Review of AdS/CFT Integrability, Chapter III.2: Exact World-Sheet S-Matrix

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Abstract. We review the derivation of the S-matrix for planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory and type IIB superstring theory on an $AdS_5 \times S^5$ background. After deriving the S-matrix for the $su(2)$ and $su(3)$ sectors at the one-loop level based on coordinate Bethe ansatz, we show how $su(2|2)$ symmetry leads to the exact asymptotic S-matrix up to an overall scalar function. We then briefly review the spectrum of bound states by relating these states to simple poles of the S-matrix. Finally, we review the derivation of the asymptotic Bethe equations, which can be used to determine the asymptotic multiparticle spectrum.
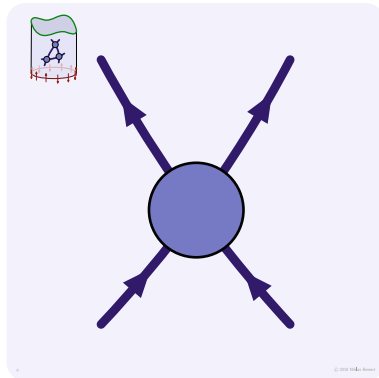
1. Introduction

S-matrices are quantum mechanical probability amplitudes between incoming and outgoing on-shell particle states. Exact factorized S-matrices have played a key role in the development of integrable models [1]. Indeed, starting from an exact S-matrix, it is in principle possible to compute the asymptotic spectrum, finite-size effects (Lüscher corrections, thermodynamic Bethe ansatz), form factors, and correlation functions non-perturbatively.

As reviewed in many articles in this volume, planar four-dimensional $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory and its holographic dual, type IIB superstring theory on $AdS_5 \times S^5$, are believed to be quantum integrable. The world-sheet and spin-chain S-matrix have been derived based on an $su(2|2)^2$ symmetry in [2–9] and will be reviewed here. This S-matrix has been confirmed by various checks. One of these checks is that the all-loop asymptotic Bethe ansatz equations (BAEs) [10] can be derived from the exact factorized S-matrix using either nested Bethe ansatz or algebraic Bethe ansatz methods [3,4,11,12]. As a warm up, we first review the computation of the one-loop S-matrix in the $su(2)$ and $su(3)$ sectors, based on a direct coordinate Bethe ansatz, using integrable spin-chain Hamiltonians whose eigenvalues are the anomalous dimensions of scalar operators in planar $\mathcal{N} = 4$ SYM. Using the S-matrices, we show how the bound-state spectrum can be constructed. Finally, we show how imposing periodicity on the asymptotic multiparticle wavefunction leads to the asymptotic Bethe equations, which can be used to determine the asymptotic multiparticle spectrum.

The outline of this chapter is as follows: In Section 2 we review the derivation of the exact $\mathcal{N} = 4$ SYM S-matrix, first by coordinate Bethe ansatz for one-loop order, and then by utilizing $su(2|2)$ symmetry for all-loop order. We also discuss the spectrum of bound states. In Section 3 we review the derivation of the asymptotic Bethe equations, first for the $su(2)$ and $su(3)$ sectors, and then for the full theory.

2. Exact S-Matrix

2.1. COORDINATE BETHE ANSATZ

For the planar $\mathcal{N} = 4$ SYM theory, we are interested in SYM composite operators,

$$\mathrm{Tr}[\mathscr{O}_1 \mathscr{O}_2 \cdots \mathscr{O}_L], \quad \mathscr{O}_i \in \{D^n \Phi, D^n \Psi, D^n F\}, \tag{2.1}$$

where all operators are at the same spacetime point. It is useful to associate the composite operators with state vectors of a quantum spin chain. The BPS operator $\mathrm{Tr}[Z^L]$, where Z is one of the scalars Φ, is the vacuum state $|0\rangle$. This choice of vacuum breaks the global $psu(2, 2|4)$ symmetry down to $su(2|2) \otimes su(2|2)$. Other composite operators which are obtained by replacing some Z's with certain other

SYM fields ("impurities") are mapped to excited states over the vacuum:

$$| \overset{1}{\overset{\downarrow}{Z}} \cdots Z \overset{x_1}{\overset{\downarrow}{\chi}} Z \cdots Z \overset{x_2}{\overset{\downarrow}{\chi}}' Z \cdots Z \overset{x_M}{\overset{\downarrow}{\chi}}'' Z \cdots \overset{L}{\overset{\downarrow}{Z}} \rangle \equiv \mathrm{Tr} \left[Z^{x_1-1} \chi Z^{x_2-x_1-1} \chi' \cdots \chi'' \cdots \right], \quad (2.2)$$

where

$$\chi, \chi', \chi'', \ldots \in \{\Phi_{a\dot{a}}, \Psi_{\dot{a}\alpha}, \bar{\Psi}_{a\dot{\alpha}}, D_{\alpha\dot{\alpha}}Z\}, \quad a, \dot{a} = 1, 2, \quad \alpha, \dot{\alpha} = 3, 4. \quad (2.3)$$

All other orientations for the operators $\mathscr{O}_i$ should be regarded as multiple excitations χ coincident at a single site.[1] Due to the cyclic property of the trace, the state (2.2) should be invariant under a uniform translation $x_k \to x_k + 1$. These excitation states belong to a bifundamental representation of a centrally extended $su(2|2)_L \otimes su(2|2)_R$, which should also be a symmetry of the S-matrix. The same structure can be discovered on the string world-sheet action in the light-cone gauge [13,14].

For the S-matrix, we focus on a particular class of states, namely asymptotic states, where the distances between the impurities $\chi, \chi', \ldots$, are very large:

$$1 \ll x_1 \ll x_2 \ll \cdots \ll x_M \ll L \to \infty. \quad (2.4)$$

The S-matrices are defined as amplitudes between two such asymptotic states.

To illustrate this, we derive the two-particle S-matrix directly from the spin chain using coordinate Bethe ansatz. For simplicity, we will first consider composite operators in the $su(2)$ sector where the impurities are a complex scalar field X.

The one-loop anomalous dimensions of the $su(2)$ sector are given by the Hamiltonian of the spin-1/2 ferromagnetic $su(2)$-invariant ("XXX") Heisenberg quantum spin-chain model [15]

$$\Gamma = \frac{\lambda}{8\pi^2} H, \quad H = \sum_{l=1}^{L} \left(1 - \mathscr{P}_{l,l+1} \right), \quad (2.5)$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling, and $\mathscr{P}$ is the permutation operator on $\mathscr{C}^2 \otimes \mathscr{C}^2$. We also need to impose a periodic boundary condition by identifying $L + 1 \equiv 1$.

It is obvious that the vacuum state $|0\rangle$ is an eigenstate of H with zero energy. Since $[H, S^z] = 0$, the energy eigenstates can be classified according to the number of impurities ("magnons"). One-particle excited states with momentum p are given by[2]

$$|\psi(p)\rangle = \sum_{x=1}^{L} e^{ipx} | \overset{1}{\overset{\downarrow}{Z}} \cdots \overset{x}{\overset{\downarrow}{X}} \cdots \overset{L}{\overset{\downarrow}{Z}} \rangle. \quad (2.6)$$

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[1]For example, $D\Phi$ is a superposition of Φ and DZ. More precisely, the excitations are $Z \mapsto DZ$ and $Z \mapsto \Phi$; combining these, one obtains $Z \mapsto DZ \mapsto D\Phi$, or equivalently $Z \mapsto \Phi \mapsto D\Phi$.

