L = L_2$. Having chosen to construct the Bethe states for $\mathcal{O}_1$ with the reference state $|0\rangle_+$, it is necessary to construct the Bethe states for $\mathcal{O}_2$ with the reference state $|0\rangle_-$. We now insert these results into equation (2.18) to get

$$\begin{aligned} c_{132} &= \lim_{z_l\to 0}\mathcal{N}_{132}\sum_{\substack{\beta\cup\bar\beta=\{v\}_{N_2}\\ \alpha\cup\bar\alpha=\{u\}_{N_1}}} H_1\{\alpha,\bar\alpha\}\,H_2\{\beta,\bar\beta\}\;{}_r\langle\mathcal{O}_{2,\bar\beta}|\mathcal{O}_{1,\alpha}\rangle_l\;{}_r\langle\mathcal{O}_{1,\bar\alpha}|f_+^{L_{3,l}}\rangle_l\;{}_r\langle f_-^{L_{3,r}}|\mathcal{O}_{2,\beta}\rangle_l \\ &= \lim_{z_l\to 0}\mathcal{N}_{132}\,H_2\{\varnothing,\{v\}_{N_2}\}\sum_{\alpha\cup\bar\alpha=\{u\}_{N_1}} H_1\{\alpha,\bar\alpha\}\;{}_r\langle\mathcal{O}_2|\mathcal{O}_{1,\alpha}\rangle_l\;{}_r\langle\mathcal{O}_{1,\bar\alpha}|f_+^{L_{3,l}}\rangle_l\;{}_r\langle f_-^{L_{3,r}}|f_-^{L_{2,l}}\rangle_l\,. \end{aligned} \tag{4.12}$$

In passing to the second line, we have made use of the fact that the expression vanishes unless the set β contains no Bethe roots, and we defined

$${}_r\langle\mathcal{O}_2| \equiv \langle f_+^{L_{2,r}}|\left(\prod_{j=1}^{N_2} C_r[v_j;\{z\}_{L_{2,r}}]\right)\,. \tag{4.13}$$

With the help of equation (3.7), we see that

$$H_2\{\varnothing,\{v\}_{N_2}\} = \prod_{j=1}^{N_2}\prod_{l=1}^{L_{2,l}}\frac{v_j - z_l + \eta}{v_j - z_l - \eta} \tag{4.14}$$

becomes equal to 1 in the homogeneous limit, $z_l = 0$, by virtue of the zero-momentum constraint

$$\prod_{j=1}^{N_2}\frac{v_j+\eta}{v_j-\eta} = 1\,, \tag{4.15}$$

which arises from the cyclicity of the trace in $\mathcal{O}_2$.[12] The remaining sum over partitions in equation (4.12) can be performed by using ${}_r\langle\mathcal{O}_{1,\bar\alpha}|f_+^{L_{3,l}}\rangle_l = {}_l\langle f_-^{L_{3,l}}|\mathcal{O}_{1,\bar\alpha}\rangle_r$. Noting also that ${}_r\langle f_-^{L_{3,r}}|f_-^{L_{2,l}}\rangle_l = 1$, we obtain

$$\begin{aligned} c_{132} &= \lim_{z_l\to 0}\mathcal{N}_{132}\sum_{\alpha\cup\bar\alpha=\{u\}_{N_1}} H_1\{\alpha,\bar\alpha\}\ {}_r\langle\mathcal{O}_2|\mathcal{O}_{1,\alpha}\rangle_l\ {}_l\langle f_-^{L_{3,l}}|\mathcal{O}_{1,\bar\alpha}\rangle_r \\ &= \lim_{z_l\to 0}\mathcal{N}_{132}\ {}_l\langle f_-^{L_{3,l}}|\otimes\ {}_r\langle\mathcal{O}_2|\mathcal{O}_1\rangle. \end{aligned} \tag{4.16}$$

We observe that this expression vanishes unless

$$L_2 - N_2 = L_1 + L_3 - N_1 \geq 0\,. \tag{4.17}$$

Indeed, the factor ${}_r\langle\mathcal{O}_2|\mathcal{O}_{1,\alpha}\rangle_l$ in the first line of equation (4.16) vanishes unless $|\alpha|$ (the number of Bethe roots in α) is given by $|\alpha| = N_2$. It follows that $|\bar\alpha| = N_1 - N_2$. Moreover, the two states in the factor ${}_l\langle f_-^{L_{3,l}}|\mathcal{O}_{1,\bar\alpha}\rangle_r$ should have the same S^3 eigenvalue; hence,

$$L_{1,r} - |\bar\alpha| = -L_{3,l}\,, \tag{4.18}$$

which then implies equation (4.17). The sum over $\mathcal{O}_2$ in equation (2.11) can therefore be understood as the sum over all L_2 and N_2 satisfying the constraint (4.17). The scalar product in the second line of equation (4.16) is a restricted Slavnov scalar product

$$c_{132}^{(0)} = \mathcal{N}_{132}^{\rm hom}\, S^{\rm hom}(\{u\}_{N_1},\{v\}_{N_2})\,, \tag{4.19}$$

where $\{u\}_{N_1},\{v\}_{N_2}$ are the Bethe roots corresponding to operators $\mathcal{O}_1,\mathcal{O}_2$, respectively. In appendix B we obtain an expression (B.25) for the restricted Slavnov scalar product, which in the homogeneous limit $z_l\to 0$ becomes

