

$\mathcal{N} = 6$ super Chern-Simons theory S -matrix and all-loop Bethe ansatz equations

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ABSTRACT: We propose the exact S -matrix for the planar limit of the $\mathcal{N} = 6$ super Chern-Simons theory recently proposed by Aharony, Bergman, Jafferis, and Maldacena for the $\text{AdS}_4/\text{CFT}_3$ correspondence. Assuming $\text{SU}(2|2)$ symmetry, factorizability and certain crossing-unitarity relations, we find the S -matrix including the dressing phase. We use this S -matrix to formulate the asymptotic Bethe ansatz. Our result for the Bethe-Yang equations and corresponding Bethe ansatz equations confirms the all-loop Bethe ansatz equations recently conjectured by Gromov and Vieira.

KEYWORDS: AdS-CFT Correspondence, Chern-Simons Theories.

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1. Introduction

The AdS/CFT correspondence [1–3], which has led to many exciting developments in the duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills (YM) theory, is now being extended into AdS_4/CFT_3 [4]. A most promising candidate is $\mathcal{N} = 6$ super Chern-Simons (CS) theory with $SU(N) \times SU(N)$ gauge symmetry and level k . This model, which was first proposed by Aharony, Bergman, Jafferis, and Maldacena [5], is believed to be dual to M-theory on $AdS_4 \times S^7/Z_k$. Furthermore, in the planar limit of $N, k \rightarrow \infty$ with a fixed value of 't Hooft coupling $\lambda = N/k$, the $\mathcal{N} = 6$ CS is believed to be dual to type IIA superstring theory on $AdS_4 \times CP^3$. This model contains two sets of scalar fields transforming in bifundamental representations of $SU(N) \times SU(N)$ along with respective superpartner fermions and non-dynamic CS gauge fields. (For some subsequent developments, see [6, 7].)

The integrability of the planar $\mathcal{N} = 6$ CS was first discovered by Minahan and Zarembo [8] in the leading two-loop-order perturbative computation of the anomalous dimensions of gauge-invariant composite operators. They found that the dilatation operator for the scalar operators is an integrable Hamiltonian of an $SU(4)$ spin chain with sites alternating between the fundamental and anti-fundamental representations. They obtained corresponding two-loop Bethe ansatz equations (BAEs) using algebraic Bethe ansatz results for an inhomogeneous spin chain with different representations developed in [9]. They then conjectured two-loop BAEs for all operators (including the fermions) corresponding to the full $OSp(2, 2|6)$ superconformal group.

More recently, all-loop BAEs for the $\mathcal{N} = 6$ CS were conjectured by Gromov and Vieira [12] based on the perturbative result [8] and the classical integrability in the large-coupling limit discovered in [13–15].

The purpose of this note is to propose an exact S -matrix for the $\mathcal{N} = 6$ CS, and to derive the all-loop BAEs [12] from this S -matrix. A factorizable S -matrix has played an important role in the developments of AdS_5/CFT_4 . Indeed, an S -matrix describing the scattering of excitations of the dynamic spin chain corresponding to planar $\mathcal{N} = 4$

YM has been proposed by Beisert [16], and a related S -matrix describing the scattering of world-sheet excitations of the $AdS_5 \times S^5$ superstring sigma model has been proposed by Arutyunov, Frolov and Zamaklar (AFZ) [17]. These S -matrices have been derived from the assumption that the excitations are described by a Zamolodchikov-Faddeev (ZF) algebra [18, 19], and that they have a centrally extended $su(2|2) \oplus su(2|2)$ symmetry [16]. The AFZ “string” S -matrix obeys the standard Yang-Baxter equation, while Beisert’s S -matrix obeys a twisted (dynamical) Yang-Baxter equation. The S -matrices are a very useful tool to overcome shortcomings from the Bethe ansatz approach such as the wrapping problem [20] and for full quantization of the string theory.