[2]The invariance of states by a shift of one site (noted earlier) implies that the total momentum should vanish. Therefore, a one-particle state with nonvanishing momentum is not allowed in a strict sense. The one- or two-particle states which we consider here can be thought of as part of an infinitely long chain where these particles are asymptotically separated from other particles.

One can easily check that (2.6) is an eigenstate of H with eigenvalue $E = \varepsilon(p)$, where

$$\varepsilon(p) = 4\sin^2(p/2). \tag{2.7}$$

Two-particle eigenstate can be written as

$$|\psi(p_1, p_2)\rangle = A_{XX}(12)|X(p_1)X(p_2)\rangle + A_{XX}(21)|X(p_2)X(p_1)\rangle, \tag{2.8}$$

$$|X(p_i)X(p_j)\rangle = \sum_{x_1 < x_2} e^{i(p_i x_1 + p_j x_2)} |\overset{1}{\underset{\downarrow}{Z}} \cdots \overset{x_1}{\underset{\downarrow}{X}} \cdots \overset{x_2}{\underset{\downarrow}{X}} \cdots \overset{L}{\underset{\downarrow}{Z}}\rangle. \tag{2.9}$$

Now we impose that these states satisfy

$$H|\psi\rangle = E(p_1, p_2)|\psi\rangle \tag{2.10}$$

and find that

$$E = \varepsilon(p_1) + \varepsilon(p_2), \tag{2.11}$$

where $\varepsilon(p)$ is given by (2.7). This leads to the $X - X$ scattering amplitude given by

$$A_{XX}(21) = S(p_2, p_1)A_{XX}(12), \tag{2.12}$$

$$S(p_2, p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}, \tag{2.13}$$

where $u_j = u(p_j)$ and

$$u(p) = \frac{1}{2}\cot(p/2). \tag{2.14}$$

We now consider the more complicated case where there are two different types of complex scalar fields, namely, X and Y. This is the so-called $su(3)$ sector, which is closed only at one loop. The ($su(3)$-invariant) Hamiltonian is again given by (2.5), except now $\mathscr{P}$ is the permutation operator on $\mathscr{C}^3 \otimes \mathscr{C}^3$. The two-particle eigenstates with one particle of each type are of the form

$$\begin{aligned}|\psi\rangle = {} &A_{XY}(12)|X(p_1)Y(p_2)\rangle + A_{XY}(21)|X(p_2)Y(p_1)\rangle \\ &+ A_{YX}(12)|Y(p_1)X(p_2)\rangle + A_{YX}(21)|Y(p_2)X(p_1)\rangle,\end{aligned} \tag{2.15}$$

$$|\phi_1(p_i)\phi_2(p_j)\rangle = \sum_{x_1 < x_2} e^{i(p_i x_1 + p_j x_2)} |\overset{1}{\underset{\downarrow}{Z}} \cdots \overset{x_1}{\underset{\downarrow}{\phi_1}} \cdots \overset{x_2}{\underset{\downarrow}{\phi_2}} \cdots \overset{L}{\underset{\downarrow}{Z}}\rangle. \tag{2.16}$$

Applying the Hamiltonian on $|\psi\rangle$ and imposing the condition (2.10), one finds that the amplitudes should be related by (see e.g. [16])

$$\begin{pmatrix} A_{XY}(21) \\ A_{YX}(21) \end{pmatrix} = \begin{pmatrix} R(p_2, p_1) & T(p_2, p_1) \\ T(p_2, p_1) & R(p_2, p_1) \end{pmatrix} \begin{pmatrix} A_{XY}(12) \\ A_{YX}(12) \end{pmatrix}, \tag{2.17}$$

where the transmission and reflection amplitudes are given by

$$T(p_2, p_1) = \frac{u_2 - u_1}{u_2 - u_1 - i}, \quad R(p_2, p_1) = \frac{i}{u_2 - u_1 - i}, \tag{2.18}$$

respectively. Combining Equations (2.13) and (2.17), one can construct an $su(2)$-invariant S-matrix which connects two amplitudes related by momentum exchange as follows:

$$\begin{pmatrix} A_{XX}(21) \\ A_{XY}(21) \\ A_{YX}(21) \\ A_{YY}(21) \end{pmatrix} = \mathbf{S} \cdot \begin{pmatrix} A_{XX}(12) \\ A_{YX}(12) \\ A_{XY}(12) \\ A_{YY}(12) \end{pmatrix} = \begin{pmatrix} S & & & \\ & T & R & \\ & R & T & \\ & & & S \end{pmatrix} \begin{pmatrix} A_{XX}(12) \\ A_{YX}(12) \\ A_{XY}(12) \\ A_{YY}(12) \end{pmatrix}. \tag{2.19}$$

At higher loops, the $su(2)$ sector remains closed, but the Hamiltonian becomes longer ranged. Integrability persists, but only in a perturbative sense [17]. Correspondingly, one must introduce a perturbative asymptotic Bethe ansatz, and in particular, an asymptotic S-matrix [2,18]. That is, in contrast to the one-loop case (XXX model) where the S-matrix is "local," for higher loops the S-matrix is only asymptotic: it applies only to in-going and out-going particles which are widely separated.

2.2. YANG–BAXTER EQUATION AND ZF ALGEBRA

It is not practical to extend the above approach to all loops and to all sectors of planar $\mathcal{N} = 4$ SYM. Fortunately, there is an alternative approach – based on symmetry – to derive an exact asymptotic S-matrix which is valid for any value of 't Hooft coupling constant. To this end, it is convenient to introduce Zamolodchikov–Faddeev (ZF) operators [1,19] to define particle states. Using the ZF operators, one can reformulate the derivation of the S-matrix into an algebraic problem. In Equation (2.16), we have introduced an asymptotic two-particle state as a superposition of plane waves. Now we express these states in terms of creation (ZF) operators acting on the vacuum state as follows:

$$|\phi_1(p_i)\phi_2(p_j)\rangle \equiv A^\dagger_{\phi_1}(p_i) A^\dagger_{\phi_2}(p_j)|0\rangle. \tag{2.20}$$

As can be noticed in (2.1), the ZF operators corresponding to the elementary fields of $\mathcal{N} = 4$ SYM can be denoted by $A^\dagger_{i\dot{i}}$, where the index $i = (a, \alpha) = 1, 2, 3, 4$ and similarly for $\dot{i}$. A very remarkable feature of the AdS/CFT S-matrix is that it is factorized into a tensor product of two identical S-matrices, one acting on the index i and the other on $\dot{i}$:

$$\mathbb{S} = S \otimes \dot{S}. \tag{2.21}$$

A natural way to describe the factorized S-matrix is to introduce "quark" ZF operators $A^\dagger_i$ and identify $A^\dagger_{i\dot{i}}$ with the tensor product of the quark ZF operators by

$$A_{ii}^{\dagger}(p) = A_{i}^{\dagger}(p) \otimes A_{i}^{\dagger}(p). \tag{2.22}$$

By the factorization property, it is enough now to consider only $A_{i}^{\dagger}$ sector for our discussion.

