# The scaling supersymmetric Yang–Lee model with boundary

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## Abstract

We define the scaling supersymmetric Yang–Lee model with boundary as the $(1, 3)$ perturbation of the superconformal minimal model $\mathcal{SM}(2/8)$ (or equivalently, the $(1, 5)$ perturbation of the conformal minimal model $\mathcal{M}(3/8)$) with a certain conformal boundary condition. We propose the corresponding boundary $S$ matrix, which is not diagonal for general values of the boundary parameter. We argue that the model has an integral of motion corresponding to an unbroken supersymmetry, and that the proposed $S$ matrix commutes with a similar quantity. We also show by means of a boundary TBA analysis that the proposed boundary $S$ matrix is consistent with massless flow away from the ultraviolet conformal boundary condition. © 2001 Elsevier Science B.V. All rights reserved.

## 1. Introduction

A $(1 + 1)$-dimensional massive integrable quantum field theory without boundary (i.e., on the full line $x \in (-\infty, \infty)$) is characterized by its factorizable bulk scattering $(S)$ matrix [1]. It can also be characterized as a perturbation [2] of a bulk conformal field theory (CFT) [3]. For example, a perturbed minimal model is the renormalization group infrared trivial fixed point of the action

$$A = A_{\text{CFT}} + \lambda \int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{\infty} dx \; \Phi_{(\Delta, \Delta)}(x, y), \tag{1.1}$$

where $A_{\text{CFT}}$ is the action of a $c < 1$ minimal model $\mathcal{M}(p/q)$, $\Phi_{(\Delta, \Delta)}$ is a spinless degenerate primary field with (right, left) conformal dimensions $(\Delta, \Delta)$ which is relevant

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($\Delta < 1$) and "integrable", and $\lambda$ is a parameter of dimension [length]$^{2\Delta-2}$. One link between these two descriptions is provided by the thermodynamic Bethe Ansatz (TBA), by means of which the central charge of the CFT can be computed from the $S$ matrix [4,5]. The integer-spin and fractional-spin [6,7] integrals of motion of an integrable field theory are manifested in both its $S$ matrix and perturbed CFT descriptions. These features of integrable field theory are by now relatively well understood, due to the great number of examples which have been worked out in detail. (See, e.g., [8] and references therein.)

For an integrable field theory with boundary (say, on the half-line $x \in (-\infty, 0]$), the above framework has a nontrivial generalization [9]. The theory is characterized by a factorizable *boundary* scattering matrix, together with the bulk $S$ matrix. It can also be described as a perturbation of a boundary CFT. The boundary generalization of (1.1) is given by

$$A = A_{\text{CFT+CBC}} + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx\, \Phi_{(\Delta,\Delta)}(x,y) + \lambda_B \int_{-\infty}^{\infty} dy\, \Phi_{(\Delta)}(y). \tag{1.2}$$

The boundary CFT is specified [10] by a conformal boundary condition (CBC), which for $c < 1$ minimal models corresponds to a cell $(n, m)$ of the Kac table. A CBC is also characterized by the so-called boundary entropy or ground-state degeneracy ($g$) factor [11], which (roughly speaking) is a measure of the number of bulk vacua which are compatible with a given CBC. This is well illustrated in the unitary minimal models [9,12]. As can be seen from (1.2), the boundary CFT in general has perturbations by both bulk ($\Phi_{(\Delta,\Delta)}$) and boundary ($\Phi_{(\Delta)}$) relevant primary fields. The boundary parameter $\lambda_B$ has dimension [length]$^{\Delta-1}$. Note that the boundary perturbation has the same conformal dimension as the bulk perturbation, and therefore, presumably it is integrable [9]. Furthermore, the CBC and the boundary perturbation must be compatible [9,10]. By means of a "boundary TBA" [13–15], ratios of $g$ factors of the boundary CFT can be computed from the bulk and boundary $S$ matrices. (See also [16,17].) Fractional-spin integrals of motion should be manifested in both the boundary $S$ matrix and the perturbed CFT descriptions [18]. These features of integrable field theory with boundary have been studied in relatively few examples and are less well understood, in comparison to the case without boundary.

In an effort to provide more such examples, we consider here the boundary version of the bulk scaling supersymmetric Yang–Lee (SYL) model [19–21]. This model is arguably the simplest nontrivial supersymmetric quantum field theory. Its spectrum consists of one Boson and one Fermion of equal mass, and the bulk $S$ matrix is factorizable and has $N = 1$ supersymmetry. This model is the supersymmetric generalization of the scaling Yang–Lee (YL) model [4,22,23], which describes the scaling region near the Yang–Lee singularity of the two-dimensional Ising model [24,25]. The SYL model is the first member of an infinite family of integrable models with $N = 1$ supersymmetry [19].

In particular, we define the boundary SYL model as a perturbed boundary CFT, and we propose the corresponding boundary $S$ matrix, which is not diagonal for general values of the boundary parameter. We support this picture by identifying a supersymmetry-like integral of motion, and by studying massless boundary flow using the boundary TBA.

Some related work was done by Moriconi and Schoutens in [26]. These authors proposed two *diagonal* boundary $S$ matrices for the boundary SYL model, without reference to any specific boundary conditions. For a special value of the boundary parameter, our boundary $S$ matrix differs from one of theirs by a CDD factor.

The outline of this article is as follows. In Section 2, we briefly review some necessary results about the YL model, and we clarify a few subtleties of the boundary TBA. In Section 3, we review some necessary results about the bulk SYL model. We also recall the useful observation [27] that the critical SYL model can be formulated as either the superconformal minimal model $\mathcal{SM}(2/8)$ or the conformal minimal model $\mathcal{M}(3/8)$. This is completely analogous to the well-known fact that the tricritical Ising model can be formulated as either $\mathcal{SM}(3/5)$ or $\mathcal{M}(4/5)$. One consequence of this fact is that the SYL model can be regarded, following [28,29], as a restriction of the ZMS model [30–32], as we discuss in an appendix. Section 4 is the heart of the paper. There we first define the boundary SYL model as a perturbed boundary CFT, and we argue that it has an integral of motion corresponding to an unbroken supersymmetry. We then propose the boundary $S$ matrix for the boundary SYL model. Our approach is to restrict the boundary $S$ matrix of the boundary supersymmetric sinh-Gordon model [15], by imposing the various boundary bootstrap constraints [9]. We then show that the proposed boundary $S$ matrix commutes with a supersymmetry-like charge. Finally, we perform a boundary TBA analysis, and show that the proposed boundary $S$ matrix is consistent with massless flow away from the ultraviolet conformal boundary condition. In Section 5 we present a brief discussion of our results.

## 2. The YL model

We now briefly recall the basic results of the scaling Yang–Lee model which we shall need in subsequent sections to formulate the supersymmetric generalization. We also clarify a few subtleties of the boundary TBA.

### 2.1. Bulk

The critical behavior of the Yang–Lee singularity is described [33] by the minimal model $\mathcal{M}(2/5)$. This is a (nonunitary) CFT with central charge $c = -22/5$. There are only two irreducible representations of the Virasoro algebra, and the corresponding conformal dimensions $\Delta_{(n,m)}$ of the primary fields are organized into a Kac table in Table 1.

The scaling Yang–Lee model (without boundary) is defined [22] by the perturbed action (1.1), where the CFT is $\mathcal{M}(2/5)$, and $\Delta = \Delta_{(1,3)} = -1/5$. Arguments developed by

Table 1
Kac table for $\mathcal{M}(2/5)$

| | | | |
|---|---|---|---|
| $0$ | $-\frac{1}{5}$ | $-\frac{1}{5}$ | $0$ |

Zamolodchikov [2] imply that this model is integrable. The spectrum consists of a single particle of mass $m$, with energy $E = m \cosh\theta$ and momentum $P = m \sinh\theta$, where $\theta$ is the rapidity. The two-particle $S$ matrix for particles with rapidities $\theta_1$ and $\theta_2$ is given by [22]

$$S_{\text{YL}}(\theta) = \frac{\sinh\theta + i \sin\left(\frac{2\pi}{3}\right)}{\sinh\theta - i \sin\left(\frac{2\pi}{3}\right)}, \tag{2.1}$$

where $\theta = \theta_1 - \theta_2$. This $S$ matrix has a direct (s) channel pole at $\theta = i2\pi/3$, since the particle is a bound state of itself. Hence, the $S$ matrix obeys the bootstrap equation

$$S_{\text{YL}}\left(\theta + \frac{i\pi}{3}\right) S_{\text{YL}}\left(\theta - \frac{i\pi}{3}\right) = S_{\text{YL}}(\theta). \tag{2.2}$$

The TBA analysis [4] demonstrates that this $S$ matrix correctly reproduces the central charge of the unperturbed CFT. The YL model can be regarded [23] as a restriction of the sine-Gordon model in which the solitons are projected out and only the first breather remains. Indeed, the $S$ matrix (2.1) coincides with that of the first sine-Gordon breather [1, 34], with $\gamma = 16\pi/3$.

## 2.2. Boundary

Following [14,35], we consider the boundary YL model which is defined by the perturbed action (1.2), where the CFT is $\mathcal{M}(2/5)$, the CBC corresponds to the cell $(1, 3)$ of the Kac table, and $\Delta = \Delta_{(1,3)} = -1/5$. The $(1, 3)$ conformal boundary condition and the $(1, 3)$ boundary perturbation are compatible, since the fusion rule coefficient $N_{(1,3)(1,3)}^{(1,3)}$ is nonvanishing. The boundary $S$ matrix $\mathsf{S}_{\text{YL}}(\theta; b)$ is given by [35] [1]

$$\mathsf{S}_{\text{YL}}(\theta; b) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{4}{2}\right)^{-1}\left(\frac{1-b}{2}\right)^{-1}\left(\frac{1+b}{2}\right)\left(\frac{5-b}{2}\right)\left(\frac{5+b}{2}\right)^{-1}, \tag{2.3}$$

where

$$(x) \equiv \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi x}{6}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi x}{6}\right)}, \tag{2.4}$$

and $b$ is a parameter which is related to $\lambda_B$. This $S$ matrix obeys the boundary bootstrap equation [9]

$$\mathsf{S}_{\text{YL}}\left(\theta + \frac{i\pi}{3}; b\right) S_{\text{YL}}(2\theta) \mathsf{S}_{\text{YL}}\left(\theta - \frac{i\pi}{3}; b\right) = \mathsf{S}_{\text{YL}}(\theta; b). \tag{2.5}$$

This model can be regarded as a restriction of the boundary sine-Gordon model. Indeed, the boundary $S$ matrix (2.3) coincides with that of the first sine-Gordon breather [36] with $\gamma = 16\pi/3$, and with the parameters $\eta$, $\vartheta$ of [9] taking the values [14] $\eta = \frac{\pi}{4}(b+4)$, $i\vartheta = \frac{\pi}{4}(b+2)$.