As discussed in more detail below, $\mathcal{N} = 6$ CS has two sets of excitations, namely A -particles and B -particles, each of which form a four-dimensional representation of $SU(2|2)$. We propose an S -matrix with the following structure:¹

$$\begin{aligned} S^{AA}(p_1, p_2) &= S^{BB}(p_1, p_2) = S_0(p_1, p_2) \widehat{S}(p_1, p_2), \\ S^{AB}(p_1, p_2) &= S^{BA}(p_1, p_2) = \widetilde{S}_0(p_1, p_2) \widehat{S}(p_1, p_2), \end{aligned}$$

where \widehat{S} is the matrix part determined by the $SU(2|2)$ symmetry, and is essentially the same as that found for $\mathcal{N} = 4$ YM in [17]. An important difference arises in the dressing phases S_0, \widetilde{S}_0 due to the fact that the A - and B -particles are related by complex conjugation. Crossing symmetry [22] relates the two phases and gives different dressing phases for the S -matrices. These dressing phases play a crucial role in the process of deriving the Bethe-Yang equations.

The outline of this paper is as follows. In section 2, we use the bulk ZF algebra and crossing relations to construct the S -matrix. In section 3, all-loop asymptotic BAEs are derived from diagonalizing the Bethe-Yang matrix. We conclude in section 4 with a brief discussion of our results.

2. Excitations and S -matrix

We recall [5] that the $\mathcal{N} = 6$ CS theory has a pair of scalar fields A_i ($i = 1, 2$) in the bifundamental representation $(\mathbf{N}, \bar{\mathbf{N}})$ of the $SU(N) \times SU(N)$ gauge group, and another pair of scalar fields B_i ($i = 1, 2$) in the conjugate representation $(\bar{\mathbf{N}}, \mathbf{N})$. The vacuum is given by the infinite chain [7, 10]

$$\text{tr}(A_1 B_1 A_1 B_1 \dots). \tag{2.1}$$

The vacuum preserves an $SU(2|2)$ subgroup of $OSp(2, 2|6)$. There are two types of elementary excitations: “ A -particles”, which correspond to replacing A_1 by $A_2, B_2^\dagger, (\psi_{B_2}^\dagger)_\alpha$; and “ B -particles”, which correspond to replacing B_1 by $A_2^\dagger, B_2, (\psi_{A_2}^\dagger)_\alpha$. We therefore identify the B -particles as charge conjugates of the A -particles.

It is convenient to represent these A -particles and B -particles by Zamolodchikov-Faddeev operators $A_i^\dagger(p)$ and $B_i^\dagger(p)$ ($i = 1, \dots, 4$), respectively. When acting on the vacuum

¹This structure is similar to that of the S -matrix proposed in [21] for the $O(3)$ sigma model with $\theta = \pi$, where the two types of particles are the left-movers and right-movers.

state $|0\rangle$, these operators create corresponding asymptotic particle states of momentum p and energy E given by [7, 10, 11]

$$E = \sqrt{\frac{1}{4} + 4g^2 \sin^2 \frac{p}{2}}, \quad (2.2)$$

where g is a function of the 't Hooft coupling

$$g = h(\lambda), \quad (2.3)$$

with $h(\lambda) \sim \lambda$ for small λ , and $h(\lambda) \sim \sqrt{\lambda/2}$ for large λ .

We define the A - A S -matrix by

$$A_i^\dagger(p_1) A_j^\dagger(p_2) = S^{AA}{}_{ij}{}^{i'j'}(p_1, p_2) A_{j'}^\dagger(p_2) A_{i'}^\dagger(p_1). \quad (2.4)$$

The $SU(2|2)$ symmetry implies that, up to a scalar factor, this S -matrix is the same as the one for $\mathcal{N} = 4$ YM theory. Hence,

$$S^{AA}(p_1, p_2) = S_0(p_1, p_2) \widehat{S}(p_1, p_2), \quad (2.5)$$

where $\widehat{S}(p_1, p_2)$ is the $SU(2|2)$ S -matrix [17] with g given by (2.3). It satisfies the Yang-Baxter equation, as well as unitarity

$$\widehat{S}_{12}(p_1, p_2) \widehat{S}_{21}(p_2, p_1) = \mathbb{I} \quad (2.6)$$

and the crossing relation [22]

$$\widehat{S}_{12}^{t_2}(p_1, p_2) C_2 \widehat{S}_{12}(p_1, \bar{p}_2) C_2^{-1} = \widehat{S}_{12}^{t_1}(p_1, p_2) C_1 \widehat{S}_{12}(\bar{p}_1, p_2) C_1^{-1} = f(p_1, p_2) \mathbb{I}, \quad (2.7)$$

where C is the charge conjugation matrix, and

$$f(p_1, p_2) = \frac{\left(\frac{1}{x_1^+} - x_2^-\right) (x_1^+ - x_2^+)}{\left(\frac{1}{x_1^-} - x_2^-\right) (x_1^- - x_2^+)}. \quad (2.8)$$

As usual,

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}, \quad \frac{x^+}{x^-} = e^{ip}, \quad (2.9)$$

and $x^\pm(\bar{p}) = 1/x^\pm(p)$. Moreover, $S_0(p_1, p_2)$ in (2.5) is a scalar factor which is yet to be determined.