The bulk S-matrix elements $S_{i\,j}^{i'j'}(p_1, p_2)$ define the ZF algebra relation

$$A_{i}^{\dagger}(p_1)\, A_{j}^{\dagger}(p_2) = S_{i\,j}^{i'j'}(p_1, p_2)\, A_{j'}^{\dagger}(p_2)\, A_{i'}^{\dagger}(p_1), \tag{2.23}$$

where summation over repeated indices is always understood. It is convenient to arrange these matrix elements into a 16×16 matrix S as follows:

$$S = S_{i\,j}^{i'j'}\, e_{i\,i'} \otimes e_{j\,j'}, \tag{2.24}$$

where e_{ij} is the usual elementary 4×4 matrix whose (i, j) matrix element is 1, and all others are zero.

As is well known [1], starting from $A_{i}^{\dagger}(p_1)\, A_{j}^{\dagger}(p_2)\, A_{k}^{\dagger}(p_3)$, one can arrive at linear combinations of $A_{k''}^{\dagger}(p_3)\, A_{j''}^{\dagger}(p_2)\, A_{i''}^{\dagger}(p_1)$ by applying the relation (2.23) three times, in two different ways. The consistency condition is the Yang–Baxter equation,

$$S_{12}(p_1, p_2)\, S_{13}(p_1, p_3)\, S_{23}(p_2, p_3) = S_{23}(p_2, p_3)\, S_{13}(p_1, p_3)\, S_{12}(p_1, p_2). \tag{2.25}$$

We use the standard convention $S_{12} = S \otimes \mathbb{I}$, $S_{23} = \mathbb{I} \otimes S$, and $S_{13} = \mathscr{P}_{12}\, S_{23}\, \mathscr{P}_{12}$, where $\mathscr{P}_{12} = \mathscr{P} \otimes \mathbb{I}$, $\mathscr{P} = e_{i\,j} \otimes e_{j\,i}$ is the permutation matrix, and $\mathbb{I}$ is the four-dimensional identity matrix. The ZF algebra (2.23) also implies the bulk unitarity equation

$$S_{12}(p_1, p_2)\, S_{21}(p_2, p_1) = \mathbb{I}, \tag{2.26}$$

where $S_{21} = \mathscr{P}_{12}\, S_{12}\, \mathscr{P}_{12}$.

Solving the Yang–Baxter equation can be complicated. Fortunately, as we shall see below, $su(2|2)$ symmetry suffices to determine the AdS/CFT S-matrix (in the fundamental representation) – there is no need to solve the Yang–Baxter equation, as it is automatically satisfied.

2.3. CENTRALLY EXTENDED $su(2|2)$

The centrally extended $su(2|2)$ algebra consists of the rotation generators $\mathbb{L}_{a}{}^{b}, \mathbb{R}_{\alpha}{}^{\beta}$, the supersymmetry generators $\mathbb{Q}_{\alpha}{}^{a}, \mathbb{Q}_{a}^{\dagger\alpha}$, and the central elements $\mathbb{C}, \mathbb{C}^{\dagger}, \mathbb{H}.$[3] Latin

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[3]The central charge $\mathbb{H}$ is identified as the world-sheet Hamiltonian. The additional central charges $\mathbb{C}$ and $\mathbb{C}^{\dagger}$, which are necessary for having momentum-dependent representations with the appropriate energy, also appear in the off-shell symmetry algebra of the gauge-fixed sigma model [14].

indices $a, b, \ldots$ take values $\{1, 2\}$, while Greek indices $\alpha, \beta, \ldots$ take values $\{3, 4\}$. These generators have the following nontrivial commutation relations [3,4,9]:

$$
\begin{aligned}
&\left[\mathbb{L}_a{}^b, \mathbb{J}_c\right] = \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c, \quad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma\right] = \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma, \\
&\left[\mathbb{L}_a{}^b, \mathbb{J}^c\right] = -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c, \quad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma\right] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma, \\
&\left\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\right\} = \varepsilon_{\alpha\beta}\varepsilon^{ab}\mathbb{C}, \quad \left\{\mathbb{Q}_a^{\dagger\alpha}, \mathbb{Q}_b^{\dagger\beta}\right\} = \varepsilon^{\alpha\beta}\varepsilon_{ab}\mathbb{C}^\dagger, \\
&\left\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_b^{\dagger\beta}\right\} = \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \frac{1}{2}\delta_b^a \delta_\alpha^\beta \mathbb{H},
\end{aligned}
\tag{2.27}
$$

where $\mathbb{J}_i (\mathbb{J}^i)$ denotes any lower (upper) index of a generator, respectively.

The action of the bosonic generators on the ZF operators is given by

$$
\begin{aligned}
&\left[\mathbb{L}_a{}^b, A_c^\dagger(p)\right] = (\delta_c^b \delta_a^d - \frac{1}{2}\delta_a^b \delta_c^d) A_d^\dagger(p), \quad \left[\mathbb{L}_a{}^b, A_\gamma^\dagger(p)\right] = 0, \\
&\left[\mathbb{R}_\alpha{}^\beta, A_\gamma^\dagger(p)\right] = (\delta_\gamma^\beta \delta_\alpha^\delta - \frac{1}{2}\delta_\alpha^\beta \delta_\gamma^\delta) A_\delta^\dagger(p), \quad \left[\mathbb{R}_\alpha{}^\beta, A_c^\dagger(p)\right] = 0.
\end{aligned}
\tag{2.28}
$$

The operator relations for supersymmetry generators[4]

$$
\begin{aligned}
\mathbb{Q}_\alpha{}^a A_b^\dagger(p) &= \mathrm{e}^{-ip/2}\left[a(p)\delta_b^a A_\alpha^\dagger(p) + A_b^\dagger(p)\mathbb{Q}_\alpha{}^a\right], \\
\mathbb{Q}_\alpha{}^a A_\beta^\dagger(p) &= \mathrm{e}^{-ip/2}\left[b(p)\varepsilon_{\alpha\beta}\varepsilon^{ab} A_b^\dagger(p) - A_\beta^\dagger(p)\mathbb{Q}_\alpha{}^a\right], \\
\mathbb{Q}_a^{\dagger\alpha} A_b^\dagger(p) &= \mathrm{e}^{ip/2}\left[c(p)\varepsilon_{ab}\varepsilon^{\alpha\beta} A_\beta^\dagger(p) + A_b^\dagger(p)\mathbb{Q}_a^{\dagger\alpha}\right], \\
\mathbb{Q}_a^{\dagger\alpha} A_\beta^\dagger(p) &= \mathrm{e}^{ip/2}\left[d(p)\delta_\beta^\alpha A_a^\dagger(p) - A_\beta^\dagger(p)\mathbb{Q}_a^{\dagger\alpha}\right],
\end{aligned}
\tag{2.29}
$$

and the central charges

$$
\begin{aligned}
\mathbb{C} A_i^\dagger(p) &= \mathrm{e}^{-ip}\left[a(p)b(p)A_i^\dagger(p) + A_i^\dagger(p)\mathbb{C}\right], \\
\mathbb{C}^\dagger A_i^\dagger(p) &= \mathrm{e}^{ip}\left[c(p)d(p)A_i^\dagger(p) + A_i^\dagger(p)\mathbb{C}^\dagger\right], \\
\mathbb{H} A_i^\dagger(p) &= [a(p)d(p) + b(p)c(p)] A_i^\dagger(p) + A_i^\dagger(p)\mathbb{H},
\end{aligned}
\tag{2.30}
$$

can be used to act with the generators on multiparticle states. The ZF operators form a representation of the symmetry algebra provided $ad - bc = 1$. The representation is also unitary provided $d = a^*, c = b^*$. Acting with $\mathbb{C}$ on both sides of Equation (2.23) applied to the vacuum state, one can deduce the further constraint

$$
\begin{aligned}
&\mathrm{e}^{-ip_1}a(p_1)b(p_1) + \mathrm{e}^{-i(p_1+p_2)}a(p_2)b(p_2) \\
&= \mathrm{e}^{-ip_2}a(p_2)b(p_2) + \mathrm{e}^{-i(p_1+p_2)}a(p_1)b(p_1),
\end{aligned}
\tag{2.31}
$$