$$\begin{aligned} &S^{\rm hom}(\{u\}_{N_1},\{v\}_{N_2}) = \\ &\prod_{k=1}^{N_2}\left(\frac{v_k+\eta}{v_k-\eta}\right)^{\left(\frac{2L_1-N_1+N_2}{2}\right)}\prod_{j>k}^{N_1}\frac{1}{u_j-u_k}\prod_{j>k}^{N_2}\frac{1}{v_j-v_k}\prod_{k=1}^{N_2}\frac{1}{((v_k-\eta)v_k)^{(N_1-N_2)/2}} \\ &\qquad\times\det\left(\begin{array}{c} \mathcal{M}_{ij} \qquad\qquad 1\leq i\leq N_2\,,\quad 1\leq j\leq N_1 \\ \hline \Psi^{(i-1)}(u_j,0)\quad 1\leq i\leq (N_1-N_2)/2\,,\quad 1\leq j\leq N_1 \\ \hline \Psi^{(i-1)}(u_j+\eta,0)\ 1\leq i\leq (N_1-N_2)/2\,,\quad 1\leq j\leq N_1 \end{array}\right)\,, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} \mathcal{M}_{ij} &= \frac{\eta}{(u_j-v_i)}\left(\prod_{\substack{m=1\\ m\neq j}}^{N_1}(v_i-u_m-\eta) - \left(\frac{v_i-\eta}{v_i+\eta}\right)^{L_1}\prod_{\substack{m=1\\ m\neq j}}^{N_1}(v_i-u_m+\eta)\right)\,, \\ \Psi(u,z) &= -\frac{1}{(u-z)(u-z-\eta)}\prod_{j=1}^{N_1}(z-u_j)\,,\quad \Psi^{(j)}(u,z) = \frac{1}{j!}\frac{\partial^j}{\partial z^j}\Psi(u,z)\,. \end{aligned} \tag{4.21}$$

[12]Note that this argument can be used only when all Bethe roots of an original unsplit eigenstate belong to the same part after splitting. This is the case for the eigenstate $|\mathcal{O}_2\rangle$ in the 3-point function with one BPS-like state. In particular, the same argument cannot be used to simplify the H_i coefficients, $i\in\{1,2,3\}$, in equation (4.6). This is because in the 3-point function with three non-BPS-like states, each state is split into a right part and a left part, and the Bethe roots can appear on either part. But neither part satisfies cyclicity on its own and the zero-momentum constraint cannot be used.

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Moreover, $\mathcal{N}_{132}$ in equation (2.19) is given by

$$\mathcal{N}_{132} = \left(\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}\right)^{\frac{1}{2}} , \tag{4.22}$$

where $\mathcal{N}_i$ are given by equation (B.10). Indeed,

$$\begin{aligned} \langle \mathcal{O}_1 | \mathcal{O}_1 \rangle &= {}_+\langle 0| \prod_{j=1}^{N_1} B[u_j, \{z\}_{L_1}]^\dagger \prod_{j=1}^{N_1} B[u_j, \{z\}_{L_1}] |0\rangle_+ \\ &= \left(\prod_{j=1}^{N_1} \prod_{l=1}^{L_1} \frac{u_j^* - z_l^* - \eta}{u_j^* - z_l^* + \eta} \right) {}_+\langle 0| \prod_{j=1}^{N_1} C[u_j^*, \{z^*\}_{L_1}] \prod_{j=1}^{N_1} B[u_j, \{z\}_{L_1}] |0\rangle_+ , \end{aligned} \tag{4.23}$$

where we have used equation (3.8). The prefactor becomes 1 in the homogeneous limit due to the zero-momentum constraint. Furthermore, the set of all Bethe roots $\{u\}_{N_1}$ transforms into itself under complex conjugation. Hence,

$$\langle \mathcal{O}_1 | \mathcal{O}_1 \rangle^{\text{hom}} = \lim_{z_l \to 0} {}_+\langle 0| \prod_{j=1}^{N_1} C[u_j, \{z\}_{L_1}] \prod_{j=1}^{N_1} B[u_j, \{z\}_{L_1}] |0\rangle_+ = \mathcal{N}_1^{\text{hom}} . \tag{4.24}$$

Similar considerations apply to $\langle \mathcal{O}_2 | \mathcal{O}_2 \rangle$. Finally, we note that $\mathcal{N}_3 = 1$.

5 Discussion

We have obtained a determinant expression for the tree-level OPE structure constants in planar QCD for operators of the type (2.7), where one of them is BPS-like, see equation (4.9). Indeed, given (L_1, N_1) and L_3, the possible values of (L_2, N_2) are determined by equation (4.17); then the corresponding Bethe equations (3.10) can be solved, and the structure constants $c_{132}^{(0)}$ can be efficiently computed using equation (4.19).

In the QCD literature, operators of the form (2.7) would be classified as *"chiral odd"*. While chiral-odd operators involving quark fields play an important role in certain hadronic scattering processes [19], the purely gluonic chiral-odd operators that we have considered here (with no covariant derivatives) do not seem to have direct relevance to QCD phenomenology.

It would be interesting to generalize this work to operators with covariant derivatives, which are more relevant to phenomenology. Such operators comprise the largest sector of QCD that is known to be integrable at one loop [17, 20–23]. Another challenge is to go to higher loops (see e.g. [24, 25]).