Similarly, we define the B - B and A - B S -matrices by

$$B_i^\dagger(p_1) B_j^\dagger(p_2) = S^{BB}{}_{ij}{}^{i'j'}(p_1, p_2) B_{j'}^\dagger(p_2) B_{i'}^\dagger(p_1) \quad (2.10)$$

and

$$A_i^\dagger(p_1) B_j^\dagger(p_2) = S^{AB}{}_{ij}{}^{i'j'}(p_1, p_2) B_{j'}^\dagger(p_2) A_{i'}^\dagger(p_1), \quad (2.11)$$

respectively. Symmetry considerations suggest that

$$\begin{aligned} S^{BB}(p_1, p_2) &= S^{AA}(p_1, p_2) = S_0(p_1, p_2) \widehat{S}(p_1, p_2), \\ S^{AB}(p_1, p_2) &= S^{BA}(p_1, p_2) = \widetilde{S}_0(p_1, p_2) \widehat{S}(p_1, p_2), \end{aligned} \quad (2.12)$$

where $\widehat{S}(p_1, p_2)$ is the same $SU(2|2)$ matrix as in (2.5).

We assume that the S -matrices satisfy the usual unitarity relation,

$$S_{12}^{AA}(p_1, p_2) S_{21}^{AA}(p_2, p_1) = S_{12}^{AB}(p_1, p_2) S_{21}^{AB}(p_2, p_1) = \mathbb{I}, \quad (2.13)$$

which implies that the scalar factors should obey

$$S_0(p_1, p_2) S_0(p_2, p_1) = 1, \quad \widetilde{S}_0(p_1, p_2) \widetilde{S}_0(p_2, p_1) = 1. \quad (2.14)$$

The identification of the B -particles as charge conjugates of the A -particles suggests the crossing relations

$$S_{12}^{AA} t_2(p_1, p_2) C_2 S_{12}^{AB}(p_1, \bar{p}_2) C_2^{-1} = S_{12}^{AA} t_1(p_1, p_2) C_1 S_{12}^{AB}(\bar{p}_1, p_2) C_1^{-1} = \mathbb{I}. \quad (2.15)$$

It follows that the scalar factors should satisfy

$$S_0(p_1, p_2) \widetilde{S}_0(p_1, \bar{p}_2) = S_0(p_1, p_2) \widetilde{S}_0(\bar{p}_1, p_2) = \frac{1}{f(p_1, p_2)}. \quad (2.16)$$

Note that the crossing relations (2.15) do *not* relate S^{AA} to itself as in the $\mathcal{N} = 4$ YM theory.

We find that values of the scalar factors which are consistent with the unitarity and crossing constraints (2.14) and (2.16) are

$$\begin{aligned} S_0(p_1, p_2) &= \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}} \sigma(p_1, p_2), \\ \widetilde{S}_0(p_1, p_2) &= \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \sigma(p_1, p_2), \end{aligned} \quad (2.17)$$

where $\sigma(p_1, p_2)$ is the BES dressing factor [23], which has the properties

$$\begin{aligned} \sigma(p_1, p_2) \sigma(p_2, p_1) &= 1, \\ \sigma(\bar{p}_1, p_2) \sigma(p_1, p_2) &= \frac{x_2^-}{x_2^+} \frac{1}{f(p_1, p_2)}, \quad \sigma(p_1, \bar{p}_2) \sigma(p_1, p_2) = \frac{x_1^+}{x_1^-} \frac{1}{f(p_1, p_2)}. \end{aligned} \quad (2.18)$$

As we shall see in the following section, the expressions (2.17) for the scalar factors are crucial for obtaining the correct BAEs.