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[4]Such momentum-dependent braiding relations, which are typical for nonlocal (fractional-spin) integrals of motion, have long been used to determine S-matrices in certain integrable models, see, e.g. [21–23].

which leads to the relation $a(p)b(p) = ig(e^{ip} - 1)$, where g is a constant. It follows that the parameters can be chosen as follows [3,9,20]:

$$a = \sqrt{g}\,\eta, \quad b = \sqrt{g}\,\frac{i}{\eta}\left(\frac{x^+}{x^-} - 1\right), \quad c = -\sqrt{g}\,\frac{\eta}{x^+}, \quad d = \sqrt{g}\,\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right), \quad (2.32)$$

where

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}, \quad \frac{x^+}{x^-} = e^{ip}, \quad \eta = e^{ip/4}\sqrt{i(x^- - x^+)}. \quad (2.33)$$

Hence, for a one-particle state,

$$\mathbb{H} = -ig\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right) = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}. \quad (2.34)$$

The anomalous dimension $\mathbb{H} - 1$ matches with the weak-coupling result given by (2.5) and (2.7), provided we make the identification $g = \sqrt{\lambda}/(4\pi)$. That is, the symmetry determines the exact dispersion relation, except for the dependence on the coupling constant. See also, [24].

The S-matrix can be determined (up to a phase) by demanding that it commute with the symmetry generators. That is, starting from $\mathbb{J}\, A_i^\dagger(p_1)\, A_j^\dagger(p_2)|0\rangle$ where $\mathbb{J}$ is a symmetry generator, and assuming that $\mathbb{J}$ annihilates the vacuum state, one can arrive at linear combinations of $A_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)|0\rangle$ in two different ways, by applying the ZF relation (2.23) and the symmetry relations (2.28), (2.29) in different orders. The consistency condition is a system of linear equations for the S-matrix elements. The result for the nonzero matrix elements $S_{i\,j}^{i'\,j'}(p_1, p_2)$ is [3,9]

$$
\begin{aligned}
&S_{a\,a}^{a\,a} = A, \quad S_{\alpha\,\alpha}^{\alpha\,\alpha} = D, \\
&S_{a\,b}^{a\,b} = \frac{1}{2}(A - B), \quad S_{a\,b}^{b\,a} = \frac{1}{2}(A + B), \\
&S_{\alpha\,\beta}^{\alpha\,\beta} = \frac{1}{2}(D - E), \quad S_{\alpha\,\beta}^{\beta\,\alpha} = \frac{1}{2}(D + E), \\
&S_{a\,b}^{\alpha\,\beta} = -\frac{1}{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\, C, \quad S_{\alpha\,\beta}^{a\,b} = -\frac{1}{2}\varepsilon^{ab}\varepsilon_{\alpha\beta}\, F, \\
&S_{a\,\alpha}^{a\,\alpha} = G, \quad S_{a\,\alpha}^{a\,\alpha} = H, \quad S_{\alpha\,a}^{\alpha\,a} = K, \quad S_{\alpha\,a}^{\alpha\,a} = L,
\end{aligned}
\qquad (2.35)
$$

where $a, b \in \{1, 2\}$ with $a \neq b$; $\alpha, \beta \in \{3, 4\}$ with $\alpha \neq \beta$; and

$$
\begin{aligned}
A &= S_0 \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\,\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}, \\
B &= -S_0\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\right]\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}, \\
C &= S_0 \frac{2ix_1^- x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+(x_1^- - x_2^+)(1 - x_1^- x_2^-)}, \quad D = -S_0, \\
E &= S_0\left[1 - 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\right], \quad (2.36)
\end{aligned}
$$

$$F = S_0 \frac{2i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde{\eta}_1 \tilde{\eta}_2},$$

$$G = S_0 \frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)} \frac{\eta_1}{\tilde{\eta}_1}, \quad H = S_0 \frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)} \frac{\eta_1}{\tilde{\eta}_2},$$

$$K = S_0 \frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)} \frac{\eta_2}{\tilde{\eta}_1}, \quad L = S_0 \frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)} \frac{\eta_2}{\tilde{\eta}_2},$$

where $x_i^\pm = x^\pm(p_i)$ and

$$\eta_1 = \eta(p_1)e^{ip_2/2}, \quad \eta_2 = \eta(p_2), \quad \tilde{\eta}_1 = \eta(p_1), \quad \tilde{\eta}_2 = \eta(p_2)e^{ip_1/2}, \tag{2.37}$$

where $\eta(p)$ is given in (2.33). This S-matrix satisfies the standard Yang–Baxter equation (2.25). It also satisfies the unitarity equation (2.26), provided that the scalar factor obeys

$$S_0(p_1, p_2)\, S_0(p_2, p_1) = 1. \tag{2.38}$$

In order to determine S_0, one should impose on the full S-matrix (2.21) crossing symmetry and other physical requirements, which will be explained in the next chapter of this volume [25]. The final result is given by

$$S_0(p_1, p_2)^2 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}} \, \sigma(p_1, p_2)^2, \tag{2.39}$$

where the dressing factor $\sigma(p_1, p_2)$ is called the BES/BHL phase factor [7,8].

We remark that the above S-matrix is in fact in the "string frame" (or "basis") [9]. Starting from the spin chain one obtains the S-matrix instead in the "spin-chain frame," where (2.37) is replaced by

$$\eta_1 = \eta(p_1), \quad \eta_2 = \eta(p_2), \quad \tilde{\eta}_1 = \eta(p_1), \quad \tilde{\eta}_2 = \eta(p_2). \tag{2.40}$$

The S-matrix in the spin-chain frame satisfies a "twisted" version of the Yang–Baxter equation, rather than (2.25).

We also remark that the $su(2|2)$ S-matrix is closely related [4,11] to Shastry's R-matrix [26,27] for the Hubbard model.

2.4. BOUND STATES

So far we have considered two-particle asymptotic scattering states. The two particles carrying real momenta can be widely separated. Another interesting case occurs when the two particles are closely localized and behave as a single particle. This kind of localized state is the bound state [28,29].