This picture is supported by the boundary TBA, which implies that the boundary entropy is given (up to an additive constant) by

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[1] We make an effort to distinguish boundary quantities from the corresponding bulk quantities by using sans serif letters to denote the former, and Roman letters to denote the latter.

$$\ln g = \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} d\theta \left[ \kappa_{\mathrm{YL}}(\theta; b) - \Phi_{\mathrm{YL}}(2\theta) - \frac{1}{2}\Phi_{\mathrm{YL}}(\theta) \right] L(\theta), \tag{2.6}$$

where

$$\Phi_{\mathrm{YL}}(\theta) = \frac{1}{i}\frac{\partial}{\partial\theta}\ln S_{\mathrm{YL}}(\theta), \qquad \kappa_{\mathrm{YL}}(\theta; b) = \frac{1}{i}\frac{\partial}{\partial\theta}\ln S_{\mathrm{YL}}(\theta; b), \tag{2.7}$$

and

$$L(\theta) = \ln\left(1 + e^{-\epsilon(\theta)}\right). \tag{2.8}$$

Moreover, $\epsilon(\theta)$ is the solution of the bulk TBA equation [4]

$$\epsilon(\theta) = r\cosh\theta - \frac{1}{2\pi}(\Phi_{\mathrm{YL}} * L)(\theta), \tag{2.9}$$

where $*$ denotes convolution

$$(f * g)(\theta) = \int\limits_{-\infty}^{\infty} d\theta' f(\theta - \theta')g(\theta'), \tag{2.10}$$

and $r = mR$, with $R$ the inverse temperature. Note that our expression (2.6) for the boundary entropy differs in the third term in the brackets from the one given in Refs. [13] and [14]. This term originates from the exclusion [37,38] of the Bethe Ansatz root at zero rapidity.

For simplicity, let us consider the case of massless boundary flow.[2] That is, we consider the bulk massless scaling limit

$$m = \mu n, \qquad \theta = \hat{\theta} \mp \ln\frac{n}{2}, \quad n \to 0, \tag{2.11}$$

where $\mu$ and $\hat{\theta}$ are finite, which implies $E = \mu e^{\pm\hat{\theta}}$, $P = \pm\mu e^{\pm\hat{\theta}}$. Moreover, we consider

$$b = -3 - \frac{i6}{\pi}\left(\theta_B - \ln\frac{n}{2}\right), \quad n \to 0, \tag{2.12}$$

where the boundary scale $\theta_B$ is finite. For the sign $-$ in the limit (2.11), the boundary $S$ matrix reduces to $S(\hat{\theta} - \theta_B)^{-1}$ [14], and we obtain

$$\ln g = -\frac{2}{4\pi} \int\limits_{-\infty}^{\infty} d\hat{\theta}\, \Phi_{\mathrm{YL}}(\hat{\theta} - \theta_B)\hat{L}(\hat{\theta}), \tag{2.13}$$

where $\hat{\epsilon}(\hat{\theta}) \equiv \epsilon(\hat{\theta} - \ln\frac{n}{2})$, and $\hat{L}(\hat{\theta}) = \ln(1 + e^{-\hat{\epsilon}(\hat{\theta})})$. Note the factor of 2 appearing in (2.13), which accounts for the contribution from the sign $+$ in the limit (2.11). That is, it can be shown that right-movers and left-movers give equal contributions to the boundary entropy. In the UV limit $\theta_B \to -\infty$, the integrand is nonvanishing for $\hat{\theta} \to -\infty$; similarly, the IR limit $\theta_B \to \infty$ requires $\hat{\theta} \to \infty$. Using the results $\hat{L}(-\infty) = \ln(\frac{1+\sqrt{5}}{2})$, $\hat{L}(\infty) = 0$ which follow from the TBA equation (2.9), we obtain

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[2] The bulk-massive case seems to have several complicated issues which remain to be resolved [14].

$$\ln \frac{g^{\mathrm{UV}}}{g^{\mathrm{IR}}} = \ln\left(\frac{1+\sqrt{5}}{2}\right). \tag{2.14}$$

This is precisely the ratio of $g$ factors corresponding to the conformal boundary conditions $(1, 3)$ and $(1, 1)$

$$\ln \frac{g_{(1,3)}}{g_{(1,1)}} = \ln\left(\frac{1+\sqrt{5}}{2}\right), \tag{2.15}$$

which have been computed [14] from the $\mathcal{M}(2/5)$ modular $S$ matrix. Hence, the boundary $S$ matrix (2.3) is consistent with massless flow away from the UV conformal boundary condition; namely, from the CBC $(1, 3)$ to the CBC $(1, 1)$. In Section 4.3 we shall find a generalization to the supersymmetric case.

## 3. The bulk SYL model

We turn now to the supersymmetric generalization of the scaling Yang–Lee model, which was first defined in [19] as a perturbation of the superconformal minimal model $\mathcal{SM}(2/8)$. This (nonunitary) CFT has central charge $c = -21/4$; and the corresponding dimensions $\Delta_{(n,m)}$ of the primary superconformal fields are given in Table 2. These fields are of Neveu–Schwarz (NS) or Ramond (R) type if $n - m$ is even or odd, respectively. We recall [3] that the superconformal symmetry is generated by the right and left supercurrents $G(z)$ and $\overline{G}(\bar{z})$ of dimensions $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$, respectively. The NS fields are local with respect to $G(z)$ and $\overline{G}(\bar{z})$, while the R fields are semi-local with respect to these currents.

The action of the SYL model is given by [19]

$$A = A_{\mathcal{SM}(2/8)} + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, G_{-\frac{1}{2}} \overline{G}_{-\frac{1}{2}} \Phi_{(\Delta,\Delta)}(x, y), \tag{3.1}$$

where $\Delta = \Delta_{(1,3)} = -\frac{1}{4}$, and $G_n$ $(\overline{G}_n)$ are operators appearing in the operator expansion of the supercurrent $G(z)$ $(\overline{G}(\bar{z}))$ with $\Phi_{(\Delta,\Delta)}(z, \bar{z})$. An interesting feature of this model is that it has fractional $(\frac{1}{2})$ spin integrals of motion. Indeed, the perturbation preserves supersymmetry, since [6,19]

$$\partial_{\bar{z}} G = \partial_z \overline{\Psi}, \ \ \overline{\Psi} = \lambda(2\Delta - 1)\overline{G}_{-\frac{1}{2}} \Phi_{(\Delta,\Delta)},$$
$$\partial_z \overline{G} = \partial_{\bar{z}} \Psi, \ \ \Psi = \lambda(2\Delta - 1) G_{-\frac{1}{2}} \Phi_{(\Delta,\Delta)}. \tag{3.2}$$

The corresponding integrals of motion are given by

Table 2
Kac table for $\mathcal{SM}(2/8)$

| 0 | $-\frac{3}{32}$ | $-\frac{1}{4}$ | $-\frac{7}{32}$ | $-\frac{1}{4}$ | $-\frac{3}{32}$ | 0 |
|---|---|---|---|---|---|---|

Table 3
Kac table for $\mathcal{M}(3/8)$

| $\frac{3}{2}$ | $\frac{25}{32}$ | $\frac{1}{4}$ | $-\frac{3}{32}$ | $-\frac{1}{4}$ | $-\frac{7}{32}$ | $0$ |
|---|---|---|---|---|---|---|
| $0$ | $-\frac{7}{32}$ | $-\frac{1}{4}$ | $-\frac{3}{32}$ | $\frac{1}{4}$ | $\frac{25}{32}$ | $\frac{3}{2}$ |

$$Q = \int_{-\infty}^{\infty} dx [G(x, y) + \overline{\Psi}(x, y)],$$

$$\overline{Q} = \int_{-\infty}^{\infty} dx [\overline{G}(x, y) + \Psi(x, y)]. \tag{3.3}$$

We now recall the important observation [27] that there is an equivalent formulation of the SYL model as a perturbation of the ordinary minimal model $\mathcal{M}(3/8)$.[3] Indeed, $\mathcal{M}(3/8)$ also has central charge $c = -21/4$. The corresponding dimensions of the primary fields are given in Table 3. Note that these dimensions either coincide with those for $\mathcal{SM}(2/8)$ or else correspond to their super-descendants. Indeed, the fields of dimension $\frac{1}{4}$ and $\frac{3}{2}$ correspond to $G_{-\frac{1}{2}}\Phi_{(1,3)}$ and $G_{-\frac{1}{2}}L_{-1}\Phi_{(1,1)}$ respectively; and the field of dimension $\frac{25}{32}$ corresponds to $G_{-1}\Phi_{(1,4)}$.

The SYL model can therefore also be formulated by the action (1.1), where the CFT is the minimal model $\mathcal{M}(3/8)$, and $\Delta = \Delta_{(1,5)} = \frac{1}{4}$. This is an integrable perturbation, since [39] the $(1, 5)$ perturbation of $\mathcal{M}(p/q)$ is integrable if $2p < q$. There is a corresponding formulation of the conservation laws (3.2), with the supercurrents $G$ and $\overline{G}$ replaced by the chiral primary fields $\Phi_{(2,1),(1,1)}$ and $\Phi_{(1,1),(2,1)}$ respectively, etc.

The spectrum of the SYL model consists of one Boson and one Fermion of equal mass $m$. Following [1,9], it is convenient to introduce the Zamolodchikov operators $A_a(\theta) = \binom{b(\theta)}{f(\theta)}$ which create the corresponding Boson and Fermion asymptotic particle states,

$$|A_{a_1}(\theta_1) A_{a_2}(\theta_2) \cdots A_{a_N}(\theta_N)\rangle = A_{a_1}(\theta_1) A_{a_2}(\theta_2) \cdots A_{a_N}(\theta_N)|0\rangle. \tag{3.4}$$

This is an "in state" or "out state" if the rapidities are ordered as $\theta_1 > \theta_2 > \cdots > \theta_N$ or $\theta_1 < \theta_2 < \cdots < \theta_N$, respectively.

