Duality in $N=2$ super-Liouville theory

Changrim Ahn,[1,*] Chanju Kim,[1,†] Chaiho Rim,[2,‡] and M. Stanishkov[1,§]
[1]*Department of Physics, Ewha Womans University, Seoul 120-750, Korea*
[2]*Department of Physics, Chonbuk National University, Chonju 561-756, Korea*
(Received 8 February 2004; published 27 May 2004)

In this paper we consider a strong-weak coupling duality of the $N=2$ super-Liouville field theory (SLFT). Without the self-duality found in other Liouville theories, the $N=2$ SLFT, we claim, is associated with a ''dual'' action by a transformation $b\rightarrow 1/b$ where b is the coupling constant. To justify our conjecture, we compute the reflection amplitudes (or two-point functions) of the Neveu-Schwarz and Ramond operators of the $N=2$ SLFT based on the conjectured dual action and show that the results are consistent with known results.

DOI: 10.1103/PhysRevD.69.106011 PACS number(s): 11.25.Hf, 11.55.Ds

I. INTRODUCTION

Two-dimensional Liouville field theory (LFT) appears naturally in the context of 2D quantum gravity and string theories [1,2]. This theory has been extended to the supersymmetric Liouville theories to accommodate world-sheet supersymmetries appearing in the string theories. These Liouville-type theories are also interesting for quantum field theoretical properties as well. They possess both conformal symmetry and the strong-weak coupling duality. The strong-weak coupling duality has been attracting much attention recently to understand nonperturbative aspects of the various quantum field theories rigorously. For example, the seminal work of Seiberg and Witten on the supersymmetric Yang-Mills theories in $3+1$ dimensions is based on the intuitive observation that the gauge theories in the strong coupling limit are described by a weak coupling region of some effective action [3]. More recently, similar duality arises in the string theory context to understand strings and branes nonperturbatively. This duality also arises in statistical systems such as the 2D Ising model.

For the LFT and the $N=1$ super-LFT (SLFT), the strong-weak coupling duality appears as a quantum symmetry closely connected to (super)conformal symmetries. It has been observed that the background charge is renormalized to $Q=b+1/b$ by quantum corrections and the theories preserve the quantum conformal symmetries [2,4]. This means the two LFTs are invariant under $b\rightarrow 1/b$—i.e., self-dual. These two symmetries are essential to determine exact correlation functions for the LFT [5] and the $N=1$ SLFT [6]. Now let us consider the duality symmetry of the $N=2$ SLFT. With or without conformal symmetry, two-dimensional models with $N=2$ supersymmetry show an interesting feature: namely, the nonrenormalization. The parameters in the supersymmetric action do not change in all orders of perturbative calculations. This means that the $N=2$ SLFT maintains conformal symmetry without renormalization of the background charge and loses self-duality.

Our main proposal in this paper is that the $N=2$ SLFT still shows an interesting duality behavior. Under the dual transformation $b\rightarrow 1/b$, the theory maps to a ''dual'' action which is another $N=2$ super conformal field theory (CFT). The $N=2$ SLFT with a strong coupling can be described by the dual action perturbatively. We compute the reflection amplitudes (the two-point functions) of the theory using functional relations derived from the actions. This procedure provides an exact relation between two parameters $\mu,\tilde{\mu}$, which fixes the relations between the two actions completely. To check the self-consistency of our proposal, we compare the reflection amplitudes derived from the conjecture action with some independent result derived in a totally different context.

CFTs with $N=2$ supersymmetry have been actively studied mainly due to possible applications to string theories. In particular, the $N=2$ SLFT has been conjectured to be dual to a fermionic black hole based on the coset $SL(2)/U(1)$ [7]. This proposal was further explored in [8]. We will relate our ''dual'' theory to the fermionic black hole theory by comparing the reflection amplitudes and explicit Lagrangians.