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A Coordinate Bethe ansatz and $\mathcal{F}$-conjugation

In order to properly formulate $\mathcal{F}$-conjugation in the algebraic Bethe ansatz formalism, it is necessary to first formulate it in the coordinate Bethe ansatz formalism.

We begin by reviewing the coordinate Bethe ansatz for spin-1, which has been discussed in [26, 27]. For simplicity, we consider the homogeneous case $z_l = 0$, and restrict to states with just two excitations, which are given by

$$|\{u_1, u_2\}\rangle^{co} = \sum_{1\le n_1\le n_2\le L} \left(e^{i(p_1 n_1 + p_2 n_2)} + S(p_2, p_1)\, e^{i(p_2 n_1 + p_1 n_2)}\right) |n_1, n_2\rangle\,. \tag{A.1}$$

Here $|n_1, n_2\rangle$ is given by [27]

$$|n_1, n_2\rangle = e^-_{n_1} e^-_{n_2} |f_+^L\rangle\,, \qquad e^- = \begin{pmatrix} 0 & 0 & 0 \\ 2^{1/2} & 0 & 0 \\ 0 & 2^{-1/2} & 0 \end{pmatrix}\,, \tag{A.2}$$

and

$$S(p_2, p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}\,, \qquad e^{ip_j} = \frac{u_j + i}{u_j - i}\,. \tag{A.3}$$

The expression (A.1) is almost the same as for the spin-$\frac{1}{2}$ case [15], the main difference is that now the summation includes $n_1 = n_2$.

We define $\mathcal{F}$-conjugation by

$$\mathcal{F} \circ |n_1, n_2\rangle = \langle L + 1 - n_2, L + 1 - n_1|\, \hat{\mathcal{C}}^{\otimes L}\,, \tag{A.4}$$

where

$$\hat{\mathcal{C}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \hat{\mathcal{C}}^\dagger\,, \tag{A.5}$$

which has the properties

$$\hat{\mathcal{C}}|f_\pm\rangle = |f_\mp\rangle, \qquad \hat{\mathcal{C}}|f_0\rangle = |f_0\rangle, \qquad \hat{\mathcal{C}}^2 = 1, \qquad \hat{\mathcal{C}}^{\otimes L} B(u)\, \hat{\mathcal{C}}^{\otimes L} = C(u)\,. \tag{A.6}$$

The definition (A.4) is consistent with equation (2.17), and is a generalization of the definition for the spin-$\frac{1}{2}$ case [15]. It follows, as in the spin-$\frac{1}{2}$ case, that $\mathcal{F}$-conjugation of the coordinate Bethe ansatz state (A.1) is given by

$$\mathcal{F} \circ |\{u_1, u_2\}\rangle^{co} = e^{i(L+1)(p_1+p_2)} S(p_2, p_1)\, {}^{co}\langle\{u_1^*, u_2^*\}|\hat{\mathcal{C}}^{\otimes L}\,, \tag{A.7}$$

where ${}^{co}\langle\{u_1, u_2\}| \equiv (|\{u_1, u_2\}\rangle^{co})^\dagger$.

We now proceed to translate this result to the algebraic Bethe ansatz. One can show that the algebraic and coordinate Bethe ansatz states are related (in our normalization) by

$$|\{u_1,u_2\}\rangle^{al} = -\frac{(u_1-u_2+i)}{(u_1+i)(u_2+i)(u_1-u_2)}|\{u_1,u_2\}\rangle^{co}\,, \tag{A.8}$$

generalizing the known spin-$\frac{1}{2}$ result [15, 28]. The corresponding hermitian-conjugate result is

$$^{al}\langle\{u_1,u_2\}| \equiv \left(|\{u_1,u_2\}\rangle^{al}\right)^\dagger = -\frac{(u_1^*-u_2^*-i)}{(u_1^*-i)(u_2^*-i)(u_1^*-u_2^*)}\,{}^{co}\langle\{u_1,u_2\}|\,, \tag{A.9}$$

and therefore

$$^{co}\langle\{u_1^*,u_2^*\}| = -\frac{(u_1-i)(u_2-i)(u_1-u_2)}{(u_1-u_2-i)}\,{}^{al}\langle\{u_1^*,u_2^*\}|\,. \tag{A.10}$$

Using equations (A.8), (A.7), (A.10) and (A.3), we obtain

$$\mathcal{F}\circ|\{u_1,u_2\}\rangle^{al} = \prod_{j=1}^{2}\left(\frac{u_j+i}{u_j-i}\right)^L\,{}^{al}\langle\{u_1^*,u_2^*\}|\hat{\mathcal{C}}^{\otimes L}\,. \tag{A.11}$$

Since $|\{u_1,u_2\}\rangle^{al} = B(u_1)B(u_2)|f_+^L\rangle$, with the help of equation (3.8) we see that

$$^{al}\langle\{u_1^*,u_2^*\}| = \prod_{j=1}^{2}\left(\frac{u_j-i}{u_j+i}\right)^L\,\langle f_+^L|C(u_1)C(u_2)\,. \tag{A.12}$$

We conclude that $\mathcal{F}$-conjugation of an algebraic Bethe ansatz state is given by

$$\mathcal{F}\circ\left(B(u_1)B(u_2)|f_+^L\rangle\right) = \langle f_+^L|C(u_1)C(u_2)\hat{\mathcal{C}}^{\otimes L} = \langle f_-^L|B(u_1)B(u_2)\,. \tag{A.13}$$