3. Asymptotic Bethe ansatz

We now proceed to formulate the asymptotic Bethe ansatz for the $\mathcal{N} = 6$ CS theory. The analysis is similar to the one for the $\mathcal{N} = 4$ YM theory [16, 24], and here we follow closely the latter reference. We consider a set of N_A A -particles with momenta p_i^A ($i = 1, \dots, N_A$)

and N_B B -particles with momenta p_i^B ($i = 1, \dots, N_B$) which are widely separated on a ring of length L' . Quantization conditions for these momenta follow from imposing periodic boundary conditions on the wavefunction. Taking a particle with momentum p_k^A around the ring leads to the Bethe-Yang equations

$$e^{-ip_k^A L'} = \Lambda(\lambda = p_k^A, \{p_i^A, p_i^B\}), \quad k = 1, \dots, N_A, \quad (3.1)$$

where $\Lambda(\lambda, \{p_i^A, p_i^B\})$ are the eigenvalues of the transfer matrix^{2,3}

$$t(\lambda, \{p_i^A, p_i^B\}) = \text{str}_a S_{a1}^{AA}(\lambda, p_1^A) \cdots S_{aN_A}^{AA}(\lambda, p_{N_A}^A) S_{aN_A+1}^{AB}(\lambda, p_1^B) \cdots S_{aN_A+N_B}^{AB}(\lambda, p_{N_B}^B). \quad (3.2)$$

The order of the particles on the ring is irrelevant. Indeed, changing the order of the particles changes the transfer matrix by a unitary transformation, since the S -matrices satisfy the Yang-Baxter equation.

Using (2.5), (2.12), we see that the transfer matrix can be reexpressed as

$$t(\lambda, \{p_i^A, p_i^B\}) = \prod_{i=1}^{N_A} S_0(\lambda, p_i^A) \prod_{i=1}^{N_B} \tilde{S}_0(\lambda, p_i^B) \hat{t}(\lambda, \{p_i^A, p_i^B\}), \quad (3.3)$$

where

$$\hat{t}(\lambda, \{p_i^A, p_i^B\}) = \text{str}_a \hat{S}_{a1}(\lambda, p_1^A) \cdots \hat{S}_{aN_A}(\lambda, p_{N_A}^A) \hat{S}_{aN_A+1}(\lambda, p_1^B) \cdots \hat{S}_{aN_A+N_B}(\lambda, p_{N_B}^B). \quad (3.4)$$

It follows that

$$\Lambda(\lambda, \{p_i^A, p_i^B\}) = \prod_{i=1}^{N_A} S_0(\lambda, p_i^A) \prod_{i=1}^{N_B} \tilde{S}_0(\lambda, p_i^B) \hat{\Lambda}(\lambda, \{p_i^A, p_i^B\}; \{\lambda_j, \mu_j\}), \quad (3.5)$$

where $\hat{\Lambda}(\lambda, \{p_i^A, p_i^B\}; \{\lambda_j, \mu_j\})$ are the eigenvalues of $\hat{t}(\lambda, \{p_i^A, p_i^B\})$, which are given by [24]

$$\begin{aligned} \hat{\Lambda}(\lambda, \{p_i^A, p_i^B\}; \{\lambda_j, \mu_j\}) &= \prod_{i=1}^{N_A} \left[\frac{x^+(\lambda) - x^-(p_i^A) \eta(p_i^A)}{x^-(\lambda) - x^+(p_i^A) \eta(\lambda)} \right] \prod_{i=1}^{N_B} \left[\frac{x^+(\lambda) - x^-(p_i^B) \eta(p_i^B)}{x^-(\lambda) - x^+(p_i^B) \eta(\lambda)} \right] \\ &\times \prod_{j=1}^{m_1} \left[\eta(\lambda) \frac{x^-(\lambda) - x^+(\lambda_j)}{x^+(\lambda) - x^+(\lambda_j)} \right] \\ &- \prod_{i=1}^{N_A} \left[\frac{x^+(\lambda) - x^+(p_i^A)}{x^-(\lambda) - x^+(p_i^A) \eta(\lambda)} \right] \prod_{i=1}^{N_B} \left[\frac{x^+(\lambda) - x^+(p_i^B)}{x^-(\lambda) - x^+(p_i^B) \eta(\lambda)} \right] \left\{ \prod_{j=1}^{m_1} \left[\eta(\lambda) \frac{x^-(\lambda) - x^+(\lambda_j)}{x^+(\lambda) - x^+(\lambda_j)} \right] \right. \\ &\times \left. \prod_{l=1}^{m_2} \frac{x^+(\lambda) + \frac{1}{x^+(\lambda)} - \tilde{\mu}_l + \frac{i}{2g}}{x^+(\lambda) + \frac{1}{x^+(\lambda)} - \tilde{\mu}_l - \frac{i}{2g}} + \prod_{j=1}^{m_1} \left[\eta(\lambda) \frac{x^+(\lambda_j) - \frac{1}{x^+(\lambda)}}{x^+(\lambda_j) - \frac{1}{x^-(\lambda)}} \right] \prod_{l=1}^{m_2} \frac{x^-(\lambda) + \frac{1}{x^-(\lambda)} - \tilde{\mu}_l - \frac{i}{2g}}{x^-(\lambda) + \frac{1}{x^-(\lambda)} - \tilde{\mu}_l + \frac{i}{2g}} \right\} \end{aligned}$$