The two-particle $S$ matrix is defined by

$$A_{a_1}(\theta_1) A_{a_2}(\theta_2) = S_{a_1 a_2}^{b_1 b_2}(\theta_1 - \theta_2) A_{b_2}(\theta_2) A_{b_1}(\theta_1). \tag{3.5}$$

For the SYL model, the $S$ matrix is given by [19]

$$S(\theta) = S_{\text{YL}}(\theta) S_{\text{SUSY}}(\theta), \tag{3.6}$$

where $S_{\text{YL}}(\theta)$ is given by (2.1). Moreover,

$$S_{\text{SUSY}}(\theta) = Y(\theta) R(\theta), \tag{3.7}$$

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[3] As mentioned in introduction, this is completely analogous to the well-known fact that the tricritical Ising model can be formulated as either $\mathcal{SM}(3/5)$ or $\mathcal{M}(4/5)$.

where $R(\theta)$ is the $4 \times 4$ matrix [4]

$$
R(\theta) = \begin{pmatrix} a_+(\theta) & 0 & 0 & d(\theta) \\ 0 & b & c(\theta) & 0 \\ 0 & c(\theta) & b & 0 \\ d(\theta) & 0 & 0 & a_-(\theta) \end{pmatrix},
\tag{3.8}
$$

with

$$
a_\pm(\theta) = \pm 1 + \frac{2i \sin \frac{\pi}{3}}{\sinh \theta}, \quad b = 1, \quad c = \frac{i \sin \frac{\pi}{3}}{\sinh \frac{\theta}{2}}, \quad d = \frac{\sin \frac{\pi}{3}}{\cosh \frac{\theta}{2}}.
\tag{3.9}
$$

The scalar factor $Y(\theta)$ is given by

$$
Y(\theta) = \frac{\sinh \frac{\theta}{2}}{\sinh \frac{\theta}{2} + i \sin \frac{\pi}{3}} \exp\left( \int_0^\infty \frac{dt}{t} \frac{\sinh(it\theta/\pi) \sinh \frac{2t}{3} \sinh \frac{t}{3}}{\cosh t \cosh^2 \frac{t}{2}} \right),
\tag{3.10}
$$

which we find has the following infinite-product representation:

$$
\begin{aligned}
Y(\theta) = {}& \frac{\Gamma\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}\right)\Gamma\left(1 + \frac{i\theta}{2\pi}\right)} \\
&\times \prod_{k=0}^\infty \left\{ \frac{\Gamma\left(\frac{3}{2} + k - \frac{i\theta}{2\pi}\right)^2 \Gamma\left(1 + k + \frac{i\theta}{2\pi}\right)^2}{\Gamma\left(\frac{1}{2} + k + \frac{i\theta}{2\pi}\right)^2 \Gamma\left(1 + k - \frac{i\theta}{2\pi}\right)^2} \right. \\
&\times \left. \frac{\Gamma\left(\frac{2}{3} + k - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{5}{6} + k + \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{3} + k - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{7}{6} + k + \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{5}{3} + k + \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{5}{6} + k - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{4}{3} + k + \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{7}{6} + k - \frac{i\theta}{2\pi}\right)} \right\}.
\end{aligned}
\tag{3.11}
$$

It is convenient to denote the total scalar factor by $Z(\theta)$

$$
\begin{aligned}
Z(\theta) &= S_{\mathrm{YL}}(\theta)\, Y(\theta) \\
&= \frac{\sinh \frac{\theta}{2}}{\sinh \frac{\theta}{2} - i \sin \frac{\pi}{3}} \exp\left( -\int_0^\infty \frac{dt}{t} \frac{\sinh(it\theta/\pi) \sinh \frac{4t}{3} \sinh \frac{t}{3}}{\cosh t \cosh^2 \frac{t}{2}} \right).
\end{aligned}
\tag{3.12}
$$

Hence, the SYL bulk $S$ matrix is given by

$$
S(\theta) = Z(\theta) R(\theta),
\tag{3.13}
$$

where the matrix $R(\theta)$ is given by Eqs. (3.8), (3.9). TBA analysis [20,21] shows that this $S$ matrix correctly reproduces the central charge of the unperturbed CFT.

In analogy with the YL model, the SYL model can be regarded as a restriction of the supersymmetric sine-Gordon (SSG) model in which the solitons are projected out and only the first breather multiplet remains. Indeed, the $S$ matrix is that of the first SSG

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[4] Our conventions are such that if $A$ and $B$ are matrices with matrix elements $A_{a_1 a_2}$ and $B_{b_1 b_2}$, then the tensor product $C = A \otimes B$ has matrix elements $C_{a_1 b_1}^{a_2 b_2} = A_{a_1 a_2} B_{b_1 b_2}$.

breather [40,41] with $\alpha = 1/3$. In particular, it coincides with the expression for the $S$ matrix of the supersymmetric sinh-Gordon model given in [15] with $B = -1/3$.

In view of the alternative formulation of SYL as the $(1, 5)$ perturbation of $\mathcal{M}(3/8)$, the SYL model can also be regarded as a restriction [28,29] of the Zhiber–Mikhailov–Shabat model [30–32]. Details of this identification are given in Appendix A.

We recall [6,19] that the supersymmetry charges are assumed to act as follows: on one-particle states,

$$QA_a(\theta) = q_{ab}(\theta)A_b(\theta), \qquad q(\theta) = \sqrt{m}\, e^{\frac{\theta}{2}} \begin{pmatrix} 0 & e^{\frac{i\pi}{4}} \\ e^{-\frac{i\pi}{4}} & 0 \end{pmatrix},$$

$$\overline{Q}A_a(\theta) = \bar{q}_{ab}(\theta)A_b(\theta), \qquad \bar{q}(\theta) = \sqrt{m}\, e^{-\frac{\theta}{2}} \begin{pmatrix} 0 & e^{-\frac{i\pi}{4}} \\ e^{\frac{i\pi}{4}} & 0 \end{pmatrix} \tag{3.14}$$

and on multiparticle states,

$$Q|A_{a_1}(\theta_1)\cdots A_{a_N}(\theta_N)\rangle$$
$$= \sum_{l=1}^{N}\left(\prod_{k=1}^{l-1}(-1)^{F_{a_k}}\right)|A_{a_1}(\theta_1)\cdots A_{a_{l-1}}(\theta_{l-1})(QA_{a_l}(\theta_l))A_{a_{l+1}}(\theta_{l+1})\cdots A_{a_N}(\theta_N)\rangle,$$

$$\overline{Q}|A_{a_1}(\theta_1)\cdots A_{a_N}(\theta_N)\rangle$$
$$= \sum_{l=1}^{N}\left(\prod_{k=1}^{l-1}(-1)^{F_{a_k}}\right)|A_{a_1}(\theta_1)\cdots A_{a_{l-1}}(\theta_{l-1})(\overline{Q}A_{a_l}(\theta_l))A_{a_{l+1}}(\theta_{l+1})\cdots A_{a_N}(\theta_N)\rangle, \tag{3.15}$$

where $(-1)^F$ is $+1$ for a Boson and $-1$ for a Fermion. These charges obey the supersymmetry algebra

$$Q^2 = E + P, \quad \overline{Q}^2 = E - P,$$
$$\{Q, \overline{Q}\} = 0, \quad \{Q, (-1)^F\} = \{\overline{Q}, (-1)^F\} = 0. \tag{3.16}$$

It can be shown [19] that the SYL $S$ matrix commutes with the supersymmetry charges $Q$ and $\overline{Q}$, as well as with $(-1)^F$.

To conclude this section, we demonstrate that the above $S$ matrix satisfies the bulk bootstrap equations. We do this in preparation for our investigation in Section 4.2 of the boundary bootstrap equations, which will help determine the boundary $S$ matrix. Near the direct-channel pole at $\theta = i2\pi/3$, the bulk $S$ matrix is given by

$$S(\theta) \simeq -\frac{i\sqrt{3}c^2}{\theta - \frac{i2\pi}{3}} \begin{pmatrix} 3 & 0 & 0 & \sqrt{3} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \sqrt{3} & 0 & 0 & 1 \end{pmatrix}, \tag{3.17}$$

where

$$c = \exp\left(-\frac{1}{2}\int_0^\infty \frac{dt}{t}\frac{\sinh^2(2t/3)\sinh(t/3)}{\cosh t \cosh^2(t/2)}\right).$$

Hence, the nonvanishing three-particle couplings are given by

$$f_{bb}^{b} = ic\sqrt{3\sqrt{3}}, \qquad f_{ff}^{b} = f_{bf}^{f} = f_{fb}^{f} = ic\sqrt[4]{3}, \tag{3.18}$$

where $b$ and $f$ denote Boson and Fermion, respectively. Using the infinite-product representation for the scalar factor $Y(\theta)$ (3.11), one can prove the identity

$$\frac{Y\left(\theta + \frac{i\pi}{3}\right)Y\left(\theta - \frac{i\pi}{3}\right)}{Y(\theta)} = \frac{2\sinh\left(\frac{\theta}{2} - \frac{i\pi}{6}\right)\cosh\left(\frac{\theta}{2} + \frac{i\pi}{6}\right)}{\sinh\theta}. \tag{3.19}$$

Recalling the YL bootstrap relation (2.2), it follows that the total scalar factor $Z(\theta)$ (3.12) satisfies

$$\frac{Z\left(\theta + \frac{i\pi}{3}\right)Z\left(\theta - \frac{i\pi}{3}\right)}{Z(\theta)} = \frac{2\sinh\left(\frac{\theta}{2} - \frac{i\pi}{6}\right)\cosh\left(\frac{\theta}{2} + \frac{i\pi}{6}\right)}{\sinh\theta}. \tag{3.20}$$

With the help of this identity, it is now straightforward to verify the bulk bootstrap equations

$$f_{a_1 a_2}^{c} S_{ca_3}^{bb_3}(\theta) = f_{c_1 c_2}^{b} S_{a_1 c_3}^{c_1 b_3}\left(\theta + \frac{i\pi}{3}\right) S_{a_2 a_3}^{c_2 c_3}\left(\theta - \frac{i\pi}{3}\right). \tag{3.21}$$

## 4. The boundary SYL model

We now address the main problems of defining the boundary SYL model and determining its boundary $S$ matrix.