B Scalar products

B.1 Izergin's determinant

To define the scalar product of two spin-chain states that are not eigenstates of the Hamiltonian, we need Izergin's determinant expression [29] for Korepin's *"domain wall partition function"* [30]. For two sets of variables $\{x\}$ and $\{y\}$ of cardinality $|x|=|y|=\ell$, Izergin's determinant expression $Z\{x,y\}$ is

$$Z\{x,y\} = \frac{\prod_{i,j=1}^{\ell}(x_i-y_j+\eta)}{\prod_{1\leqslant i<j\leqslant\ell}(x_j-x_i)(y_i-y_j)}\det\left(\frac{1}{(x_i-y_j+\eta)(x_i-y_j)}\right)_{1\leqslant i,j\leqslant\ell}\,, \tag{B.1}$$

where $\eta=\frac{i}{2}$.

B.2 The generic scalar product in an XXX spin-$\frac{1}{2}$ chain

For a length-L periodic XXX spin-$\frac{1}{2}$ chain, we consider *1.* Two non-BPS-like states in the space of states of the spin chain, $|\mathcal{O}_i\{u\}\rangle$ and $|\mathcal{O}_j\{v\}\rangle$, $|u| = |v| = N \leqslant L/2$, that are *not* Bethe eigenstates of the Hamiltonian $\mathcal{H}_{\frac{1}{2}}$, that is, $\{u\}$ and $\{v\}$ do *not* satisfy Bethe equations, and *2.* The set of all possible partitions of each of $\{u\}$ and $\{v\}$ into two disjoint subsets

$$\{u\} = \{u_1\} \cup \{u_2\}, \quad \{v\} = \{v_1\} \cup \{v_2\}, \quad 0 \leqslant |u_1| = |v_1| \leqslant N, \quad 0 \leqslant |u_2| = |v_2| \leqslant N\,, \tag{B.2}$$

where $\{u_1\} = \{u_{1,1}, u_{1,2}, \cdots, u_{1,|u_1|}\}$, *etc.* Following [30, 31], the scalar product $\langle\mathcal{O}_j\{v\}|\mathcal{O}_i\{u\}\rangle$, is

$$\begin{aligned}\langle\mathcal{O}_j\{v\}|\mathcal{O}_i\{u\}\rangle = &\sum_{\{u_1\}\cup\{u_2\},\{v_1\}\cup\{v_2\}} \left(\prod_{\{u_1\}} a^{\frac{1}{2}}[u_1, \{z\}_L] \prod_{\{v_2\}} a^{\frac{1}{2}}[v_2, \{z\}_L]\right) \\ &\times \left(\prod_{i=1}^{|v_1|}\prod_{j=1}^{|v_2|} f(v_{1,i}, v_{2,j})\right) \left(\prod_{i=1}^{|u_2|}\prod_{j=1}^{|u_1|} f(u_{2,i}, u_{1,j})\right) Z\{v_1, u_1\} Z\{u_2, v_2\}\,,\end{aligned} \tag{B.3}$$

where the sum is over all partitions of $\{u\}$ and $\{v\}$ into two disjoint subsets, and

$$a^{\frac{1}{2}}[x, \{z\}_L] = \prod_{i=1}^{L} \frac{x - z_i + \eta}{x - z_i}, \quad f(x_i, y_j) = \frac{x_i - y_j + \eta}{x_i - y_j}\,. \tag{B.4}$$

B.3 The generic scalar product in an XXX spin-1 chain

In the case of a length-L XXX spin-1 chain case, the generic scalar product has the same form as in equation (B.3), but with the following extra conditions. *1.* We start from a spin-$\frac{1}{2}$ chain with $2L$ sites. *2.* We set the inhomogeneities

$$z_{2i+2} = z_{2i+1} + \eta, \quad i \in \{0, 1, \cdots, (L-1)\}\,, \tag{B.5}$$

as required by fusion. *3.* We take the inhomogeneities w_i, $i \in \{1, 2, \cdots, L\}$ of the L-sites of the spin-1 chain to be those of the odd-indexed sites of the original spin-$\frac{1}{2}$ chain, $w_i = z_{2i-1}$. *4.* We change $a^{\frac{1}{2}}[x, \{z\}_L]$ to $a^1[x, \{w\}_L]$ defined by

$$a^1[x, \{w\}_L] = \prod_{i=1}^{L} \frac{x - w_i + \eta}{x - w_i - \eta}, \quad \eta = \frac{i}{2}\,. \tag{B.6}$$

while all other factors remain unchanged as they have no dependence on the inhomogeneities. The result is the generic scalar product for the spin-1 chain.