²As emphasized by Martins and Melo [24], in order to properly implement periodic boundary conditions, it is necessary to use the graded S -matrix (which is related to the non-graded S -matrix by the factor $\mathcal{P}^{(g)} \mathcal{P}$, where $\mathcal{P}^{(g)}$ and \mathcal{P} are the graded and non-graded permutation matrices, respectively), and take the supertrace (instead of the ordinary trace) of the monodromy matrix.

³Following [24], we denote the spectral parameter of the transfer matrix by λ , which should not be confused with the 't Hooft coupling!

$$\begin{aligned}
& + \prod_{i=1}^{N_A} \left[\frac{x^+(\lambda) - x^+(p_i^A)}{x^-(\lambda) - x^+(p_i^A)} \frac{1 - \frac{1}{x^-(\lambda)x^+(p_i^A)}}{1 - \frac{1}{x^-(\lambda)x^-(p_i^A)}} \frac{\eta(p_i^A)}{\eta(\lambda)} \right] \prod_{i=1}^{N_B} \left[\frac{x^+(\lambda) - x^+(p_i^B)}{x^-(\lambda) - x^+(p_i^B)} \frac{1 - \frac{1}{x^-(\lambda)x^+(p_i^B)}}{1 - \frac{1}{x^-(\lambda)x^-(p_i^B)}} \frac{\eta(p_i^B)}{\eta(\lambda)} \right] \\
& \times \prod_{j=1}^{m_1} \left[\eta(\lambda) \frac{x^+(\lambda_j) - \frac{1}{x^+(\lambda)}}{x^+(\lambda_j) - \frac{1}{x^-(\lambda)}} \right], \tag{3.6}
\end{aligned}$$

where $\eta(\lambda) = e^{i\lambda/2}$. The corresponding BAEs are given by

$$\begin{aligned}
e^{i(P^A+P^B)/2} \prod_{i=1}^{N_A} \frac{x^+(\lambda_j) - x^-(p_i^A)}{x^+(\lambda_j) - x^+(p_i^A)} \prod_{i=1}^{N_B} \frac{x^+(\lambda_j) - x^-(p_i^B)}{x^+(\lambda_j) - x^+(p_i^B)} &= \prod_{l=1}^{m_2} \frac{x^+(\lambda_j) + \frac{1}{x^+(\lambda_j)} - \tilde{\mu}_l + \frac{i}{2g}}{x^+(\lambda_j) + \frac{1}{x^+(\lambda_j)} - \tilde{\mu}_l - \frac{i}{2g}}, \\
& j = 1, \dots, m_1, \\
\prod_{j=1}^{m_1} \frac{\tilde{\mu}_l - x^+(\lambda_j) - \frac{1}{x^+(\lambda_j)} + \frac{i}{2g}}{\tilde{\mu}_l - x^+(\lambda_j) - \frac{1}{x^+(\lambda_j)} - \frac{i}{2g}} &= \prod_{\substack{k=1 \\ k \neq l}}^{m_2} \frac{\tilde{\mu}_l - \tilde{\mu}_k + \frac{i}{g}}{\tilde{\mu}_l - \tilde{\mu}_k - \frac{i}{g}}, \quad l = 1, \dots, m_2, \tag{3.7}
\end{aligned}$$