As a first example, let us consider again the $su(2)$ sector at one loop. In terms of

$$x = \frac{x_1 + x_2}{2}, \quad r = x_2 - x_1, \quad p_{1,2} = \frac{p}{2} \pm k, \tag{2.41}$$

we can reexpress the two-particle state (2.8) as

$$|\psi\rangle = \sum_{x,\,r} e^{ipx} \left(A_{XX}(12)e^{-ikr} + A_{XX}(21)e^{ikr} \right) |Z \cdots \overbrace{XZ \cdots ZX}^{r} \cdots Z\rangle. \qquad (2.42)$$

Notice that $r > 0$ by definition. To have a localized wave, the amplitude should decay exponentially as the distance r increases. This can be satisfied if we take $k = iq(q > 0)$ and $A_{XX}(12) = 0$. From Equation (2.12) this leads to a condition that $S(p_2, p_1)$ should have a pole. In other words, a simple pole of the S-matrix corresponds to a bound state. In terms of u-variables, this condition is satisfied by $u_{2,1} = u \pm i/2$ as one can see from (2.13). This is an example of a so-called string solution, of size 2. Following a similar procedure, one can find that the higher bound-state poles of the S-matrices can be obtained when the particles carry momenta

$$u_j^{(n)} = u + i\,\frac{2j - n - 1}{2}, \quad j = 1, \ldots, n. \qquad (2.43)$$

This is a string of size n. The energy of this particle can be obtained from (2.7)

$$\varepsilon_n(u) = \frac{n}{u^2 + n^2/4}. \qquad (2.44)$$

Now consider the more complicated case of the $su(3)$ sector, for which the two-particle eigenstates are given by (2.15) and (2.16). By the same argument as above, the localized state is possible when $u_2 - u_1 = i$. This leads to $A_{XY}(12) = A_{YX}(12) = 0$ from (2.17) and $A_{XY}(21) = A_{YX}(21)$ because the residues of T and R in (2.18) are the same. Therefore, the localized state can be written as

$$|\psi\rangle \sim \sum_{x,\,r} e^{ipx} e^{ikr} \left[|Z \cdots \overbrace{XZ \cdots ZY}^{r} \cdots Z\rangle + |Z \cdots \overbrace{YZ \cdots ZX}^{r} \cdots Z\rangle \right], \qquad (2.45)$$

where X and Y appear symmetrically.

The bound states for generic value of 't Hooft coupling constant can be constructed in a similar way. Combining two factors of the amplitude A (2.36) with (2.39), the S-matrix of the $su(2)$ sector (in the spin-chain frame) is given by

$$S(p_1, p_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}} \sigma(p_1, p_2)^2. \qquad (2.46)$$

This amplitude has two simple poles at $x_1^- = x_2^+$ and $x_1^- = 1/x_2^+$. Let us consider first the former case for general higher-order bound states where simple poles appear

$$x_1^- = x_2^+, \quad x_2^- = x_3^+, \ldots, x_{n-1}^- = x_n^+. \qquad (2.47)$$

With these bound-state conditions, one can easily show that the momentum (p) and energy ($\mathbb{H}$) are given by

$$\frac{X^+}{X^-} = e^{ip}, \qquad X^+ + \frac{1}{X^+} - X^- - \frac{1}{X^-} = \frac{in}{g} \tag{2.48}$$

$$\mathbb{H} = -ig\left(X^+ - \frac{1}{X^+} - X^- + \frac{1}{X^-}\right) = \sqrt{n^2 + 16g^2 \sin^2\frac{p}{2}}, \tag{2.49}$$

and satisfy the BPS (shortening) condition in (2.48) if we identify

$$X^- \equiv x_n^-, \qquad \text{and} \qquad X^+ \equiv x_1^+. \tag{2.50}$$

The other pole at $x_1^- = 1/x_2^+$ cannot satisfy this condition and leads to non-BPS states.

The situation for the full $su(2|2)$ S-matrix is more complicated even though the locations of poles are the same as in the $su(2)$ sector. The M-particle bound states belong to an atypical totally symmetric representation of the centrally extended $su(2|2)$ algebra. This representation has dimension $2M|2M$ and can be realized on the graded vector space where the basis is given by

- $M + 1$ bosonic states: symmetric in $a_i : |e_{a_1 \cdots a_M}\rangle$, where $a_i = 1, 2$ are bosonic indices.
- $M - 1$ bosonic states: symmetric in $a_i : |e_{a_1 \cdots a_{M-2}\alpha_1\alpha_2}\rangle$, where $\alpha_i = 3, 4$ are fermionic indices.
- $2M$ fermionic states: symmetric in $a_i : |e_{a_1 \cdots a_{M-1}\alpha}\rangle$, where $\alpha = 3, 4$.

An efficient realization of this representation is to introduce [20] a vector space of analytic functions of two bosonic variables w_a and two fermionic variables θ_α. For example, the 8-dimensional states for $M = 2$ can be given by

$$|e_1\rangle = \frac{w_1 w_1}{\sqrt{2}}, \quad |e_2\rangle = w_1 w_2, \quad |e_3\rangle = \frac{w_2 w_2}{\sqrt{2}}, \quad |e_4\rangle = \theta_3\theta_4,$$

$$|e_5\rangle = w_1\theta_3, \quad |e_6\rangle = w_1\theta_4, \quad |e_7\rangle = w_2\theta_3, \quad |e_8\rangle = w_2\theta_4. \tag{2.51}$$

The $su(2|2)$ generators can be represented by differential operators on this vector space as follows:

$$\mathbb{L}_a{}^b = w_a \frac{\partial}{\partial w_b} - \frac{1}{2}\delta_a^b w_c \frac{\partial}{\partial w_c}, \quad \mathbb{R}_\alpha{}^\beta = \theta_\alpha \frac{\partial}{\partial\theta_\beta} - \frac{1}{2}\delta_\alpha^\beta \theta_\gamma \frac{\partial}{\partial\theta_\gamma},$$

$$\mathbb{Q}_\alpha{}^a = a\,\theta_\alpha \frac{\partial}{\partial w_a} + b\,\varepsilon^{ab}\varepsilon_{\alpha\beta} w_b \frac{\partial}{\partial\theta_\beta}, \quad \mathbb{Q}_a^{\dagger\alpha} = d\,w_a \frac{\partial}{\partial\theta_\alpha} + c\,\varepsilon_{ab}\varepsilon^{\alpha\beta}\theta_\beta \frac{\partial}{\partial w_b},$$

$$\mathbb{C} = ab\left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial\theta_\alpha}\right), \qquad \mathbb{C}^\dagger = cd\left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial\theta_\alpha}\right),$$

$$\mathbb{H} = (ad + bc)\left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial\theta_\alpha}\right). \tag{2.52}$$

From this, it is straightforward to evaluate how the generators act on the bound states.

In contrast with the case of the fundamental representation reviewed in the previous subsection, the $su(2|2)$ symmetry is not enough to determine the bound-state S-matrix completely. A very important observation is that the fundamental bulk S-matrix (2.35) has a remarkable Yangian symmetry $Y(su(2|2))$ [30,31] which can be used to completely determine the two-particle [20,32] and general l-particle bound state bulk S-matrices [33]. It is fortunate that such a general way of generating higher-dimensional S-matrices has been found, since the fusion procedure does not seem to work for AdS/CFT S-matrices [20].