B.4 The Slavnov scalar product

Let us first consider the matrix element

$$S_N(\{u\}_N, \{v\}_N, \{z\}_L) = \langle 0| \prod_{j=1}^{N} C[v_j; \{z\}_L] \prod_{k=1}^{N} B[u_k; \{z\}_L] |0\rangle\,, \tag{B.7}$$

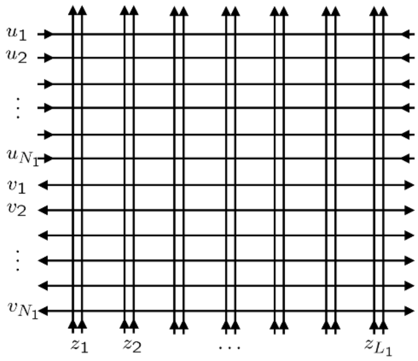

Figure 2. 2D lattice configuration for the Slavnov determinant. Double vertical lines denote spin-1 quantum spaces with double up-arrows for the f_+ state. Horizontal lines with incoming spin-$\frac{1}{2}$ arrows denote B operators, while those with outgoing arrows denote C operators. If we impose $u_i = v_i$ $(i = 1, \ldots, N_1)$, then this configuration depicts the Gaudin norm.

where $\{u\}_N = \{u_1, \ldots, u_N\}$ (but not necessarily $\{v\}_N = \{v_1, \ldots, v_N\}$) satisfy the Bethe equations (3.10), and $|0\rangle \equiv |0\rangle_+$. In the 2-dimensional vertex-model description, this scalar product is represented by figure 2. It follows from Slavnov [32] that this matrix element is given by[13]

$$S_N(\{u\}_N, \{v\}_N, \{z\}_L) = \left(\prod_{j>i}^{N} \frac{1}{v_j - v_i} \frac{1}{u_i - u_j} \right) \det M_{lk} \,, \tag{B.8}$$

where the $N \times N$ matrix M_{lk} is given by

$$M_{lk} = \frac{\eta}{u_k - v_l} \left(\prod_{\substack{m=1 \\ m \neq k}}^{N} (v_l - u_m - \eta) \prod_{j=1}^{L} \frac{v_l - z_j + \eta}{v_l - z_j - \eta} - \prod_{\substack{m=1 \\ m \neq k}}^{N} (v_l - u_m + \eta) \right) . \tag{B.9}$$

B.5 Gaudin norm

For the special case that $\{v_i\}$ coincide with $\{u_i\}$, the scalar product (B.7) reduces to the Gaudin norm [30, 33, 34]

$$\mathcal{N}(\{u\}_N, \{z\}_L) = \langle 0| \prod_{j=1}^{N} C[u_j; \{z\}_L] \prod_{k=1}^{N} B[u_k; \{z\}_L] |0\rangle = \eta^N \left(\prod_{j \neq k} \frac{u_j - u_k - \eta}{u_j - u_k} \right) \det \Phi' \,, \tag{B.10}$$

where Φ' is an $N \times N$ matrix given by

$$\Phi'_{jk} = \frac{\partial}{\partial u_k} \log \left(\prod_{l=1}^{L} \frac{u_j - z_l + \eta}{u_j - z_l - \eta} \prod_{m \neq j} \frac{u_j - u_m - \eta}{u_j - u_m + \eta} \right) . \tag{B.11}$$

[13] We identify $-ic$ in [32] with η.

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B.6 Restricted Slavnov scalar product

We now show how to restrict the Slavnov scalar product (B.7)–(B.9) (with $N = N_1$ and $L = L_1$) to obtain equation (B.25). The basic trick [16, 35, 36] is to set the *"extra"* v-variables equal to inhomogeneities:

$$v_{N_1-2j+1} = z_j, \qquad v_{N_1-2j+2} = z_j + \eta, \qquad j = 1, \dots, \frac{1}{2}(N_1 - N_2), \quad N_2 < N_1 . \tag{B.12}$$

However, since the expression (B.9) for M_{lk} then becomes singular, it is convenient to first change normalization. Using a tilde to denote quantities in the new normalization, we see that

$$\tilde{R}^{(\frac{1}{2},1)}(u,v) = \alpha(u,v)\, R^{(\frac{1}{2},1)}(u,v) \tag{B.13}$$

implies that

$$\tilde{B}[u;\{z\}_L] = \prod_{l=1}^{L} \alpha(u,z_l)\, B[u;\{z\}_L] \,, \quad \tilde{C}[u;\{z\}_L] = \prod_{l=1}^{L} \alpha(u,z_l)\, C[u;\{z\}_L] \,. \tag{B.14}$$

Hence,

$$\tilde{S}_{N_1} \equiv \langle 0| \prod_{j=1}^{N_1} \tilde{C}[v_j;\{z\}_{L_1}] \prod_{k=1}^{N_1} \tilde{B}[u_k;\{z\}_{L_1}]|0\rangle = \left(\prod_{j=1}^{N_1} \prod_{l=1}^{L_1} \alpha(u_j,z_l)\alpha(v_j,z_l) \right) S_{N_1} \,. \tag{B.15}$$

We choose the normalization factor

$$\alpha(u,v) = \frac{u-v-\eta}{u-v+\eta} \,, \tag{B.16}$$

which will avoid the singularity. Then

$$\tilde{S}_{N_1} = \left(\prod_{j=1}^{N_1} \prod_{l=1}^{L_1} \frac{u_j - z_l - \eta}{u_j - z_l + \eta} \right) \left(\prod_{j>i}^{N_1} \frac{1}{v_j - v_i} \frac{1}{u_i - u_j} \right) \det \tilde{M}_{lk} \,, \tag{B.17}$$

where

$$\tilde{M}_{lk} = \frac{\eta}{(u_k - v_l)} \left(\prod_{\substack{m=1\\ m\neq k}}^{N_1} (v_l - u_m - \eta) - \prod_{j=1}^{L_1} \frac{v_l - z_j - \eta}{v_l - z_j + \eta} \prod_{\substack{m=1\\ m\neq k}}^{N_1} (v_l - u_m + \eta) \right) \,. \tag{B.18}$$