II. $N=2$ SUPER-LIOUVILLE THEORY AND ITS DUAL ACTION

The action of the $N=2$ SLFT at the flat background is given by

$$\mathcal{A}_{\rm I}=\int d^2z\left[\int d^4\theta S S^{\dagger}+\mu\int d^2\theta e^{bS}+\text{c.c.}\right],\qquad(1)$$

where S is a scalar superfield satisfying

$$D_-S=\bar{D}_-S=0,\quad D_+S^{\dagger}=\bar{D}_+S^{\dagger}=0.\qquad(2)$$

The Lagrangian of the $N=2$ SLFT can be expressed in terms of the component fields as follows:

*Electronic address: ahn@dante.ewha.ac.kr
†Electronic address: cjkim@ewha.ac.kr
‡Electronic address: rim@mail.chonbuk.ac.kr
§On leave of absence from Institute of Nuclear Research and Nuclear Energy, Sofia, Bulgaria. Electronic address: stanishkov@dante.ewha.ac.kr

$$\mathcal{L}=\frac{1}{4\pi}[2\varphi\partial\bar{\partial}\bar{\varphi}^{\dagger}+2\varphi^{\dagger}\partial\bar{\partial}\bar{\varphi}+\psi^{\dagger}\bar{\partial}\psi+\psi\bar{\partial}\psi^{\dagger}+\bar{\psi}^{\dagger}\partial\bar{\psi}+\bar{\psi}\partial\bar{\psi}^{\dagger}]-\frac{5}{4}\pi\mu^{2}b^{2}e^{b\varphi+b\varphi^{\dagger}}+\frac{\mu b^{2}}{2}\psi^{\dagger}\bar{\psi}^{\dagger}e^{b\varphi}+\frac{\mu b^{2}}{2}\psi\bar{\psi}e^{b\varphi^{\dagger}}. \tag{3}$$

As in the LFT and the $N=1$ SLFTs, one should introduce a background charge $1/b$ so that the second term in Eq. (1) becomes the screening operator of the CFT. However, a fundamental difference arises where the background charge is unrenormalized due to the $N=2$ supersymmetry. For the LFT and the $N=1$ SLFTs, the background charge is renormalized to $Q=1/b+b$ and the theories are invariant under the dual transformation $b\to 1/b$. This self-duality plays an essential role in determining various exact correlation functions of those Liouville theories. Unrenormalized, the $N=2$ SLFT is not self-dual.

This theory is a CFT with a central charge

$$c=3+6/b^{2}. \tag{4}$$

The primary operators of the $N=2$ SLFT are classified into Neveu-Schwarz (NS) and the Ramond (R) sectors and can be written in terms of the component fields as follows [9]:

$$N_{\alpha\bar{\alpha}}=e^{\alpha\varphi^{\dagger}+\bar{\alpha}\varphi},\quad R^{\pm}_{\alpha\bar{\alpha}}=\sigma^{\pm}e^{\alpha\varphi^{\dagger}+\bar{\alpha}\varphi}, \tag{5}$$

where $\sigma^{\pm}$ are spin operators. The conformal dimensions of these fields are given by

$$\Delta^{N}_{\alpha\bar{\alpha}}=-\alpha\bar{\alpha}+\frac{1}{2b}(\alpha+\bar{\alpha}),\quad \Delta^{R}_{\alpha\bar{\alpha}}=\Delta^{N}_{\alpha\bar{\alpha}}+\frac{1}{8}. \tag{6}$$

The $U(1)$ charges are given by

$$Q^{N}_{\alpha\bar{\alpha}}=-\frac{1}{2b}(\alpha-\bar{\alpha}),\quad Q^{R\pm}_{\alpha\bar{\alpha}}=Q^{N}_{\alpha\bar{\alpha}}\pm\frac{1}{4}. \tag{7}$$

From these expressions, one can notice that

$$\alpha\to 1/b-\bar{\alpha},\quad \bar{\alpha}\to 1/b-\alpha \tag{8}$$

do not change the same conformal dimension and $U(1)$ charge. From the CFT point of view, this means that $N_{1/b-\bar{\alpha},1/b-\alpha}$ should be identified with $N_{\alpha\bar{\alpha}}$ and similarly for the (R) operators up to normalization factors. The reflection amplitudes are determined by these normalization factors.