3. Asymptotic Bethe Equations

For a system of N *free* particles on a ring of length L, the quantized momenta, and therefore the exact spectrum, are trivially determined. For particles which are not free but instead have *integrable* interactions, the problem of determining the spectrum is much more difficult, but nevertheless is still tractable. Indeed, if one knows the (asymptotic) S-matrix which satisfies the Yang–Baxter equations, then in principle it is possible to derive a set of (asymptotic) Bethe equations which determine the (asymptotic) quantized momenta, and therefore, the (asymptotic) multiparticle spectrum. These (asymptotic) Bethe equations are obtained by imposing periodicity on the (asymptotic) multiparticle wavefunction. In the AdS/CFT case, this task is technically difficult due to the matrix structure of the S-matrix and the complicated functional dependence of its matrix elements. Before addressing this problem, it is helpful to consider some simpler examples.

3.1. THE S-MATRIX IS A PHASE

As a first warm-up exercise, let us consider the simple case of a two-body (asymptotic) S-matrix which is a phase rather than a matrix.[5] An example is the magnon–magnon S-matrix in the $su(2)$ sector at one loop, which is given by (2.13), (2.14). The ZF operator $A^\dagger(p)$ does not have an internal index, and satisfies (cf., (2.23))

$$A^\dagger(p_1)\, A^\dagger(p_2) = S(p_1, p_2)\, A^\dagger(p_2)\, A^\dagger(p_1). \tag{3.1}$$

Integrability of the model implies that the multiparticle wavefunction is of the Bethe type. That is, the (asymptotic) eigenstates can be expressed as

$$|\psi\rangle = \sum_{1 \le x_{Q_1} \ll \cdots \ll x_{Q_N} \le L} \Psi^{(Q)}(x_1, \ldots, x_N)| \overset{1}{\underset{\downarrow}{Z}} \cdots \overset{x_{Q_1}}{\underset{\downarrow}{X}} \cdots \overset{x_{Q_N}}{\underset{\downarrow}{X}} \cdots \overset{L}{\underset{\downarrow}{Z}}\rangle, \tag{3.2}$$

---

[5]In this case, the Yang–Baxter equations are trivially satisfied by the S-matrix.

where the (asymptotic) N-particle wavefunction in the sector $Q = (Q_1, \ldots, Q_N)$ such that $x_{Q_1} \ll \cdots \ll x_{Q_N}$ is given by

$$\Psi^{(Q)}(x_1, \ldots, x_N) = \sum_P A^P e^{i p_P \cdot x_Q}. \tag{3.3}$$

The sum is over all permutations of $P = (P_1, \ldots, P_N)$, and $p_P \cdot x_Q = \sum_{k=1}^N p_{P_k} x_{Q_k}$. Also, the coordinate-independent amplitudes A^P are related to each other according to

$$A^P \sim A^\dagger(p_{P_1}) \cdots A^\dagger(p_{P_N}). \tag{3.4}$$

For example, for $N = 2$, the wavefunction in the sector $x_1 \ll x_2$ is given by

$$\Psi^{(12)}(x_1, x_2) = A^{12} e^{i(p_1 x_1 + p_2 x_2)} + A^{21} e^{i(p_2 x_1 + p_1 x_2)}, \quad x_1 \ll x_2. \tag{3.5}$$

Since

$$A^{21} \sim A^\dagger(p_2) A^\dagger(p_1) = S(p_2, p_1) \, A^\dagger(p_1) A^\dagger(p_2) \sim S(p_2, p_1) \, A^{12}, \tag{3.6}$$

we recover the previous results (2.8), (2.9), (2.12) upon identifying

$$A_{XX}(12) = A^{12}, \quad A_{XX}(21) = A^{21}. \tag{3.7}$$

We consider a system of N widely-separated particles on a ring of length L. Periodicity of the wavefunction $\Psi(x_1, \ldots, x_N)$ in (say) the first coordinate,

$$\Psi(1, x_2, \ldots, x_N) = \Psi(L + 1, x_2, \ldots, x_N), \tag{3.8}$$

implies a relationship between the wavefunctions in the sectors $x_1 \ll \cdots \ll x_N$ and $x_2 \ll \cdots \ll x_N \ll x_1$:

$$\Psi^{(1 \cdots N)}(1, x_2, \ldots, x_N) = \Psi^{(2 \cdots N1)}(L + 1, x_2, \ldots, x_N). \tag{3.9}$$

According to (3.3), the wavefunctions in these two sectors are given by

$$
\begin{aligned}
\Psi^{(1 \cdots N)}(1, x_2, \ldots, x_N) &= A^{1 \cdots N} e^{i(p_1 + p_2 x_2 + \cdots + p_N x_N)} + \cdots, \\
\Psi^{(2 \cdots N1)}(L + 1, x_2, \ldots, x_N) &= A^{2 \cdots N1} e^{i(p_1 L + p_1 + p_2 x_2 + \cdots + p_N x_N)} + \cdots,
\end{aligned} \tag{3.10}
$$

where we have displayed only the terms which depend on the particular combination $p_2 x_2 + \cdots + p_N x_N$. In view of the periodicity condition (3.9), the coefficients $A^{1 \cdots N}$ and $A^{2 \cdots N1}$ in (3.10) must be related as follows:

$$A^{1 \cdots N} = A^{2 \cdots N1} e^{i p_1 L}. \tag{3.11}$$

There is another relation between the coefficients $A^{1 \cdots N}$ and $A^{2 \cdots N1}$ which follows from (3.4). Indeed, it is easy to see that

$$
\begin{aligned}
A^{1 \cdots N} &\sim A^\dagger(p_1) A^\dagger(p_2) \cdots A^\dagger(p_N) \\
&= \prod_{j=2}^N S(p_1, p_j) A^\dagger(p_2) \cdots A^\dagger(p_N) A^\dagger(p_1) \sim \prod_{j=2}^N S(p_1, p_j) A^{2 \cdots N1},
\end{aligned} \tag{3.12}
$$

where we have used (3.1) to move $A^\dagger(p_1)$ to the right successively past all the other ZF operators. The two relations (3.11) and (3.12) imply that

$$\prod_{j=2}^{N} S(p_1, p_j) = e^{ip_1 L}. \tag{3.13}$$

Examining the terms in the ellipsis in (3.10) similarly leads to the (asymptotic) Bethe equations for all the momenta,

$$\prod_{\substack{j=1 \\ j \neq k}}^{N} S(p_k, p_j) = e^{ip_k L}, \quad k = 1, \ldots, N. \tag{3.14}$$

For a "local" S-matrix such as the one for the spin-1/2 ferromagnetic Heisenberg chain, these equations are exact for finite L; at least in principle one can solve these equations for the momenta and therefore compute the exact finite-L spectrum,

$$\mathbf{P} = \sum_{k=1}^{N} p_k, \quad \mathbf{E} = \sum_{k=1}^{N} \varepsilon(p_k), \tag{3.15}$$

where $\varepsilon(p)$ is the one-particle dispersion relation (see, e.g. (2.7)). For an asymptotic S-matrix such as the one for AdS/CFT, the asymptotic Bethe equations can be used to determine the spectrum only asymptotically.[6]

3.2. THE S-MATRIX IS A 4×4 MATRIX

As a second warm-up exercise, we consider a solution of the Yang–Baxter equations which is a 4×4 matrix. For simplicity, we further restrict the S-matrix to be $su(2)$-invariant. Hence, we take

$$S_{jk}^{j'k'}(p_1, p_2) = \frac{1}{u_1 - u_2 - i} \left[(u_1 - u_2)\delta_j^{j'}\delta_k^{k'} + i\delta_j^{k'}\delta_k^{j'} \right], \tag{3.16}$$

where again $u_j = u(p_j)$ and $u(p)$ is given by (2.14). This is in fact the magnon–magnon S-matrix in the $su(3)$ sector which we discussed earlier (2.19). The ZF operator now has an internal index which can take the values 1 and 2, and satisfies (2.23). As we shall see, the analysis is similar to the one in Section 3.1. The new feature is the internal symmetry, which is handled neatly by introducing the transfer matrix (3.25).