We are now ready to *"freeze"*, or *"restrict"*, the scalar product $\tilde{S}_{N_1}$ by setting $\{v_{N_2+1},\dots,v_{N_1}\}$ to the values in equation (B.12), to obtain $\tilde{S}_{\text{restricted}}$, which is (see figure 3)

$$\begin{aligned}\tilde{S}_{\text{restricted}} = \prod_{j=1}^{N_1} \prod_{l=1}^{L_1} \frac{u_j - z_l - \eta}{u_j - z_l + \eta} \prod_{N_1 \geq j > k \geq 1} \frac{1}{u_j - u_k} \prod_{N_2 \geq j > k \geq 1} \frac{1}{v_j - v_k} \\ \times \prod_{\frac{1}{2}(N_1 - N_2) \geq j > k \geq 1} \frac{1}{(z_j - z_k)^2 (z_j - z_k - \eta)(z_j - z_k + \eta)} \\ \times \prod_{j=1}^{\frac{1}{2}(N_1-N_2)} \prod_{k=1}^{N_2} \frac{1}{(z_j - v_k + \eta)(z_j - v_k)} \det \mathcal{M}_{lk} \,,\end{aligned} \tag{B.19}$$

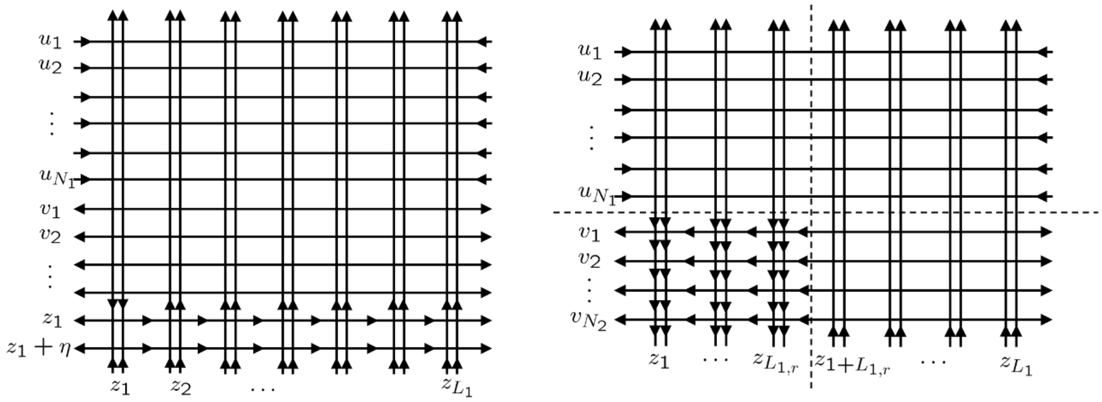

Figure 3. On the left, we *"freeze"* the two bottom rows of figure 2 by imposing equation (B.12). These two frozen rows are then eliminated. By repeating this procedure, we freeze out and eliminate the $N_1 - N_2$ bottom rows, thereby obtaining the figure on the right. The spins in the bottom-left part of the remaining lattice are in fact completely fixed. After also removing this part, we obtain the restricted Slavnov determinant in figure 4.

where $\mathcal{M}_{lk}$ is an $N_1 \times N_1$ matrix, which for $l \leq N_2$ is the same as (B.18), namely

$$\mathcal{M}_{lk} = \frac{\eta}{(u_k - v_l)} \left(\prod_{\substack{m=1\\ m\neq k}}^{N_1} (v_l - u_m - \eta) - \prod_{\substack{m=1\\ m\neq k}}^{N_1} (v_l - u_m + \eta) \prod_{j=1}^{L_1} \frac{v_l - z_j - \eta}{v_l - z_j + \eta} \right) , \; l \leq N_2 ; \tag{B.20}$$

and for $l > N_2$,

$$\begin{aligned} \mathcal{M}_{N_2+2j-1,k} &= \frac{1}{(u_k - z_j)} \left(\prod_{\substack{n=1\\ n\neq k}}^{N_1} (z_j - u_n - \eta) - \prod_{l=1}^{L_1} \frac{z_j - z_l - \eta}{z_j - z_l + \eta} \prod_{\substack{n=1\\ n\neq k}}^{N_1} (z_j - u_n + \eta) \right) , \\ \mathcal{M}_{N_2+2j,k} &= \frac{1}{(u_k - z_j - \eta)} \prod_{\substack{n=1\\ n\neq k}}^{N_1} (z_j - u_n) , \qquad j = 1, \ldots, \frac{1}{2}(N_1 - N_2) . \end{aligned} \tag{B.21}$$