where

$$P^A = \sum_{i=1}^{N_A} p_i^A, \quad P^B = \sum_{i=1}^{N_B} p_i^B. \tag{3.8}$$

The Bethe-Yang equations for the A -particles (3.1) therefore take the form

$$\begin{aligned}
e^{ip_k^A(-L' + \frac{N^A+N^B}{2} - \frac{m_1}{2})} &= e^{i(P^A+P^B)/2} \prod_{\substack{i=1 \\ i \neq k}}^{N_A} \left[\frac{x^+(p_k^A) - x^-(p_i^A)}{x^-(p_k^A) - x^+(p_i^A)} \right] \left[\frac{1 - \frac{1}{x^+(p_k^A)x^-(p_i^A)}}{1 - \frac{1}{x^-(p_k^A)x^+(p_i^A)}} \sigma(p_k^A, p_i^A) \right] \\
&\times \prod_{i=1}^{N_B} \sigma(p_k^A, p_i^B) \prod_{j=1}^{m_1} \left[\frac{x^-(p_k^A) - x^+(\lambda_j)}{x^+(p_k^A) - x^+(\lambda_j)} \right], \quad k = 1, \dots, N_A. \tag{3.9}
\end{aligned}$$

In obtaining the result (3.9), we have used the expressions (3.5), (3.6) for the eigenvalues of the transfer matrix (note that only the first term in (3.6) survives after setting $\lambda = p_k^A$), as well as our expressions (2.17) for the scalar factors. Note that the scalar factor \tilde{S}_0 produces a cancelation, so that $A - B$ scattering in (3.9) is given only by the dressing factor; and the scalar factor S_0 produces a contribution which allows the $A - A$ scattering to be expressed in terms of $u_{4,j}$ (see (3.12) below).

Similarly, the Bethe-Yang equations for the B -particles become

$$\begin{aligned}
e^{ip_k^B(-L' + \frac{N^A+N^B}{2} - \frac{m_1}{2})} &= e^{i(P^A+P^B)/2} \prod_{\substack{i=1 \\ i \neq k}}^{N_B} \left[\frac{x^+(p_k^B) - x^-(p_i^B)}{x^-(p_k^B) - x^+(p_i^B)} \right] \left[\frac{1 - \frac{1}{x^+(p_k^B)x^-(p_i^B)}}{1 - \frac{1}{x^-(p_k^B)x^+(p_i^B)}} \sigma(p_k^B, p_i^B) \right] \\
&\times \prod_{i=1}^{N_A} \sigma(p_k^B, p_i^A) \prod_{j=1}^{m_1} \left[\frac{x^-(p_k^B) - x^+(\lambda_j)}{x^+(p_k^B) - x^+(\lambda_j)} \right], \quad k = 1, \dots, N_B, \tag{3.10}
\end{aligned}$$

since they are formulated in terms of the same transfer matrix $\hat{t}(\lambda, \{p_i^A, p_i^B\})$ (3.4).

The BAEs (3.7) and Bethe-Yang equations (3.9), (3.10) can be mapped to the all-loop BAEs of Gromov and Vieira [12]. To this end, we make the following identifications:⁴

$$\begin{aligned}
 x^\pm(p_k^A) &= x_{4,k}^\pm, & k &= 1, \dots, K_4 \equiv N_A, \\
 x^\pm(p_k^B) &= x_{\bar{4},k}^\pm, & k &= 1, \dots, K_{\bar{4}} \equiv N_B, \\
 x^+(\lambda_j) &= \frac{1}{x_{1,j}}, & j &= 1, \dots, K_1, \\
 x^+(\lambda_{K_1+j}) &= x_{3,j}, & j &= 1, \dots, K_3, \quad K_1 + K_3 \equiv m_1, \\
 \tilde{\mu}_j &= \frac{u_{2,j}}{g}, & j &= 1, \dots, K_2 \equiv m_2,
 \end{aligned} \tag{3.11}$$

and also define

$$u_{4,j} = x_{4,j}^+ + \frac{1}{x_{4,j}^+} - \frac{i}{2} = x_{4,j}^- + \frac{1}{x_{4,j}^-} + \frac{i}{2} \tag{3.12}$$

(similarly for $u_{\bar{4},j}$), and $u_{i,j} = \frac{1}{g} \left(x_{i,j} + \frac{1}{x_{i,j}} \right)$ for $i = 1, 3$. Indeed, imposing the zero-momentum condition [8, 12]

$$P^A + P^B = \sum_{j=1}^{K_4} p_{4,j} + \sum_{j=1}^{K_{\bar{4}}} p_{\bar{4},j} = 0, \tag{3.13}$$

the Bethe-Yang equations (3.9) become

$$\begin{aligned}
 e^{ip_{4,k} \left(-L' + \frac{K_4 + K_{\bar{4}} + K_1 - K_3}{2} \right)} &= \prod_{\substack{j=1 \\ j \neq k}}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \sigma(u_{4,k}, u_{4,j}) \prod_{j=1}^{K_{\bar{4}}} \sigma(u_{4,k}, u_{\bar{4},j}) \\
 &\times \prod_{j=1}^{K_1} \frac{1 - \frac{1}{x_{4,k}^- x_{1,j}}}{1 - \frac{1}{x_{4,k}^+ x_{1,j}}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}}, \quad k = 1, \dots, K_4.
 \end{aligned} \tag{3.14}$$