The (asymptotic) eigenstates can now be expressed as

$$|\psi\rangle = \sum_{1 \leq x_{Q_1} \ll \cdots \ll x_{Q_N} \leq L} \sum_{i_1, \ldots, i_N = 1}^{2} \Psi_{i_1 \cdots i_N}^{(Q)}(x_1, \ldots, x_N) | \overset{1}{Z} \cdots \overset{x_{Q_1}}{\phi_{i_1}} \cdots \overset{x_{Q_N}}{\phi_{i_N}} \cdots \overset{L}{Z} \rangle, \tag{3.17}$$

---

[6]Nevertheless, it is possible to obtain at least a part of the exact spectrum by other means [34].

where the (asymptotic) N-particle wavefunction in the sector $Q = (Q_1, \ldots, Q_N)$ is given by[7]

$$\Psi^{(Q)}_{i_1 \cdots i_N}(x_1, \ldots, x_N) = \sum_P A^{P|Q}_{i_1 \cdots i_N} e^{i p_P \cdot x_Q} \tag{3.18}$$

and

$$A^{P|Q}_{i_1 \cdots i_N} \sim A^{\dagger}_{i_{Q_1}}(p_{P_1}) \cdots A^{\dagger}_{i_{Q_N}}(p_{P_N}), \tag{3.19}$$

cf. (3.2)–(3.4). For $N = 2$ in the sector $x_1 \ll x_2$, upon identifying

$$A_{\phi_i \phi_j}(12) = A^{12|12}_{ij}, \quad A_{\phi_i \phi_j}(21) = A^{21|12}_{ij} \tag{3.20}$$

where $\phi_1 = X, \phi_2 = Y$, we recover the previous results (2.15)–(2.19).[8]

Proceeding as before, we see that the periodicity of the wavefunction in the first coordinate,

$$\Psi_{i_1 \cdots i_N}(1, x_2, \ldots, x_N) = \Psi_{i_1 \cdots i_N}(L + 1, x_2, \ldots, x_N) \tag{3.21}$$

implies a relationship between the wavefunctions in the sectors $x_1 \ll \cdots \ll x_N$ and $x_2 \ll \cdots \ll x_N \ll x_1$:

$$\Psi^{(1 \cdots N)}_{i_1 \cdots i_N}(1, x_2, \ldots, x_N) = \Psi^{(2 \cdots N 1)}_{i_1 \cdots i_N}(L + 1, x_2, \ldots, x_N). \tag{3.22}$$

This leads to the following relationship between coefficients:

$$A^{1 \cdots N|1 \cdots N}_{i_1 \cdots i_N} = A^{2 \cdots N 1|2 \cdots N 1}_{i_1 \cdots i_N} e^{i p_1 L}. \tag{3.23}$$

We now proceed to generate from (3.19) another relation between these two coefficients. Using (2.23) to move $A^{\dagger}_{i_1}(p_1)$ to the right successively past all the other ZF operators, we obtain

$$\begin{aligned}
A^{1 \cdots N|1 \cdots N}_{i_1 \cdots i_N} &\sim A^{\dagger}_{i_1}(p_1) A^{\dagger}_{i_2}(p_2) \cdots A^{\dagger}_{i_N}(p_N) \\
&= S^{a_2 i'_2}_{i_1 i_2}(p_1, p_2) S^{a_3 i'_3}_{a_2 i_3}(p_1, p_3) \cdots S^{i'_1 i'_N}_{a_{N-1} i_N}(p_1, p_N) A^{\dagger}_{i'_2}(p_2) \cdots A^{\dagger}_{i'_N}(p_N) A^{\dagger}_{i'_1}(p_1) \\
&\sim S^{a_2 i'_2}_{i_1 i_2}(p_1, p_2) S^{a_3 i'_3}_{a_2 i_3}(p_1, p_3) \cdots S^{i'_1 i'_N}_{a_{N-1} i_N}(p_1, p_N) A^{2 \cdots N 1|2 \cdots N 1}_{i'_1 \cdots i'_N}.
\end{aligned} \tag{3.24}$$

---

[7]The original papers include [35–38]. Here, we follow the appendix in [39].

[8]For example,

$$\begin{aligned}
A_{XY}(21) = A^{21|12}_{12} &\sim A^{\dagger}_1(p_2) A^{\dagger}_2(p_1) = S^{12}_{12} A^{\dagger}_2(p_1) A^{\dagger}_1(p_2) + S^{21}_{12} A^{\dagger}_1(p_1) A^{\dagger}_2(p_2) \\
&\sim S^{12}_{12} A^{12|12}_{21} + S^{21}_{12} A^{12|12}_{12} = T A_{YX}(12) + R A_{XY}(12),
\end{aligned}$$

which is in agreement with (2.17). Here, the arguments (p_2, p_1) of all the S-matrix elements have been suppressed for brevity.

It is very convenient to introduce the so-called (inhomogeneous) transfer matrix

$$t^{i'_1 \cdots i'_N}_{i_1 \cdots i_N}(p; p_1, \ldots, p_N) \equiv S^{a_1 i'_1}_{a_N i_1}(p, p_1) S^{a_2 i'_2}_{a_1 i_2}(p, p_2) \cdots S^{a_N i'_N}_{a_{N-1} i_N}(p, p_N). \tag{3.25}$$

Its value at $p = p_1$ is proportional to the coefficient of $A^{2 \cdots N1 | 2 \cdots N1}_{i'_1 \cdots i'_N}$ in (3.24),

$$t^{i'_1 \cdots i'_N}_{i_1 \cdots i_N}(p_1; p_1, \ldots, p_N) = -S^{a_2 i'_2}_{i_1 i_2}(p_1, p_2) S^{a_3 i'_3}_{a_2 i_3}(p_1, p_3) \cdots S^{i'_1 i'_N}_{a_{N-1} i_N}(p_1, p_N), \quad (3.26)$$

since $S^{i'j'}_{ij}(p, p) = -\delta^{j'}_i \delta^{i'}_j$, as one can see from (3.16).