We now observe that $\det \mathcal{M}_{lk}$ does not change if we add to $\mathcal{M}_{N_2+2j-1,k}$ any k-independent factor times $\mathcal{M}_{N_2+2j,k}$. The second term of $\mathcal{M}_{N_2+2j-1,k}$ can therefore be dropped, since it can be written as

$$-\frac{1}{(u_k - z_j)(z_j - u_k + \eta)} \prod_{l=1}^{L_1} \frac{z_j - z_l - \eta}{z_j - z_l + \eta} \prod_{n=1}^{N_1} (z_j - u_n + \eta) ,$$

which is a k-independent factor times $\mathcal{M}_{N_2+2j,k}$. In short, for $l > N_2$, $\mathcal{M}_{lk}$ is given by

$$\mathcal{M}_{N_2+2j-1,k} = \frac{1}{(u_k - z_j)} \prod_{\substack{n=1\\ n\neq k}}^{N_1} (z_j - u_n - \eta) , \tag{B.22}$$

$$\mathcal{M}_{N_2+2j,k} = \frac{1}{(u_k - z_j - \eta)} \prod_{\substack{n=1\\ n\neq k}}^{N_1} (z_j - u_n) , \qquad j = 1, \ldots, \frac{1}{2}(N_1 - N_2) . \tag{B.23}$$

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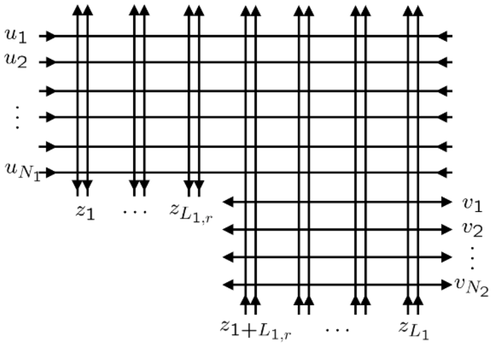

Figure 4. 2D lattice representation of the restricted Slavnov determinant, which is a scalar product between $|\mathcal{O}_1\rangle$ (where the B operators with arguments $\{u\}_{N_1}$ act on all the quantum spaces $1\ldots, L_1$) and ${}_r\langle\mathcal{O}_2|$ (where the C operators with arguments $\{v\}_{N_2}$ act only on the quantum spaces $1+L_{1,r},\ldots,L_1$).

With the help of the vertex-model correspondence, we can make the identification

$$\tilde{S}_{\text{restricted}} = \langle 1,\ldots,\frac{1}{2}(N_1-N_2)|\prod_{j=1}^{N_1}\tilde{C}[v_j;\{z\}_{L_1}]\prod_{k=1}^{N_1}\tilde{B}[u_k;\{z\}_{L_1}]|0\rangle\,, \tag{B.24}$$

where $|1,\ldots,\frac{1}{2}(N_1-N_2)\rangle$ is the state with down-spins at the sites $1,\ldots,\frac{1}{2}(N_1-N_2)$ and up-spins at the remaining $L_1-\frac{1}{2}(N_1-N_2)$ sites. See figure 4. Finally, returning to the original normalization using equation (B.14), we obtain

$$\begin{aligned}
&S[\{u\}_{N_1},\{v\}_{N_2},\{z\}_{L_1}]\\
&\quad=\langle 1,\ldots,\frac{1}{2}(N_1-N_2)|\prod_{j=1}^{N_2}C[v_j;\{z\}_{L_1}]\prod_{k=1}^{N_1}B[u_k;\{z\}_{L_1}]|0\rangle\\
&\quad=\left(\prod_{l=\frac{1}{2}(N_1-N_2)+1}^{L_1}\prod_{k=1}^{N_2}\frac{v_k-z_l+\eta}{v_k-z_l-\eta}\right)\prod_{N_1\geq j>k\geq 1}\frac{1}{u_j-u_k}\prod_{N_2\geq j>k\geq 1}\frac{1}{v_j-v_k}\\
&\qquad\times\prod_{\frac{1}{2}(N_1-N_2)\geq j>k\geq 1}\frac{1}{(z_j-z_k)^2(z_j-z_k-\eta)(z_j-z_k+\eta)}\\
&\qquad\times\prod_{j=1}^{\frac{1}{2}(N_1-N_2)}\prod_{k=1}^{N_2}\frac{1}{(z_j-v_k+\eta)(z_j-v_k)}\det\mathcal{M}_{lk}\,.
\end{aligned}\tag{B.25}$$

References

[1] J. Minahan and K. Zarembo, *The Bethe ansatz for $N=4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208] [INSPIRE].

[2] N. Beisert and M. Staudacher, *The $N=4$ SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042] [INSPIRE].

[3] N. Beisert and M. Staudacher, *Long-range $PSU(2,2|4)$ Bethe ansatze for gauge theory and strings*, Nucl. Phys. **B 727** (2005) 1 [hep-th/0504190] [INSPIRE].

[4] N. Beisert et al., *Review of AdS/CFT integrability: an overview*, Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982] [INSPIRE].

[5] G. Ferretti, R. Heise and K. Zarembo, *New integrable structures in large-N QCD*, Phys. Rev. **D 70** (2004) 074024 [hep-th/0404187] [INSPIRE].

[6] A. Zamolodchikov and V. Fateev, *Model factorized S matrix and an integrable Heisenberg chain with spin. 1* (in Russian), Sov. J. Nucl. Phys. **32** (1980) 298 [INSPIRE].

[7] P. Kulish, N.Y. Reshetikhin and E. Sklyanin, *Yang-Baxter equation and representation theory. 1*, Lett. Math. Phys. **5** (1981) 393 [INSPIRE].

[8] P. Kulish and E. Sklyanin, *Quantum spectral transform method. Recent developments*, Lect. Notes Phys. **151** (1982) 61.