Without self-duality, it is possible that there exists a "dual" action to Eq. (1) whose perturbative (weak coupling) behaviors describe the $N=2$ SLFT in the strong coupling region. This action should be another CFT. Our proposal for the dual action is as follows:

$$\mathcal{A}_{\rm II}=\int d^{2}z\int d^{4}\theta[SS^{\dagger}+\tilde{\mu}e^{(S+S^{\dagger})/b}], \tag{9}$$

with background charge b. The $N=2$ supersymmetry is preserved because $S+S^{\dagger}$ is a $N=2$ scalar superfield. One can see that this action is conformally invariant because the interaction term is a screening operator. Our conjecture is that the two actions $\mathcal{A}_{\rm I}$ and $\mathcal{A}_{\rm II}$ are equivalent. To justify this conjecture, we will compute the reflection amplitudes based on these actions and will compare with some independent results.

III. REFLECTION AMPLITUDES

As mentioned above, the reflection amplitudes of the Liouville-type CFT is defined by the linear transformations between different exponential fields, corresponding to the same primary field of chiral algebra. In this paper, we apply the method developed in [10] to derive the reflection amplitudes of the $N=2$ SLFT. For simplicity, we will restrict ourselves to the case of $\alpha=\bar{\alpha}$ in Eq. (5) where the $U(1)$ charge of the (NS) operators becomes 0. We will refer to this case as the "neutral" sector. (From now on, we will suppress the second indices $\bar{\alpha}$.) The physical states in this sector are given by

$$\alpha=\frac{1}{2b}+iP, \tag{10}$$

where P is a real parameter. This parameter is transformed by $P\to -P$ under the reflection relation (8) and can be thought of as a "momentum" which is reflected off from a potential wall.

Two-point functions of the same operators can be expressed as

$$\langle N_{\alpha}(z,\bar{z})N_{\alpha}(0,0)\rangle=\frac{D^{N}(\alpha)}{|z|^{4\Delta^{N}_{\alpha}}}, \tag{11}$$

$$\langle R^{+}_{\alpha}(z,\bar{z})R^{-}_{\alpha}(0,0)\rangle=\frac{D^{R}(\alpha)}{|z|^{4\Delta^{R}_{\alpha}}}, \tag{12}$$

where $\Delta^{N}_{\alpha},\Delta^{R}_{\alpha}$ are given by Eq. (6). The reflection amplitudes are given by the normalization factors $D^{N}(\alpha),D^{R}(\alpha)$ and should satisfy

$$D^{N}(\alpha)D^{N}(1/b-\alpha)=1,\quad D^{R}(\alpha)D^{R}(1/b-\alpha)=1. \tag{13}$$

To find these amplitudes explicitly, we consider the operator product expansions (OPEs) with degenerate operators.

The (NS) and (R) degenerate operators in the neutral sector are $N_{\alpha_{nm}}$ and $R^{\pm}_{\alpha_{nm}}$ with integers n,m and

$$\alpha_{nm}=\frac{1-n}{2b}-\frac{mb}{2},\quad n,m\geqslant 0. \tag{14}$$

The OPE of a (NS) field with a degenerate operator $N_{-b/2}$ is simply given by

$$N_{\alpha}N_{-b/2}=N_{\alpha-b/2}+C^{N}_{-}(\alpha)N_{\alpha+b/2}. \tag{15}$$

Here the structure constant can be obtained from the screening integral as follows:

$$C_-^N(\alpha)=\kappa_1\gamma(1-\alpha b)\gamma(1/2-\alpha b-b^2/2)\gamma(-1/2+\alpha b)\gamma(\alpha b+b^2/2), \tag{16}$$

where

$$\kappa_1=\frac{\mu^2b^4\pi^2}{2}\gamma(-b^2-1)\gamma\left(1+\frac{b^2}{2}\right)\gamma\left(\frac{b^2}{2}+\frac{3}{2}\right),$$

with $\gamma(x)=\Gamma(x)/\Gamma(1-x)$ as usual.