### 4.1. Definition of the model as a perturbed CFT

As in the bulk case, we can define the boundary SYL model in either of two ways. One way is to define the model as a perturbation of the superconformal minimal model $\mathcal{SM}(2/8)$ (cf., Eq. (3.1))

$$A = A_{\mathcal{SM}(2/8)+\text{SCBC}(1,3)} + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx\, G_{-\frac{1}{2}}\overline{G}_{-\frac{1}{2}}\Phi_{(\Delta,\Delta)}(x,y)$$

$$+ \lambda_B \int_{-\infty}^{\infty} dy\, G_{-\frac{1}{2}}\Phi_{(\Delta)}(y), \tag{4.1}$$

where $\Delta = \Delta_{(1,3)} = -\frac{1}{4}$. Indeed, the arguments of [9] suggest that this boundary perturbation is integrable. Following [10], we observe that for the boundary CFT, superconformal invariance requires that the stress-energy tensors and supercurrents obey the boundary conditions

$$\left(T - \overline{T}\right)\big|_{x=0} = 0, \qquad \left(G - \overline{G}\right)\big|_{x=0} = 0. \tag{4.2}$$

We assume that for a superconformal minimal model, a superconformal boundary condition (SCBC) corresponds to a cell of the Kac table, which in (4.1) we take to be (1, 3). (See below.)

Although for the case with boundary the supersymmetry charges

$$Q = \int_{-\infty}^{0} dx \big[ G(x, y) + \overline{\Psi}(x, y) \big], \qquad \overline{Q} = \int_{-\infty}^{0} dx \big[ \overline{G}(x, y) + \Psi(x, y) \big] \qquad (4.3)$$

(cf., Eq. (3.3)) are not conserved, it is plausible that some combination of these charges (plus a possible boundary term) survives. Indeed, following [9], let us first consider the massless case $\lambda = 0$, and compute the operator product expansion $[G(y + ix) - G(y - ix)]G_{-\frac{1}{2}}\Phi_{(\Delta)}(y')$. We conclude that the quantity

$$\mathsf{Q} = \int_{-\infty}^{0} dx \big[ G(x, y) + \overline{G}(x, y) \big] + \Theta(y), \qquad (4.4)$$

with $\Theta(y) \propto \lambda_B (1 - 2\Delta)\Phi_{(\Delta)}(y)$ is an integral of motion. It is plausible that, for the general massive case $\lambda \neq 0$, this becomes

$$\mathsf{Q} = Q + \overline{Q} + \Theta, \qquad (4.5)$$

where $Q$ and $\overline{Q}$ are given in (4.3).

Alternatively, we can define the boundary SYL model as a perturbation of the minimal model $\mathcal{M}(3/8)$. That is, we can define the model by the action (1.2), where the CFT is $\mathcal{M}(3/8)$, $\Delta = \Delta_{(1,5)} = \frac{1}{4}$, and the CBC is either $(1, 3)$, $(1, 4)$, or $(1, 5)$. Indeed, these three conformal boundary conditions are compatible with the $(1, 5)$ boundary perturbation, since the corresponding fusion rule coefficients $N^{(1,5)}_{(1,3)\,(1,3)}$, $N^{(1,5)}_{(1,4)\,(1,4)}$ and $N^{(1,5)}_{(1,5)\,(1,5)}$ are all nonvanishing, as can be seen from Table 4. Presumably, only the CBC $(1, 3)$ preserves superconformal invariance, since only for this CBC does the corresponding dimension $\Delta_{(1,3)} = -\frac{1}{4}$ appear in the $\mathcal{SM}(2/8)$ Kac Table 2. Hence, here we shall consider only the CBC $(1, 3)$, for which case the corresponding action is presumably equivalent to (4.1).

We have obtained the $\mathcal{M}(3/8)$ fusion rule coefficients given in Table 4 using the corresponding modular $S$ matrix. Indeed, we recall (see, e.g., [42]) that for $\mathcal{M}(p/q)$ the modular $S$ matrix elements are given by

Table 4
Fusion rule coefficients for $\mathcal{M}(3/8)$. Here we list all the triplets $(i, j, k)$ with $i \leqslant j \leqslant k$ for which $N^{(1,i)}_{(1,j)(1,k)}$ is nonvanishing, and in fact, equal to 1. Note that $N^{(1,i)}_{(1,j)(1,k)}$ is symmetric under the interchange of any pair of indices $(i, j, k)$

| | | | | |
|---|---|---|---|---|
| (1, 1, 1) | (2, 2, 3) | (3, 3, 3) | (4, 4, 5) | (5, 5, 5) |
| (1, 2, 2) | (2, 3, 4) | (3, 3, 5) | (4, 4, 7) | |
| (1, 3, 3) | (2, 4, 5) | (3, 4, 4) | (4, 5, 6) | |
| (1, 4, 4) | (2, 5, 6) | (3, 4, 6) | | |
| (1, 5, 5) | (2, 6, 7) | (3, 5, 5) | | |
| (1, 6, 6) | | (3, 5, 7) | | |
| (1, 7, 7) | | (3, 6, 6) | | |

$$S_{(r,s)\,(r',s')} = 2\sqrt{\frac{2}{pq}}\,(-1)^{rs'+r's+1}\sin\frac{\pi q r r'}{p}\sin\frac{\pi p s s'}{q},$$

$$1 \leqslant r, r' \leqslant p-1, \qquad 1 \leqslant s, s' \leqslant q-1. \tag{4.6}$$

Setting $r = r' = 1$, for $\mathcal{M}(3/8)$ we obtain the result

$$S = \begin{pmatrix}
-\frac{1}{2}\sin\frac{3\pi}{8} & \frac{1}{2\sqrt{2}} & \frac{1}{2}\sin\frac{\pi}{8} & -\frac{1}{2} & \frac{1}{2}\sin\frac{\pi}{8} & \frac{1}{2\sqrt{2}} & -\frac{1}{2}\sin\frac{3\pi}{8} \\
\frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2\sqrt{2}} \\
\frac{1}{2}\sin\frac{\pi}{8} & \frac{1}{2\sqrt{2}} & \frac{1}{2}\sin\frac{3\pi}{8} & \frac{1}{2} & \frac{1}{2}\sin\frac{3\pi}{8} & \frac{1}{2\sqrt{2}} & \frac{1}{2}\sin\frac{\pi}{8} \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2}\sin\frac{\pi}{8} & -\frac{1}{2\sqrt{2}} & \frac{1}{2}\sin\frac{3\pi}{8} & -\frac{1}{2} & \frac{1}{2}\sin\frac{3\pi}{8} & -\frac{1}{2\sqrt{2}} & \frac{1}{2}\sin\frac{\pi}{8} \\
\frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2} & -\frac{1}{2\sqrt{2}} \\
-\frac{1}{2}\sin\frac{3\pi}{8} & -\frac{1}{2\sqrt{2}} & \frac{1}{2}\sin\frac{\pi}{8} & \frac{1}{2} & \frac{1}{2}\sin\frac{\pi}{8} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2}\sin\frac{3\pi}{8}
\end{pmatrix}, \tag{4.7}$$

where the matrix element $(s, s')$ corresponds to $S_{(1,s)\,(1,s')}$. This matrix is real, symmetric, and unitary, $S\,S^\dagger = S^2 = \mathbb{I}$. Finally, the Verlinde formula [44] implies that the fusion rule coefficients are given by

$$N_{(1,j)(1,k)}^{(1,i)} = \sum_{l=1}^{7} \frac{S_{(1,i)(1,l)} S_{(1,j)(1,l)} S_{(1,k)(1,l)}}{S_{(1,1)(1,l)}}. \tag{4.8}$$

We close this subsection with the computation of $g$ factors for the various conformal boundary conditions, which also relies on the modular $S$ matrix. As shown in [10,14], the $g$ factor for the CBC $(1, s)$ is given by

$$g_{(1,s)} = \frac{S_{\Omega\,(1,s)}}{\sqrt{|S_{\Omega\,0}|}}, \tag{4.9}$$

where 0 denotes the conformal vacuum (which has the property $N_{0\,(1,j)}^{(1,i)} = \delta_j^i$), and $\Omega$ is the state of lowest dimension. For $\mathcal{M}(3/8)$, 0 is $(1, 1)$ and $\Omega$ is $(1, 3)$. In this way, we obtain

$$g_{(1,4)} = \frac{1}{\sqrt{2\sin\frac{\pi}{8}}}, \quad g_{(1,3)} = g_{(1,5)} = \frac{\sin\frac{3\pi}{8}}{\sqrt{2\sin\frac{\pi}{8}}},$$

$$g_{(1,2)} = g_{(1,6)} = \frac{1}{2\sqrt{\sin\frac{\pi}{8}}}, \quad g_{(1,1)} = g_{(1,7)} = \sqrt{\frac{1}{2}\sin\frac{\pi}{8}}. \tag{4.10}$$

It should also be possible to compute $g$ factors from the $\mathcal{SM}(2/8)$ modular $S$ matrix. However, we do not attempt this here. [5]

### 4.2. Boundary S matrix

The boundary $S$ matrix $\mathsf{S}(\theta)$ is defined as [9]

---

[5] It is not clear how to compute the $\mathcal{SM}(2/8)$ modular $S$ matrix directly from the coset $su(2)_2 \oplus su(2)_m/su(2)_{2+m}$ with $m = -4/3$ [43], since an additional coset field seems to be required.