[9] L. Takhtajan, *The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins*, Phys. Lett. **A 87** (1982) 479 [INSPIRE].

[10] H.M. Babujian, *Exact solution of the isotropic Heisenberg chain with arbitrary spins: thermodynamics of the model*, Nucl. Phys. **B 215** (1983) 317 [INSPIRE].

[11] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three point functions of chiral operators in $D = 4$, $N = 4$ SYM at large-N*, Adv. Theor. Math. Phys. **2** (1998) 697 [hep-th/9806074] [INSPIRE].

[12] K. Okuyama and L.-S. Tseng, *Three-point functions in $N = 4$ SYM theory at one-loop*, JHEP **08** (2004) 055 [hep-th/0404190] [INSPIRE].

[13] R. Roiban and A. Volovich, *Yang-Mills correlation functions from integrable spin chains*, JHEP **09** (2004) 032 [hep-th/0407140] [INSPIRE].

[14] L.F. Alday, J.R. David, E. Gava and K. Narain, *Structure constants of planar $N = 4$ Yang-Mills at one loop*, JHEP **09** (2005) 070 [hep-th/0502186] [INSPIRE].

[15] J. Escobedo, N. Gromov, A. Sever and P. Vieira, *Tailoring three-point functions and integrability*, JHEP **09** (2011) 028 [arXiv:1012.2475] [INSPIRE].

[16] O. Foda, *$N = 4$ SYM structure constants as determinants*, JHEP **03** (2012) 096 [arXiv:1111.4663] [INSPIRE].

[17] N. Beisert, G. Ferretti, R. Heise and K. Zarembo, *One-loop QCD spin chain and its spectrum*, Nucl. Phys. **B 717** (2005) 137 [hep-th/0412029] [INSPIRE].

[18] V. Braun, G. Korchemsky and D. Mueller, *The uses of conformal symmetry in QCD*, Prog. Part. Nucl. Phys. **51** (2003) 311 [hep-ph/0306057] [INSPIRE].

[19] R. Jaffe and X.-D. Ji, *Chiral odd parton distributions and Drell-Yan processes*, Nucl. Phys. **B 375** (1992) 527 [INSPIRE].

[20] V.M. Braun, S.E. Derkachov and A. Manashov, *Integrability of three particle evolution equations in QCD*, Phys. Rev. Lett. **81** (1998) 2020 [hep-ph/9805225] [INSPIRE].

[21] V.M. Braun, S.E. Derkachov, G. Korchemsky and A. Manashov, *Baryon distribution amplitudes in QCD*, Nucl. Phys. **B 553** (1999) 355 [hep-ph/9902375] [INSPIRE].

[22] A.V. Belitsky, *Fine structure of spectrum of twist — Three operators in QCD*, Phys. Lett. **B 453** (1999) 59 [hep-ph/9902361] [INSPIRE].

JHEP06(2012)168

[23] C. Ahn, R.I. Nepomechie and J. Suzuki, *The QCD spin chain S matrix*, Nucl. Phys. **B 798** (2008) 402 [arXiv:0711.2415] [INSPIRE].

[24] A. Belitsky, G. Korchemsky and D. Mueller, *Integrability of two-loop dilatation operator in gauge theories*, Nucl. Phys. **B 735** (2006) 17 [hep-th/0509121] [INSPIRE].

[25] N. Gromov and P. Vieira, *Quantum integrability for three-point functions*, arXiv:1202.4103 [INSPIRE].

[26] A. Lima-Santos, *Bethe ansatz for nineteen vertex models*, J. Phys. **A 32** (1999) 1819 [hep-th/9807219] [INSPIRE].

[27] N. Crampé, E. Ragoucy and L. Alonzi, *Coordinate Bethe ansatz for spin s XXX model*, *SIGMA* **7** (2011) 6 [arXiv:1009.0408].

[28] A. Ovchinnikov, *Coordinate space wave function from the algebraic Bethe ansatz for the inhomogeneous six-vertex model*, Phys. Lett. **A 374** (2010) 1311 [arXiv:1001.2672] [INSPIRE].

[29] A.G. Izergin, *Partition function of the six-vertex model in a finite volume*, Sov. Phys. Dokl. **32** (1987) 878.

[30] V. Korepin, *Calculation of norms of Bethe wave functions*, Commun. Math. Phys. **86** (1982) 391 [INSPIRE].

[31] V. Korepin, N. Bogoliubov, and A. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, Cambridge U.K. (1993).

[32] N. Slavnov, *Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz*, Theor. Math. Phys. **79** (1989) 502.

[33] M. Gaudin, *Diagonalisation of a class of spin hamiltonians*, J. Phys-Paris **37** (1976) 1087.

[34] M. Gaudin, B.M. McCoy and T.T. Wu, *Normalization sum for the Bethe's hypothesis wave functions of the Heisenberg-Ising chain*, Phys. Rev. **D 23** (1981) 417.

[35] N. Kitanine, J. Maillet and V. Terras, *Form factors of the XXZ Heisenberg spin/2 finite chain*, Nucl. Phys. **B 554** (1999) 647 [math-ph/9807020].

[36] M. Wheeler, *An Izergin-Korepin procedure for calculating scalar products in six-vertex models*, Nucl. Phys. **B 852** (2011) 468 [arXiv:1104.2113] [INSPIRE].