To use this OPE, we consider a three-point function $\langle N_{\alpha+b/2}N_\alpha N_{-b/2}\rangle$ and take the OPE by $N_{-b/2}$ either on $N_{\alpha+b/2}$ or on N_α using Eq. (15). This leads to a functional equation

$$C_-^N(\alpha)D^N(\alpha+b/2)=D^N(\alpha). \tag{17}$$

This functional equation can determine the (NS) reflection amplitude in the following form:

$$D^N(\alpha)=\left(\frac{\kappa_1}{b^4}\right)^{-2\alpha/b}\gamma(2\alpha/b-1/b^2)\frac{\gamma(b\alpha+1/2)}{\gamma(b\alpha)}f(\alpha), \tag{18}$$

with an aribitrary function $f(\alpha)$ satisfying $f(\alpha)=f(\alpha+b)$. To fix this unknown function, we need an additional functional equation. It is natural that this relation is provided by the dual action $\mathcal{A}_{\rm II}$.

For this purpose, we consider OPEs with another degenerate operator: namely,

$$N_\alpha R^+_{-1/2b}=R^+_{\alpha-1/2b}+\tilde{C}_-^N(\alpha)R^+_{\alpha+1/2b}, \tag{19}$$

$$R^-_\alpha R^+_{-1/2b}=N_{\alpha-1/2b}+\tilde{C}_-^R(\alpha)N_{\alpha+1/2b}. \tag{20}$$

The structure constants can be computed by the screening integrals using the dual action $\mathcal{A}_{\rm II}$, which is equivalent to $\mathcal{A}_{\rm I}$. The results are

$$\tilde{C}_-^N(\alpha)=\kappa_2(b)\frac{\gamma(2\alpha/b-1/b^2)}{\gamma(2\alpha/b)}, \tag{21}$$

$$\tilde{C}_-^R(\alpha)=\kappa_2(b)\frac{\gamma(2\alpha/b-1/b^2+1)}{\gamma(2\alpha/b+1)}, \tag{22}$$

where

$$\kappa_2(b)=\tilde{\mu}\pi\gamma\left(\frac{1}{b^2}+1\right). \tag{23}$$

These results are consistent with the $N=2$ superminimal CFT results [9].

Now we consider the three-point functions $\langle R^-_{\alpha+1/2b}N_\alpha R^+_{-1/2b}\rangle$ and $\langle N_{\alpha+1/2b}R^-_\alpha R^+_{-1/2b}\rangle$. Taking an OPE with $R^+_{-1/2b}$ on one of two other operators in the correlation functions and using the OPE relations (19) and (20), we obtain an independent set of functional relations as follows:

$$\tilde{C}_-^N(\alpha)D^R(\alpha+1/2b)=D^N(\alpha), \tag{24}$$

$$\tilde{C}_-^R(\alpha)D^N(\alpha+1/2b)=D^R(\alpha). \tag{25}$$

Solving for the $D^N(\alpha)$, we find that a most general solution of Eq. (24) is

$$D^N(\alpha)=\kappa_2^{-2\alpha b}\frac{\Gamma^2(\alpha b+1/2)}{\Gamma^2(\alpha b)}\gamma(2\alpha/b-1/b^2)g(\alpha), \tag{26}$$

where $g(\alpha)$ is another arbitrary function satisfying $g(\alpha)=g(\alpha+1/b)$. Combining Eqs. (18) and (26), and requiring the normalization $D^N(\alpha=1/2b)=1$, we can determine the (NS) reflection amplitude completely as follows:

$$D^N(\alpha)=-\frac{2}{b^2}\kappa_2^{-2\alpha b+1}\gamma\left(\frac{2\alpha}{b}-\frac{1}{b^2}\right)\gamma\left(\alpha b+\frac{1}{2}\right)\gamma(1-\alpha b), \tag{27}$$

where two parameters in the actions, μ and $\tilde{\mu}$, are related by

$$\left(\frac{\kappa_1}{b^4}\right)^{1/b}=\kappa_2^b. \tag{28}$$

The (R) reflection amplitude can be obtained by Eq. (24):

$$D^R(\alpha)=-\frac{b^2}{2}\kappa_2^{-2\alpha b+1}\gamma\left(\frac{2\alpha}{b}-\frac{1}{b^2}+1\right)\times\gamma\left(-\alpha b+\frac{1}{2}\right)\gamma(\alpha b). \tag{29}$$