$$A_a(\theta)B = S_a^b(\theta)A_b(-\theta)B, \tag{4.11}$$

where here $B$ is the so-called boundary creation operator. We now try to determine $\mathsf{S}(\theta)$ for the boundary SYL model (4.1). By analogy with the bulk SYL model, as well as with the boundary YL model, we expect that the boundary $S$ matrix of the boundary SYL model should be some reduction of that of the boundary supersymmetric sine-Gordon model [45], or equivalently, the boundary supersymmetric sinh-Gordon model [15]. We therefore consider

$$\mathsf{S}(\theta) = \mathsf{S}_{\mathrm{YL}}(\theta; b)\mathsf{S}_{\mathrm{SUSY}}(\theta; \phi), \tag{4.12}$$

where the scalar factor $\mathsf{S}_{\mathrm{YL}}(\theta; b)$ is given by (2.3), and $\mathsf{S}_{\mathrm{SUSY}}(\theta; \phi)$ is given by

$$\mathsf{S}_{\mathrm{SUSY}}(\theta; \phi) = \mathsf{Y}(\theta; \phi)\mathsf{R}(\theta; \phi), \tag{4.13}$$

where $\mathsf{R}(\theta; \phi)$ is the $2 \times 2$ matrix

$$\mathsf{R}(\theta; \phi) = \begin{pmatrix} \mathcal{A}_+ & \mathcal{B} \\ \mathcal{B} & \mathcal{A}_- \end{pmatrix}, \tag{4.14}$$

with matrix elements

$$\mathcal{A}_\pm = \cosh\frac{\theta}{2}\, G_+ \pm i \sinh\frac{\theta}{2}\, G_-, \qquad \mathcal{B} = -i \sinh\theta, \tag{4.15}$$

where

$$G_+ = \mathfrak{r}\left(\sinh\phi + \frac{e^\phi \sinh^2\frac{\theta}{2}}{1 - \sin\frac{\pi}{3}}\right), \qquad G_- = \mathfrak{r}\left(\cosh\phi + \frac{e^\phi \sinh^2\frac{\theta}{2}}{1 - \sin\frac{\pi}{3}}\right),$$

$$\mathfrak{r} = \left(\frac{2(1 - \sin\frac{\pi}{3})}{\sin\frac{\pi}{3}}\right)^{1/2}. \tag{4.16}$$

Moreover, $\mathsf{Y}(\theta; \phi)$ is a scalar factor given by

$$\mathsf{Y}(\theta; \phi) = \mathsf{Y}_0(\theta)\mathsf{Y}_1(\theta; \phi)F(\theta; \phi), \tag{4.17}$$

where

$$
\begin{aligned}
\mathsf{Y}_0(\theta) &= \frac{i}{\sqrt{2}\sinh(\frac{\theta}{2} + \frac{i\pi}{4})} \\
&\quad \times \exp\left(-\frac{1}{2}\int_0^\infty \frac{dt}{t}\, \frac{\sinh(2it\theta/\pi)\sinh(2t/3)\sinh(t/3)}{\cosh^2 t \cosh^2(t/2)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Y}_1(\theta; \phi) &= \frac{1}{\mathfrak{r}\sinh\phi}\, \frac{\sin(\frac{\pi}{12} - \frac{\zeta}{2})\sin(\frac{\pi}{12} + \frac{\zeta}{2})}{\sin(\frac{\pi}{12} - \frac{\zeta}{2} - \frac{i\theta}{2})\sin(\frac{\pi}{12} + \frac{\zeta}{2} - \frac{i\theta}{2})} \\
&\quad \times \exp\left(-2\int_0^\infty \frac{dt}{t}\, \frac{\sinh(it\theta/\pi)\sinh(t/3)\cosh(t\zeta/\pi)}{\sinh t \cosh(t/2)}\right),
\end{aligned}
\tag{4.18}
$$

and $\zeta$ is a function of $\phi$ defined by

$$\cos\zeta = 1 - e^{-2\phi}\left(1 - \sin\frac{\pi}{3}\right). \tag{4.19}$$

The exponential factors of $Y_0(\theta)$ and $Y_1(\theta;\phi)$ do not have zeros or poles in the physical strip $0 \leqslant \mathrm{Im}\,\theta \leqslant \frac{\pi}{2}$, provided $|\zeta| < 2\pi/3$. Finally, $F(\theta;\phi)$ is a CDD-like factor obeying

$$F(\theta;\phi)F(-\theta;\phi) = 1, \qquad F\left(\frac{i\pi}{2} + \theta; \phi\right) = F\left(\frac{i\pi}{2} - \theta; \phi\right), \tag{4.20}$$

which is still to be determined.

The above expression for $S_{\mathrm{SUSY}}$ essentially coincides with the one for the supersymmetric sinh-Gordon model given in [15] with $B = -1/3$, $\varepsilon = +1$, $\varphi = \phi + \frac{i\pi}{2}$ with $\phi$ real, and $r = -i\mathfrak{r}$. The only differences lie in the CDD factor $F(\theta;\phi)$ (which is absent from [15]) and the factor $Y_1$: the expression given here is an analytic continuation of the one given in [15]. The former does not diverge for $\theta = \pm\frac{i\pi}{3}$, which is important for implementing the boundary bootstrap equations, as we shall see below (4.27).

The alert reader will have noticed that, while the boundary SYL action (4.1) contains only one boundary parameter (namely, $\lambda_B$), the above boundary $S$ matrix seems to contain two parameters, namely, $b$ and $\phi$. The key point to realize is that these two parameters are *not* independent. By demanding that the boundary $S$ matrix satisfy the various constraints [9] arising from the existence of boundary and bulk bound states, we shall determine the relation between $\phi$ and $b$ (4.26), as well as the CDD factor $F(\theta;\phi)$ (4.33).

We begin by considering the constraints due to boundary bound states. In general [9], let $iv_{0a}^\alpha$ be the position of a pole of the boundary $S$ matrix in the physical strip associated with the excited boundary state $|\alpha\rangle_B$, which can be interpreted as a boundary bound state of particle $A_a$ with the boundary ground state $|0\rangle_B$. Near this pole, the boundary $S$ matrix has the form

$$S_a^b(\theta) \simeq \frac{i}{2}\frac{g_{a0}^\alpha g_\alpha^{b0}}{\theta - iv_{0a}^\alpha}, \tag{4.21}$$

where $g_{a0}^\alpha$ are boundary-particle couplings.

We assume that (as in the bulk) the SYL boundary $S$ matrix inherits its pole structure from the YL boundary $S$ matrix (2.3). Therefore, it has [46] two boundary bound state poles, corresponding to excited boundary states $|1\rangle_B$, $|2\rangle_B$, with [6]

$$v_0^1 = \frac{\pi(b+1)}{6}, \qquad v_0^2 = \frac{\pi(b-1)}{6}. \tag{4.22}$$

It follows from the condition (4.21) and the form (4.14) of the $S$ matrix that for $\theta = iv_0^\alpha$,

$$\mathcal{A}_+ \propto \left(g_\alpha^{b0}\right)^2, \quad \mathcal{A}_- \propto \left(g_\alpha^{f0}\right)^2, \quad \mathcal{B} \propto g_\alpha^{b0} g_\alpha^{f0}, \tag{4.23}$$

where the indices $b$ and $f$ again denote Boson and Fermion, respectively. Hence, we arrive at the important constraint

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[6] The subscript $a$ of $v_{0a}^\alpha$ can be dropped, since YL has only one type of particle.

$$\frac{\mathcal{A}_+ \mathcal{A}_-}{\mathcal{B}^2}\bigg|_{\theta = i v_0^\alpha} = 1. \tag{4.24}$$

This equation gives a relation between the boundary parameter $\phi$ and $v_0^\alpha$. As shown in Appendix B, the relation can be expressed most succinctly in terms of the parameter $\zeta$ defined in (4.19):

$$\zeta = v_0^\alpha \pm \frac{\pi}{6}. \tag{4.25}$$

The above relation can hold for both poles (4.22) only if

$$\zeta = \frac{\pi b}{6}. \tag{4.26}$$

Eq. (4.26) is the desired relation between $\phi$ and $b$. The restriction $|\zeta| < 2\pi/3$ which we found above implies $|b| < 4$.

We now consider the constraints due to bulk bound states. In view of the direct-channel pole of the SYL bulk $S$ matrix at $\theta = i2\pi/3$, the following boundary bootstrap relations must hold [9]

$$f_d^{ab} \mathsf{S}_c^d(\theta) = f_c^{b_1 a_1} \mathsf{S}_{a_1}^{a_2}\left(\theta + \frac{i\pi}{3}\right) S_{b_1 a_2}^{b_2 a}(2\theta) \mathsf{S}_{b_2}^b\left(\theta - \frac{i\pi}{3}\right). \tag{4.27}$$

Using infinite-product representations for the scalar factors $\mathsf{Y}_0(\theta)$, $\mathsf{Y}_1(\theta; \phi)$, and $Y(\theta)$, one can prove the identities

$$\frac{\mathsf{Y}_0(\theta + \frac{i\pi}{3})\mathsf{Y}_0(\theta - \frac{i\pi}{3})Y(2\theta)}{\mathsf{Y}_0(\theta)} = \frac{i\sqrt{2}\sinh\theta \sinh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\theta + \frac{i\pi}{3}\right)\cosh\left(\theta - \frac{i\pi}{3}\right)},$$

$$\frac{\mathsf{Y}_1(\theta + \frac{i\pi}{3}; \phi)\mathsf{Y}_1(\theta - \frac{i\pi}{3}; \phi)}{\mathsf{Y}_1(\theta; \phi)} = \frac{1}{\tau \sinh\phi} \sin\left(\frac{\pi}{12} - \frac{\zeta}{2}\right)\sin\left(\frac{\pi}{12} + \frac{\zeta}{2}\right)$$
$$\times \frac{2(1 + 2\cos 2\zeta - 2\cosh 2\theta - 4i\cos\zeta \sinh\theta)}{\cos 3\zeta + i\sinh 3\theta}. \tag{4.28}$$

With the help of these identities, together with (2.5), one can show that the SYL boundary bootstrap relations (4.27) are satisfied, provided that the CDD factor obeys

$$\frac{F\left(\theta + \frac{i\pi}{3}; \phi\right)F\left(\theta - \frac{i\pi}{3}; \phi\right)}{F(\theta; \phi)} = \frac{\cos 3\zeta + i\sinh 3\theta}{\cos 3\zeta - i\sinh 3\theta}. \tag{4.29}$$

In addition to the boundary bootstrap relation, another constraint due to bulk bound states is stated in [9]. Namely, let $i u_{ab}^c$ be the position of the pole of the bulk $S$ matrix associated with the direct-channel bound state of $A_a A_b$ which can be interpreted as the particle $A_c$. If the particles $A_a$ and $A_b$ have equal mass, then the boundary $S$ matrix must have a pole at $\theta = i\bar{u}_{ab}^c/2$, where $\bar{u}_{ab}^c = \pi - u_{ab}^c$. Furthermore,

$$\mathsf{S}_{\bar{a}}^b(\theta) \simeq -\frac{i}{2} \frac{f_c^{ab} g^c}{\theta - \frac{i\bar{u}_{ab}^c}{2}}, \tag{4.30}$$

where $g^c$ describes the coupling of $A_c$ to the boundary. The SYL boundary $S$ matrix indeed has such a pole at $\theta = i\pi/6$. It follows from the condition (4.30) that for $\theta = i\pi/6$,

$$\mathcal{A}_+ \propto f_b^{bb} g^b, \qquad \mathcal{A}_- \propto f_b^{ff} g^b; \tag{4.31}$$

and hence [26]

$$\frac{\mathcal{A}_+}{\mathcal{A}_-}\bigg|_{\theta = \frac{i\pi}{6}} = \frac{f_b^{bb}}{f_b^{ff}} = \sqrt{3}. \tag{4.32}$$

However, this equation is satisfied for arbitrary values of $\phi$, and so does not provide any further constraints on the $S$ matrix.