We demand that $A^{2 \cdots N1 | 2 \cdots N1}_{i'_1 \cdots i'_N}$ be an eigenvector of the transfer matrix,[9]

$$t^{i'_1 \cdots i'_N}_{i_1 \cdots i_N}(p; p_1, \ldots, p_N) A^{2 \cdots N1 | 2 \cdots N1}_{i'_1 \cdots i'_N} = \Lambda(p; p_1, \ldots, p_N) A^{2 \cdots N1 | 2 \cdots N1}_{i_1 \cdots i_N}, \tag{3.27}$$

where $\Lambda(p; p_1, \ldots, p_N)$ is the corresponding eigenvalue. It follows from Equations (3.23), (3.24), (3.26), (3.27) that

$$\Lambda(p_1; p_1, \ldots, p_N) = -e^{i p_1 L}; \tag{3.28}$$

and more generally

$$\Lambda(p_k; p_1, \ldots, p_N) = -e^{i p_k L}, \qquad k = 1, \ldots, N. \tag{3.29}$$

To summarize so far: imposing periodic boundary conditions on the multiparticle wavefunction has led to the important relations (3.29). However, in order to obtain more explicit equations for the momenta, we need the eigenvalues $\Lambda(p; p_1, \ldots, p_N)$ of the transfer matrix (3.25). For the case of the S-matrix (3.16), the result is well known [40–42],

$$\Lambda(p; p_1, \ldots, p_N) = \frac{1}{\prod_{l=1}^{N}(u - u_l - i)} \left\{ \prod_{l=1}^{N}(u - u_l + i) \prod_{l=1}^{m} \left(\frac{u - \lambda_l - \frac{i}{2}}{u - \lambda_l + \frac{i}{2}} \right) \right.$$
$$\left. + \prod_{l=1}^{N}(u - u_l) \prod_{l=1}^{m} \left(\frac{u - \lambda_l + \frac{3i}{2}}{u - \lambda_l + \frac{i}{2}} \right) \right\}, \tag{3.30}$$

where the "auxiliary" Bethe roots $\lambda_1, \ldots, \lambda_m$ satisfy the Bethe ansatz equations

$$\prod_{l=1}^{N} \frac{\lambda_k - u_l + \frac{i}{2}}{\lambda_k - u_l - \frac{i}{2}} = \prod_{\substack{j=1 \\ j \neq k}}^{m} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \quad k = 1, \ldots, m. \tag{3.31}$$

---

[9]This is necessary in order to be able to satisfy (3.23). We note that the transfer matrix has the commutativity property

$$\left[t(p; p_1, \ldots, p_N), t(p'; p_1, \ldots, p_N) \right] = 0$$

by virtue of the fact that the S-matrix satisfies the Yang–Baxter equation. (See, e.g. [40]-[42].) Hence, the corresponding eigenvectors do not depend on the value of p.

Finally, substituting the result (3.30) into (3.29), we obtain

$$\prod_{l=1}^{N} \frac{u_k - u_l + i}{u_k - u_l - i} \prod_{l=1}^{m} \frac{u_k - \lambda_l - \frac{i}{2}}{u_k - \lambda_l + \frac{i}{2}} = -e^{i p_k L}, \quad k = 1, \ldots, N. \tag{3.32}$$

The coupled set of equations (3.31) and (3.32) are the sought-after (asymptotic) Bethe equations for a system of N particles on a ring of length L with the two-particle (asymptotic) S-matrix (3.16).

3.3. AdS/CFT

We are finally ready to address the AdS/CFT case, albeit only sketchily. The arguments of Section 3.2 leading to (3.29) carry through essentially unchanged.[10] The difficult step is determining the eigenvalues of the transfer matrix. Whereas for the 4×4 S-matrix (3.16) the result (3.30) is easily obtained by algebraic Bethe ansatz, for the larger AdS/CFT S-matrix (2.35), (2.36) a more general procedure (namely, *nested* algebraic Bethe ansatz) is required [11]. Alternatively, the result can be obtained by nested coordinate Bethe ansatz [3,12] or by analytic Bethe ansatz [4]. In this way, one can derive the AdS_5/CFT_4 asymptotic Bethe equations which were first conjectured in [10]. In terms of the compact notation introduced in [43], these equations are given by

$$U_0 = 1, \quad U_j(x_{j,k}) \prod_{\substack{j'=1 \\ (j',k') \neq (j,k)}}^{7} \prod_{k'=1}^{K_{j'}} \frac{u_{j,k} - u_{j',k'} + \frac{i}{2} M_{j,j'}}{u_{j,k} - u_{j',k'} - \frac{i}{2} M_{j,j'}} = 1, \quad j = 1, \ldots, 7, \tag{3.33}$$

where $u_{j,k} = g(x_{j,k} + 1/x_{j,k}), u_{j,k} \pm i/2 = g(x_{j,k}^{\pm} + 1/x_{j,k}^{\pm})$, and $M_{j,j'}$ is the Cartan matrix specified by Figure 1. Explicitly,

$$M = \begin{pmatrix} & & 1 & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & \end{pmatrix}, \tag{3.34}$$

where matrix elements which are zero are left empty. Also,

$$U_0 = \prod_{k=1}^{K_4} \frac{x_{4,k}^{+}}{x_{4,k}^{-}}, \quad U_2 = U_6 = 1,$$

$$U_1(x) = U_3^{-1}(x) = U_5^{-1}(x) = U_7(x) = \prod_{k=1}^{K_4} S_{\text{aux}}(x_{4,k}, x) \tag{3.35}$$

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[10]It is convenient to work in a graded formalism, where certain minus signs appear [11].

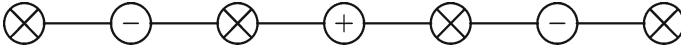*Figure 1.* Dynkin diagram of $su(2, 2|4)$.

and

$$U_4(x) = U_s(x) \left(\frac{x^-}{x^+}\right)^L \prod_{k=1}^{K_1} S_{\text{aux}}^{-1}(x, x_{1,k}) \prod_{k=1}^{K_3} S_{\text{aux}}(x, x_{3,k})$$

$$\times \prod_{k=1}^{K_5} S_{\text{aux}}(x, x_{5,k}) \prod_{k=1}^{K_7} S_{\text{aux}}^{-1}(x, x_{7,k}). \tag{3.36}$$

Moreover,

$$S_{\text{aux}}(x_1, x_2) = \frac{1 - 1/x_1^+ x_2}{1 - 1/x_1^- x_2}, \quad U_s(x) = \prod_{k=1}^{K_4} \sigma(x, x_{4,k})^2, \tag{3.37}$$

where σ is the dressing phase [8,25]. The anomalous dimensions of a state is given by

$$\Gamma = 2ig \sum_{k=1}^{K_4} \left(\frac{1}{x_{4,k}^+} - \frac{1}{x_{4,k}^-}\right). \tag{3.38}$$

For further important details such as the restrictions on the excitation numbers $K_1, \ldots, K_7$, the so-called dynamical transformations relating roots of type 1 and type 3 (and similarly, roots of type 5 and type 7), and the weak-coupling limit, the reader should consult [10,43].

Similarly, starting from the AdS_4/CFT_3 S-matrix [44], one can derive the corresponding asymptotic Bethe equations which were first conjectured in [45].

4. Concluding Remarks

The all-loop AdS_5/CFT_4 S-matrix has further important applications. In particular, it is used for computing wrapping corrections via the Lüscher formula (reviewed in [34]) and finite-size effects via thermodynamic Bethe ansatz (reviewed in [46]). A certain Drinfeld twist of this S-matrix, together with c-number diagonal twists of the boundary conditions, lead [47] to the deformed Bethe equations of Beisert and Roiban [43,48].

The $su(2|2)$ S-matrix of AdS_5/CFT_4 also plays an important role in determining the S-matrix of AdS_4/CFT_3 [44] (see also [49]). Indeed, the scattering matrices for the two types of particles ("solitons" and "antisolitons") again have the same $su(2|2)$ matrix structure; the main difference with respect to the AdS_5/CFT_4 case is in the scalar factors, which satisfy new crossing relations. As already noted, this S-matrix leads to the all-loop BAEs conjectured in [45].

Acknowledgments

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