We can rewrite these amplitudes using the momentum P defined in Eq. (10):

$$D^N(P)=\kappa_2^{-2iPb}\frac{\Gamma\left(1+\frac{2iP}{b}\right)}{\Gamma\left(1-\frac{2iP}{b}\right)}\frac{\Gamma(1+iPb)}{\Gamma(1-iPb)}\frac{\Gamma\left(\frac{1}{2}-iPb\right)}{\Gamma\left(\frac{1}{2}+iPb\right)}, \tag{30}$$

$$D^R(P)=\kappa_2^{-2iPb}\frac{\Gamma\left(1+\frac{2iP}{b}\right)}{\Gamma\left(1-\frac{2iP}{b}\right)}\frac{\Gamma(1-iPb)}{\Gamma(1+iPb)}\frac{\Gamma\left(\frac{1}{2}+iPb\right)}{\Gamma\left(\frac{1}{2}-iPb\right)}. \tag{31}$$

IV. CONSISTENCY CHECK

To justify the reflection amplitudes derived in the previous section based on the conjectured action $\mathcal{A}_{\rm II}$, we provide several consistency checks.

It has been noticed that an integrable model with two parameters proposed in [11] can have $N=2$ supersymmetry if one of the parameters takes a special value [12]. This means that one can compute the reflection amplitudes of the $N=2$ SLFT independently as a special case of those in [12]. Indeed, we have confirmed that the two results agree exactly.

Furthermore, one can check the reflection amplitude for specific values of α directly from the action. When $\alpha=1/2b-b/2$, the Coulomb integral using the action $\mathcal{A}_{\rm I}$ gives

$$\langle N_\alpha(0)N_\alpha(1)\rangle=\left(\frac{\mu b^2}{2}\right)^2\int d^2z_1d^2z_2\langle e^{\alpha(\varphi+\varphi^\dagger)}(0)\times e^{\alpha(\varphi+\varphi^\dagger)}(1)\psi^\dagger\overline{\psi}^\dagger e^{b\varphi(z_1,\bar{z}_1)}\psi\overline{\psi}e^{b\varphi^\dagger(z_2,\bar{z}_2)}\rangle=\frac{\mu^2b^3\pi^2}{\left(\alpha-\frac{1-b^2}{2b}\right)}\gamma\left(\frac{1+b^2}{2}\right)\frac{\gamma(-1-b^2)}{\gamma(-1/2-b^2/2)}. \quad (32)$$

This result agrees with Eq. (27) for $\alpha\to1/2b-b/2$. Similarly, when $\alpha\to0$, one can compute the two-point function directly from the action $\mathcal{A}_{\rm II}$ and get

$$\langle N_\alpha(0)N_\alpha(1)\rangle=-\frac{\tilde{\mu}}{b^2}\int d^2z\langle e^{\alpha(\varphi+\varphi^\dagger)}(0)e^{\alpha(\varphi+\varphi^\dagger)}(1)\times e^{(\varphi+\varphi^\dagger)/b}\partial(\varphi+\varphi^\dagger)\overline{\partial}(\varphi+\varphi^\dagger)(z,\bar{z})\rangle=-\frac{\tilde{\mu}}{b^2}\pi\alpha b=0 \quad \text{as} \quad \alpha\to0. \quad (33)$$

Here, the insertion we have considered is the only term in the action $\mathcal{A}_{\rm II}$ which can give nonvanishing contribution. Again, this result agrees with Eq. (27) for $\alpha=0$.

In the semiclassical limit $b\to0$, the reflection amplitudes can be interpreted as the quantum mechanical reflection amplitudes of the wave function of the zero modes from the exponential potential wall arising from the action (1). One can easily find that the reflection amplitude corresponding to the (NS) operator is given by

$$R^N(P)\sim\frac{\Gamma(1+2iP/b)}{\Gamma(1-2iP/b)}, \quad (34)$$

which is consistent with Eq. (31) in the limit $b\to0$.