The scalar factor $\mathsf{Y}(\theta; \phi)$ (4.17) should not have zeros or poles in the physical strip. In view of the relation (4.26), we see that the factor $\mathsf{Y}_1(\theta; \phi)$ has poles at $\theta = i(\pm\zeta - \frac{\pi}{6}) = i\pi(\pm b - 1)/6$. The pole at $\theta = i\pi(b - 1)/6$ is undesirable, since it is physical for $1 < b < 4$.[7] Fortunately, we can arrange for this pole to be canceled by a corresponding zero of the CDD factor. Indeed, a solution to the CDD constraint equations (4.20) and (4.29) which has a zero at $\theta = i\pi(b - 1)/6$ is given by

$$F(\theta; \phi) = \left(\frac{1-b}{2}\right)\left(\frac{5+b}{2}\right), \tag{4.33}$$

where we have again used the notation (2.4).

In short, the boundary $S$ matrix which we propose for the boundary SYL model (4.1) is given by Eqs. (4.12)–(4.19), (4.26), (4.33). This is one of the main results of our paper. Note that our proposed boundary $S$ matrix depends on a single independent boundary parameter $b$. The relation of this parameter to the boundary parameter $\lambda_B$ in the action (4.1) is not yet known.

One check on this proposal is provided by supersymmetry. We have suggested that the SYL model (4.1) has the integral of motion $\mathsf{Q}$ given by (4.5). We now demonstrate that our proposed boundary $S$ matrix commutes with a similar quantity. Indeed, let us assume that the supersymmetry charges $Q$ and $\overline{Q}$ act on states according to (3.14), (3.15). It is straightforward to show that the matrix $\mathsf{R}(\theta; \phi)$ (4.14) commutes with

$$\mathsf{Q} = Q + \overline{Q} + \gamma(-1)^F, \tag{4.34}$$

where here $\gamma = -\sqrt{m}\sqrt{-1 + 2/\sqrt{3}}\, e^{-\phi}$. Note that $\mathsf{Q}$ does not anticommute with $(-1)^F$, unlike usual supersymmetry charges. The appearance of $(-1)^F$ in $\mathsf{Q}$ should not be too surprising, since similar topological charges also appear in the fractional-spin integrals of motion of the boundary sine-Gordon model [18]. Presumably the operator $\Theta$ in (4.5) can be identified with $\gamma(-1)^F$. Note that $\lambda_B = 0$ (for which $\Theta$ vanishes) corresponds to $\phi = \infty$, and hence $b = 0$. For this value of $b$, the boundary $S$ matrix $\mathsf{S}(\theta)$ is diagonal. We recall that Moriconi and Schoutens proposed [26] two diagonal boundary $S$ matrices for the boundary SYL model (although without reference to any specific boundary conditions), which they designated $R_{(1)}^{[1]}$ and $R_{(2)}^{[1]}$. Our boundary $S$ matrix for $b = 0$ differs from $R_{(2)}^{[1]}$ by the CDD factor, i.e.,

$$\frac{\mathsf{S}(\theta)}{F(\theta; \phi)}\bigg|_{b=0} = R_{(2)}^{[1]}(\theta). \tag{4.35}$$

---

[7] The pole at $\theta = i\pi(-b - 1)/6$ is canceled by a corresponding zero in the factor $\left(\frac{1+b}{2}\right)$ from $\mathsf{S}_{YL}$.

### 4.3. Boundary TBA and massless boundary flow

We have defined the boundary SYL model in Section 4.1, and we have proposed the corresponding boundary $S$ matrix in Section 4.2. We shall now demonstrate that this picture is supported by the boundary TBA. Our analysis is a generalization of the one for the boundary YL model, which we briefly reviewed in Section 2.2. For simplicity, we again focus our attention on the case of massless boundary flow.

We begin by determining the massless scaling limit. We set

$$m = \mu n, \quad \theta = \hat{\theta} - \ln\frac{n}{2}, \quad \frac{i\pi}{6}(b+a) = \theta_B - \ln\frac{n}{2}, \quad n \to 0, \tag{4.36}$$

with $\mu$, $\hat{\theta}$, and $\theta_B$ real and finite. Our objective is to determine the value(s) of $a$ (also real and finite) for which the boundary $S$ matrix, in the above limit, remains finite and unitary. After some computation, we find that $a = 6$; and the resulting massless boundary $S$ matrix is given by

$$\mathsf{S}(\theta) = \mathsf{Z}(\hat{\theta} - \theta_B)\mathsf{R}(\hat{\theta} - \theta_B), \tag{4.37}$$

where

$$\mathsf{Z}(\theta) = \frac{\sinh\left(\frac{\theta}{2} - \frac{i\pi}{12}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i5\pi}{12}\right)\sinh\left(\frac{\theta}{2} + \frac{i5\pi}{12}\right)} \exp\left(-\int\limits_0^\infty \frac{dt}{t} \frac{\sinh\frac{t}{3}\sinh\left(t\left(\frac{i\theta}{\pi} - 1\right)\right)}{\sinh t \cosh\frac{t}{2}}\right), \tag{4.38}$$

and

$$\mathsf{R}(\theta) = \begin{pmatrix} \sinh\left(\frac{\theta}{2} + \frac{i\pi}{4}\right) & -\frac{i\sqrt[4]{3}}{2} \\ -\frac{i\sqrt[4]{3}}{2} & \sinh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right) \end{pmatrix}. \tag{4.39}$$

Indeed, $\mathsf{S}(\theta)$ satisfies the unitarity condition, since

$$\mathsf{Z}(\theta)\mathsf{R}(\theta)\,\mathsf{Z}(-\theta)\mathsf{R}(-\theta) = \mathbb{I}. \tag{4.40}$$

In order to formulate the TBA equations, we consider $N$ particles with real rapidities $\theta_1, \ldots, \theta_N$ in an interval of length $L$, with bulk $S$ matrix $S(\theta)$ (3.13) and boundary $S$ matrices $\mathsf{S}(\theta; b_\pm)$ (4.12), where the subscripts $\pm$ here denote the left and right boundaries. As already discussed, the bulk and boundary $S$ matrices of the SYL model essentially coincide with those for the supersymmetric sinh-Gordon model given in [15] with $B = -\frac{1}{3}$, $\varepsilon = +1$, $\varphi = \phi + \frac{i\pi}{2}$, and $r = -i\mathfrak{r}$. Hence, the Bethe Ansatz equations and the transfer matrix eigenvalues for SYL can be easily obtained from [15], to which we shall henceforth refer as I. From Eq. (I 4.14) we obtain the Bethe ansatz equations for $z_k^+$

$$\prod_{j=1}^N \frac{\tanh\left(\frac{1}{2}(z_k^+ - \theta_j)\right)}{\tanh\left(\frac{1}{2}(z_k^+ - \theta_j) + \frac{i\pi}{3}\right)} \frac{\tanh\left(\frac{1}{2}(z_k^+ + \theta_j)\right)}{\tanh\left(\frac{1}{2}(z_k^+ + \theta_j) + \frac{i\pi}{3}\right)}$$

$$= \frac{\sinh^2\left(\frac{1}{2}\left(\frac{i5\pi}{6} + z_k^+\right)\right)}{\sinh^2\left(\frac{1}{2}\left(\frac{i\pi}{6} - z_k^+\right)\right)}\left[\frac{-e^{-\phi_-}\sinh^2\left(\frac{i\pi}{12}\right) + e^{\phi_-}\sinh^2\left(\frac{1}{2}\left(\frac{i\pi}{6} - z_k^+\right)\right)}{-e^{-\phi_-}\sinh^2\left(\frac{i\pi}{12}\right) + e^{\phi_-}\sinh^2\left(\frac{1}{2}\left(\frac{i5\pi}{6} + z_k^+\right)\right)}\right][\phi_- \to \phi_+], \tag{4.41}$$

and from (I 4.15) we obtain a similar result for $z_k^-$. In view of the massless scaling limit (4.36), we set

$$\theta_j = \hat{\theta}_j - \ln\frac{n}{2}, \quad z_k^\pm = \hat{z}_k^\pm - \ln\frac{n}{2}, \quad \frac{i\pi}{6}(b_\pm + 6) = \theta_B^\pm - \ln\frac{n}{2}, \quad n \to 0, \quad (4.42)$$

and we obtain

$$\prod_{j=1}^{N} \frac{\tanh\left(\frac{1}{2}(\hat{z}_k^+ - \hat{\theta}_j)\right)}{\tanh\left(\frac{1}{2}(\hat{z}_k^+ - \hat{\theta}_j) + \frac{i\pi}{3}\right)}$$
$$= \frac{\cosh\left(\frac{1}{2}(\hat{z}_k^+ - \hat{\theta}_B^-) - \frac{i\pi}{12}\right)\cosh\left(\frac{1}{2}(\hat{z}_k^+ - \hat{\theta}_B^+) - \frac{i\pi}{12}\right)}{\cosh\left(\frac{1}{2}(\hat{z}_k^+ - \hat{\theta}_B^-) + \frac{i5\pi}{12}\right)\cosh\left(\frac{1}{2}(\hat{z}_k^+ - \hat{\theta}_B^+) + \frac{i5\pi}{12}\right)}. \quad (4.43)$$