V. RELATION TO THE FERMIONIC BLACK HOLES

It was proposed in Ref. [7] that $N=2$ SLFT is T dual to the theory of the fermionic black hole based on the coset $SL(2)/U(1)$. The purpose was to compute some physical observables in the so-called little string theory using a holographic projection. This statement was not proved but based on a few observations in [7]. The matching of the central charges in both theories requires a connection between the slope of the dilaton and the level k of the coset theory. Similarly, the compactification radius of a boson in $N=2$ SLFT is the inverse of that of the asymptotic radius of the black hole. There is a similar dualty between the bosonic black hole and the so-called sine-Liouville theory [13]. The two-point function of vertex operators $V_{j;m,\bar{m}}$ was computed in [7] based on a previous result found in [14] using the coset theory $SL(2)/U(1)$. One can check that this two-point function coincides exactly with ours in the neutral case $m=0$ provided the spin $j=-\alpha b$ and the level $k=b^2$.

The above proposal was further explored in [8] where it was argued that these two theories can be viewed as a mirror image of each other. Starting with certain gauged Wess-Zumino-Witten (WZW) model and its T dual transformation, the authors of [8] showed that these lead to the fermionic black hole and the $N=2$ SLFT, respectively, in the IR limit. The fermionic black hole is defined as a Kazama-Suzuki coset model described by a gauged supersymmetric WZW theory. After solving for the nondynamical gauge fields and performing some implicit transformations between the $SL(2)$ and $N=2$ superfields, one obtains an action of the form

$$S=\int d^2zd^4\theta K(S,S^+), \quad (35)$$

where $K(S,S^+)$ is a Kähler potential depending on the chiral superfields S,S^+ with the black hole metric:

$$\frac{1}{1-\frac{\tilde{\mu}}{b^2}e^{\varphi+\varphi^+}} \quad (36)$$

(φ and φ^+ are the first components of the superfields). At first sight this action looks different from $\mathcal{A}_{\rm II}$ which we proposed above. But if one expands interacting terms in the action perturbatively in the manner of the screening procedure, it is easy to see that the result for the physical observables is the same. This is particularly evident for the calculation of the structure constants computed in Sec. III where only the first-order perturbation (i.e., one screening insertion) is needed.

In summary, we have conjectured a strong coupling effective action which is dual to the $N=2$ SLFT. Based on this conjecture, we have computed the reflection amplitudes (the two-point functions) of the (NS) and the (R) primary fields in neutral sectors exactly. We have fixed the relation between the two parameters μ and $\tilde{\mu}$. Then, we have checked the validity of these amplitudes by comparing with an independent result along with some other consistency checks. Also we related our result to the fermionic black hole which has been conjectured to be dual to the $N=2$ SLFT. The reflection amplitudes derived in this paper can be applied to compute the effective central charge of the $N=2$ supersymmetric sinh-Gordon (or sine-Gordon) model. The comparison of this with the thermodynamic Bethe ansatz can provide another stringent check for our conjecture. We hope to report this progress in a future publication.

ACKNOWLEDGMENTS

We thank J. Teschner for helpful discussions. This work was supported in part by Korea Research Foundation, Grant No. 2002-070-C00025. M.S. is supported by the Brain Pool program from Korean Association Science and Technology.

[1] A. Polyakov, Phys. Lett. **103B**, 207 (1981).
[2] T. Curtright and C. Thorn, Phys. Rev. Lett. **48**, 1309 (1982).
[3] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994).
[4] L. O'Raifeartaigh and V.V. Sreedhar, Phys. Lett. B **461**, 66 (1999).
[5] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. **B477**, 577 (1996).
[6] R.C. Rashkov and M. Stanishkov, Phys. Lett. B **380**, 49 (1996); R.H. Poghossian, Nucl. Phys. **B496**, 451 (1997).
[7] A. Giveon and D. Kutasov, J. High Energy Phys. **10**, 034 (1999).
[8] K. Hori and A. Kapustin, J. High Energy Phys. **08**, 045 (2001).
[9] G. Mussardo, G. Sotkov, and M. Stanishkov, Int. J. Mod. Phys. A **4**, 1135 (1989).
[10] J. Teschner, Phys. Lett. B **363**, 65 (1995).
[11] V. Fateev, Nucl. Phys. **B473**, 509 (1996).
[12] P. Baseilhac and V. Fateev, Nucl. Phys. **B532**, 567 (1998).
[13] V. A. Fateev, A. B. Zamolodchikov, and Al. B. Zamolodchikov (unpublished).
[14] J. Teschner, Nucl. Phys. **B571**, 555 (2000).