Finally, setting $\hat{z}_k^+ = \hat{x}_k - \frac{i\pi}{3}$, we obtain the Bethe ansatz equations for $\hat{x}_k$ (cf., Eq. (I 4.19)),

$$\prod_{j=1}^{N} \frac{\tanh\left(\frac{1}{2}(\hat{x}_k - \hat{\theta}_j) - \frac{i\pi}{6}\right)}{\tanh\left(\frac{1}{2}(\hat{x}_k - \hat{\theta}_j) + \frac{i\pi}{6}\right)} \frac{\cosh\left(\frac{1}{2}(\hat{x}_k - \hat{\theta}_B^-) + \frac{i\pi}{4}\right)\cosh\left(\frac{1}{2}(\hat{x}_k - \hat{\theta}_B^+) + \frac{i\pi}{4}\right)}{\cosh\left(\frac{1}{2}(\hat{x}_k - \hat{\theta}_B^-) - \frac{i\pi}{4}\right)\cosh\left(\frac{1}{2}(\hat{x}_k - \hat{\theta}_B^+) - \frac{i\pi}{4}\right)} = 1,$$
$$k = 0, 1, \ldots, N. \quad (4.44)$$

The transfer matrix eigenvalues $\Lambda(\theta|\theta_1, \ldots, \theta_N)$ can be deduced from Eqs. (I 4.12), (I 4.17), (I 4.24). In the scaling limit (4.42) (with $\theta = \hat{\theta} - \ln\frac{n}{2}$), we obtain

$$\Lambda \propto \mathsf{Z}(\hat{\theta} - \theta_B^+)\mathsf{Z}(\hat{\theta} - \theta_B^-)e^{\hat{x}_0 - \frac{1}{2}(\theta_B^+ + \theta_B^-)} \prod_{k=1}^{N} \frac{Z(\hat{\theta} - \hat{\theta}_k)e^{\hat{x}_k - \hat{\theta}_k}}{\frac{1}{2}\sinh(\hat{\theta} - \hat{\theta}_k)} \prod_{k=0}^{N} \lambda_{\epsilon_k}(\hat{\theta} - \hat{\theta}_k), \quad (4.45)$$

where $Z(\theta)$ and $\mathsf{Z}(\theta)$ are given by (3.12) and (4.38), respectively;

$$\lambda_\epsilon(\theta) = \sinh\left(\frac{\theta}{2} + \frac{\epsilon i\pi}{6}\right)\cosh\left(\frac{\theta}{2} - \frac{\epsilon i\pi}{6}\right), \quad (4.46)$$

$\epsilon_k = \pm 1$ (see Eq. (I 4.20)), and $\hat{x}_k$ satisfy (4.44).

We introduce the densities $P_\pm(\hat{\theta})$ of "magnons", i.e., of real Bethe ansatz roots $\{\hat{x}_k\}$ with $\epsilon_k = \pm 1$, respectively; and also the densities $\rho_1(\hat{\theta})$ and $\tilde{\rho}(\hat{\theta})$ of particles $\{\hat{\theta}_k\}$ and holes, respectively. The Bethe ansatz equations (4.44) imply [8]

$$P_+(\hat{\theta}) + P_-(\hat{\theta}) = \frac{1}{2\pi}(\rho_1 * \Phi)(\hat{\theta}) + \frac{1}{2\pi L}\left[\Psi\left(\hat{\theta} - \theta_B^+\right) + \Psi\left(\hat{\theta} - \theta_B^-\right)\right], \quad (4.47)$$

where

$$\Phi(\theta) = \frac{1}{i}\frac{\partial}{\partial\theta}\ln\left(\frac{\tanh\left(\frac{\theta}{2} - \frac{i\pi}{6}\right)}{\tanh\left(\frac{\theta}{2} + \frac{i\pi}{6}\right)}\right) = \frac{4\cosh\theta\sin\frac{\pi}{3}}{\cosh 2\theta - \cos\frac{2\pi}{3}} = -\Phi_{\mathrm{YL}}(\theta),$$

$$\Psi(\theta) = \frac{1}{i}\frac{\partial}{\partial\theta}\ln\left(\frac{\cosh\left(\frac{\theta}{2} + \frac{i\pi}{4}\right)}{\cosh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right)}\right) = \frac{1}{\cosh\theta}, \quad (4.48)$$

and we have defined $\rho_1(\hat{\theta})$ for negative values of $\hat{\theta}$ to be equal to $\rho_1(|\hat{\theta}|)$.

---

[8] The counting function should be monotonic increasing, in order that the corresponding density (defined as the derivative of the counting function) be nonnegative.

The Yang equations (I 5.7) and the expression (4.45) for the eigenvalues imply

$$
\begin{aligned}
\rho_1(\hat{\theta}) &+ \tilde{\rho}(\hat{\theta}) \\
&= \frac{\mu}{\pi} e^{\hat{\theta}} + \frac{1}{2\pi}(\rho_1 * \Phi_Z)(\hat{\theta}) + \frac{1}{2\pi}(P_+ * \Phi_+)(\hat{\theta}) + \frac{1}{2\pi}(P_- * \Phi_-)(\hat{\theta}) \\
&\quad + \frac{1}{2\pi L}\left[\frac{\partial}{\partial\hat{\theta}}\mathrm{Im}\ln Z(\hat{\theta} - \theta_B^+) + \frac{\partial}{\partial\hat{\theta}}\mathrm{Im}\ln Z(\hat{\theta} - \theta_B^-)\right],
\end{aligned} \tag{4.49}
$$

where

$$
\Phi_Z(\theta) = \frac{\partial}{\partial\theta}\mathrm{Im}\ln Z(\theta), \qquad \Phi_\pm(\theta) = \frac{\partial}{\partial\theta}\mathrm{Im}\ln\lambda_\pm(\theta), \tag{4.50}
$$

and we have defined $P_\pm(\hat{\theta})$ for negative values of $\hat{\theta}$ to be equal to $P_\mp(|\hat{\theta}|)$. Using the fact $\Phi_\pm(\theta) = \mp\frac{1}{2}\Phi(\theta)$, and using (4.47) to eliminate $P_+$, we obtain

$$
\begin{aligned}
\rho_1(\hat{\theta}) + \tilde{\rho}(\hat{\theta}) &= \frac{\mu}{\pi}e^{\hat{\theta}} + \frac{1}{2\pi}(P_- * \Phi)(\hat{\theta}) + \frac{1}{2\pi}\left(\rho_1 * \left(\Phi_Z - \frac{1}{4\pi}\Phi * \Phi\right)\right)(\hat{\theta}) \\
&\quad + \frac{1}{2\pi L}\left[\frac{\partial}{\partial\hat{\theta}}\mathrm{Im}\ln Z(\hat{\theta} - \theta_B^+) - \frac{1}{4\pi}(\Psi * \Phi)(\hat{\theta} - \theta_B^+)\right. \\
&\quad \left. + \frac{\partial}{\partial\hat{\theta}}\mathrm{Im}\ln Z(\hat{\theta} - \theta_B^-) - \frac{1}{4\pi}(\Psi * \Phi)(\hat{\theta} - \theta_B^-)\right]. \quad (4.51)
\end{aligned}
$$

With the help of the identities

$$
\Phi_Z(\theta) - \frac{1}{4\pi}(\Phi * \Phi)(\theta) = -\Phi(\theta),
$$
$$
\frac{\partial}{\partial\theta}\mathrm{Im}\ln Z(\theta) - \frac{1}{4\pi}(\Psi * \Phi)(\theta) = 0, \tag{4.52}
$$

we obtain the simple result

$$
\rho_1(\hat{\theta}) + \tilde{\rho}(\hat{\theta}) = \frac{\mu}{\pi}e^{\hat{\theta}} + \frac{1}{2\pi}(P_- * \Phi)(\hat{\theta}) - \frac{1}{2\pi}(\rho_1 * \Phi)(\hat{\theta}). \tag{4.53}
$$

Proceeding as in I, we obtain the TBA equations [9]

$$
\begin{aligned}
re^{\hat{\theta}} &= \epsilon_1(\hat{\theta}) - \frac{1}{2\pi}\left(\Phi * (L_1 - L_2)\right)(\hat{\theta}), \\
0 &= \epsilon_2(\hat{\theta}) + \frac{1}{2\pi}(\Phi * L_1)(\hat{\theta}),
\end{aligned} \tag{4.54}
$$

where

$$
\begin{aligned}
L_i(\hat{\theta}) &= \ln\left(1 + e^{-\epsilon_i(\hat{\theta})}\right), \qquad r = \mu R, \\
\epsilon_1 &= \ln\left(\frac{\tilde{\rho}}{\rho_1}\right), \qquad\qquad \epsilon_2 = \ln\left(\frac{P_+}{P_-}\right).
\end{aligned} \tag{4.55}
$$

---

[9] This set of TBA equations is the same as for the case of periodic boundary conditions, which was first conjectured in [47] (see also [48]) and later derived from the SYL $S$ matrix in [20] and generalized in [21].

Moreover, the boundary entropy of one boundary is given (up to an additive constant) by

$$
\ln g = \frac{2}{4\pi} \int\limits_{-\infty}^{\infty} d\hat{\theta}\, \Psi(\hat{\theta} - \theta_B) L_2(\hat{\theta}),
\tag{4.56}
$$

where we have included the factor 2 in order to account for contributions from both right-movers and left-movers. In the UV limit $\theta_B \to -\infty$, the integrand is nonvanishing for $\hat{\theta} \to -\infty$; similarly, the IR limit $\theta_B \to \infty$ requires $\hat{\theta} \to \infty$. Using the results $L_2(-\infty) = \ln(2 + \sqrt{2})$, $L_2(\infty) = \ln 2$ which follow from the TBA equations (4.54), we obtain

$$
\ln \frac{g^{\mathrm{UV}}}{g^{\mathrm{IR}}} = \frac{1}{2} \ln\left(\frac{1 + \sqrt{2}}{\sqrt{2}}\right).
\tag{4.57}
$$

This is precisely the ratio of $g$ factors corresponding to the $\mathcal{M}(3/8)$ conformal boundary conditions $(1, 3)$ and $(1, 2)$

$$
\ln \frac{g_{(1,3)}}{g_{(1,2)}} = \frac{1}{2} \ln\left(\frac{1 + \sqrt{2}}{\sqrt{2}}\right),
\tag{4.58}
$$

as one can verify from Eq. (4.10). Hence, the proposed boundary $S$ matrix is consistent with massless flow away from the UV conformal boundary condition; namely, from the CBC $(1, 3)$ to the CBC $(1, 2)$. In the $\mathcal{SM}(2/8)$ description, this corresponds to the flow from the SCBC $(1, 3)$ to the SCBC $(1, 4)$. A plot of $\ln g$ as a function of $\theta_B$ is given in Fig. 1. For convenience, a constant has been added so that the UV value is $\frac{1}{2} \ln(1 + \sqrt{2})$ and the IR value is $\frac{1}{2} \ln \sqrt{2}$.
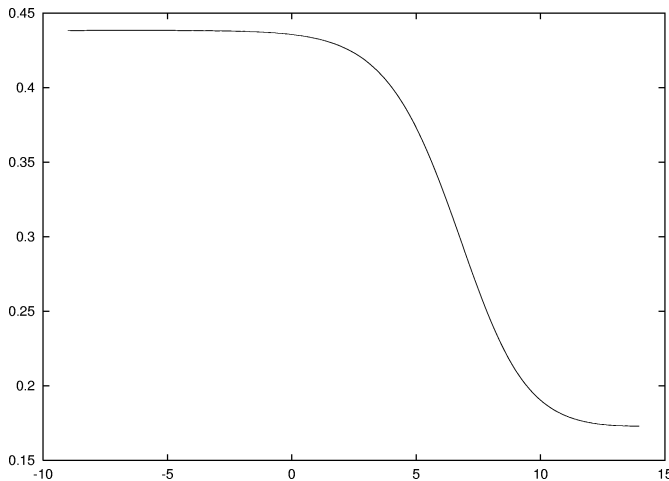


Fig. 1. Boundary entropy: $\ln g$ vs. $\theta_B$.

## 5. Discussion

We have proposed the boundary $S$ matrix (4.12)–(4.19), (4.26), (4.33) for the boundary SYL model defined by the action (4.1). Some support for this conjecture is provided by the fractional-spin integral of motion (4.5), (4.34), and by the massless boundary flow (4.57), (4.58). Several important problems remain to be solved, including the relation of the parameter $\lambda_B$ in the action to the parameter $b$ of the boundary $S$ matrix; and the identification of the operator $\Theta$ in (4.5) with the operator $\gamma(-1)^F$ in (4.34). It would also be interesting to consider other conformal boundary conditions, as well as extend the present study to the full family of integrable models with $N = 1$ supersymmetry [19,26].

## Acknowledgements

## Appendix A.  SYL model as restriction of ZMS model

Here we show that the scaling supersymmetric Yang–Lee model is a restriction of the Zhiber–Mikhailov–Shabat model [30,31], whose action is given by

$$A = \int d^2x \, (\partial_\mu \varphi)^2 + \frac{m^2}{\gamma^2} \left( e^{i\sqrt{8\gamma}\,\varphi} + e^{-i\sqrt{2\gamma}\,\varphi} \right). \tag{A.1}$$

This is the $A_2^{(2)}$ imaginary coupling affine Toda field theory, whose $S$ matrix was found by Izergin and Korepin [32]. We follow closely the paper [29] of Takács, to which we shall refer as II. (See also [49,50].)

It is useful to first recall the related work [28] of Smirnov. There it is observed that, for

$$\gamma = \frac{\pi r}{s}, \tag{A.2}$$

the ZMS model is the $(1, 2)$ perturbation of the minimal model $\mathcal{M}(r/s)$. Indeed, one can regard the first two terms in the action (A.1) as the action for $\mathcal{M}(r/s)$, and the third term as the $(1, 2)$ perturbation. The $S$ matrix of the perturbed model can be obtained as the RSOS restriction of the $A_2^{(2)}$ $S$ matrix, using the model's $U_q(sl(2))$ symmetry, where $q = e^{i\pi^2/\gamma}$.

In II, it is observed that, for

$$\gamma = 4\gamma' = \frac{4\pi r'}{s'}, \tag{A.3}$$

the ZMS model is the $(1, 5)$ perturbation of the minimal model $\mathcal{M}(r'/s')$. Indeed, one can regard the first and third terms in the action (A.1) as the action for $\mathcal{M}(r'/s')$, and the second term as the $(1, 5)$ perturbation. The $S$ matrix of the perturbed model can be obtained

as the RSOS restriction of the $A_2^{(2)}$ $S$ matrix, using the model's $U_{q'}(sl(2))$ symmetry, where $q' = e^{i\pi^2/\gamma'} = q^4$.

We have suggested in Section 3 that the SYL model can be regarded as the $(1, 5)$ perturbation of $\mathcal{M}(3/8)$. We now proceed to compute the latter's $S$ matrix following II, and we shall find that it coincides (up to a scalar factor) with Eq. (3.8). For $\mathcal{M}(3/8)$ we have $r' = 3$, $s' = 8$; hence, $q' = q = e^{2i\pi/3}$. The first positive integer $p$ for which $q'^p = \pm 1$ is $p = 3$. Hence, the maximum spin is $j_{max} = \frac{p}{2} - 1 = \frac{1}{2}$. Thus, the model contains "charged" kinks $K_{0\,\frac{1}{2}} = K_{\frac{1}{2}\,0}$ which we denote by $c$, and "neutral" kinks $K_{0\,0} = K_{\frac{1}{2}\,\frac{1}{2}}$ which we denote by $n$. Since (II 23)

$$\xi = \frac{2}{3}\left(\frac{\pi\gamma}{2\pi - \gamma}\right) = 2\pi, \tag{A.4}$$

the model contains neither breathers nor higher kinks. The $S$ matrix is expressed in terms of the rapidity variable $y = e^{\pi\theta/\xi} = e^{\theta/2}$. The $c\,c \to c\,c$ amplitude is given by (II 43)–(II 45)

$$\frac{y^2}{q} - \frac{q}{y^2} - \frac{1}{q} + q + \frac{y^2}{q^5} - \frac{q^5}{y^2} - \frac{1}{q} + q = 2i\sqrt{3} - 2\sinh\theta. \tag{A.5}$$

The $n\,n \to n\,n$ amplitude is given by (II 46), (II 40)

$$\frac{q^6 y^2 + y^2 q^8 - q^8 - q^4 y^2 + y^2 - q^{10} y^2 + y^4 q^2 - y^2 q^2}{y^2 q^5}$$
$$= 2i\sqrt{3} + 2\sinh\theta. \tag{A.6}$$

The $c\,c \to n\,n$ and $n\,n \to c\,c$ amplitudes are equal, are are given by (II 48)

$$i\frac{(q^4 - 1)(y^2 - 1)}{q^2 y} = 2\sqrt{3}\sinh\frac{\theta}{2}. \tag{A.7}$$

Finally, the $n\,c$ forward scattering and reflection amplitudes are given by (II 46), (II 40)

$$\frac{(y^2 + q^6)(y^2 - 1)}{y^2 q^3} = 2\sinh\theta \tag{A.8}$$

and

$$-\frac{(q^4 - 1)(y^2 + q^6)}{y q^5} = 2i\sqrt{3}\cosh\frac{\theta}{2}, \tag{A.9}$$

respectively. Identifying $n$ and $c$ as the Boson and Fermion (respectively) of the SYL model, we see that the above amplitudes coincide with those in $2(\sinh\theta)R(\theta)$, where $R(\theta)$ is the matrix (3.8). That is, the SYL model is indeed a restriction of the ZMS model, corresponding to the $(1, 5)$ perturbation of $\mathcal{M}(3/8)$.

## Appendix B. Solution of constraint Eq. (4.24)

Here we solve Eq. (4.24), which for simplicity we now write as

$$\left(\mathcal{A}_+\mathcal{A}_- - \mathcal{B}^2\right)\big|_{\theta=iv} = 0. \tag{B.1}$$

Using the definitions of $\mathcal{A}_+$, $\mathcal{A}_-$, and $\mathcal{B}$ given in (4.15), (4.16), and introducing the variable $t \equiv \sin^2\frac{v}{2}$, Eq. (B.1) can be brought to the form

$$\left(t - \frac{1}{2}\right)\left[t^2 + t\left(-1 + \frac{\sqrt{3}}{2} + e^{-2\phi}\left(\frac{3}{4} - \frac{\sqrt{3}}{2}\right)\right)\right.$$
$$\left. + \frac{7}{16} - \frac{\sqrt{3}}{4} + e^{-2\phi}\left(\frac{\sqrt{3}}{2} - \frac{7}{8}\right) + e^{-4\phi}\left(\frac{7}{16} - \frac{\sqrt{3}}{4}\right)\right] = 0. \tag{B.2}$$

We discard the solution $t = \frac{1}{2}$, which corresponds to a fixed value of $v$ (and hence, $b$). The two remaining solutions are $t = \frac{1}{4}(\gamma \mp \sqrt{\Delta})$, where

$$\gamma = 2 - \sqrt{3} + e^{-2\phi}\left(\sqrt{3} - \frac{3}{2}\right),$$
$$\Delta = e^{-2\phi}(2 - \sqrt{3}) + e^{-4\phi}\left(\sqrt{3} - \frac{7}{4}\right). \tag{B.3}$$

In terms of the parameter $\zeta$ defined by

$$\cos\zeta = 1 - e^{-2\phi}\left(1 - \frac{\sqrt{3}}{2}\right), \tag{B.4}$$

we have

$$\gamma = 2 - \sqrt{3}\cos\zeta, \qquad \Delta = \sin\zeta; \tag{B.5}$$

and therefore,

$$t = \frac{1}{2}\left[1 - \cos\left(\zeta \mp \frac{\pi}{6}\right)\right]. \tag{B.6}$$

Finally, recalling the definition $t = \sin^2\frac{v}{2}$, we arrive at the remarkably simple result

$$\zeta = v \pm \frac{\pi}{6}, \tag{B.7}$$

which is quoted in text (4.25).

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