This agrees with the all-loop BAEs for $x_{4,k}^\pm$ in [12], provided we identify

$$L = -L' + \frac{K_4 + K_{\bar{4}} + K_1 - K_3}{2}. \tag{3.15}$$

If we assume that $L' = -J$ (as in [24]), then (3.15) implies a relation among L , J and the number of various Bethe roots. Similarly, the Bethe-Yang equations (3.10) give the the all-loop BAEs for $x_{4,k}^\pm$ in [12].

The first set of BAEs in (3.7) imply the following two sets of equations

$$\begin{aligned}
 \prod_{i=1}^{K_4} \frac{1 - \frac{1}{x_{1,j} x_{4,i}^-}}{1 - \frac{1}{x_{1,j} x_{4,i}^+}} \prod_{i=1}^{K_{\bar{4}}} \frac{1 - \frac{1}{x_{1,j} x_{\bar{4},i}^-}}{1 - \frac{1}{x_{1,j} x_{\bar{4},i}^+}} &= \prod_{l=1}^{K_2} \frac{u_{1,j} - u_{2,l} + \frac{i}{2}}{u_{1,j} - u_{2,l} - \frac{i}{2}}, & j &= 1, \dots, K_1, \\
 \prod_{i=1}^{K_4} \frac{x_{3,j} - x_{4,i}^-}{x_{3,j} - x_{4,i}^+} \prod_{i=1}^{K_{\bar{4}}} \frac{1 - \frac{1}{x_{3,j} x_{\bar{4},i}^-}}{1 - \frac{1}{x_{3,j} x_{\bar{4},i}^+}} &= \prod_{l=1}^{K_2} \frac{u_{3,j} - u_{2,l} + \frac{i}{2}}{u_{3,j} - u_{2,l} - \frac{i}{2}}, & j &= 1, \dots, K_3,
 \end{aligned} \tag{3.16}$$

⁴In the $\mathcal{N} = 4$ YM case, a similar set of identifications has been made [24] in order to map the $SU(2|2)^2$ asymptotic BAEs to the all-loop BAEs of Beisert and Staudacher [25].

which agree with the first and third BAEs in [12]. Finally, the second set of BAEs in (3.7) imply

$$\prod_{\substack{j=1 \\ j \neq l}}^{K_2} \frac{u_{2,l} - u_{2,j} + i}{u_{2,l} - u_{2,j} - i} = \prod_{j=1}^{K_1} \frac{u_{2,l} - u_{1,j} + \frac{i}{2}}{u_{2,l} - u_{1,j} - \frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,l} - u_{3,j} + \frac{i}{2}}{u_{2,l} - u_{3,j} - \frac{i}{2}}, \quad l = 1, \dots, K_2, \quad (3.17)$$

which agrees with the second set of BAEs in [12]. In short, the asymptotic BAEs which follow from the proposed S -matrix give the full set of all-loop BAEs in [12].

4. Discussion

We have seen that, as in $\mathcal{N} = 4$ YM, $SU(2|2)$ symmetry again plays an important role in determining the factorizable S -matrix of $\mathcal{N} = 6$ CS. The scattering matrices for the two types of particles are the same, up to the dressing phases which are related by a new crossing relation. This relation seems to be weaker than the one in the AdS_5/CFT_4 case, which completely determines the phase up to the usual CDD ambiguity. Hence, more analysis is necessary to fix the dressing phases uniquely. The solution proposed here is supported by the all-loop BAEs conjectured in [12].

It should be possible to make further checks of our proposal. Perturbative computations at both small and large values of 't Hooft coupling could be done. The classical S -matrix can be computed, as was done for the giant magnon for AdS_5/CFT_4 [26]. These results may give some insights into the relation between the parameter g in the S -matrix and the 't Hooft coupling λ defined in (2.3).

One interesting application of the S -matrix is to compute finite-size effects, or the Lüscher correction. The result can be compared with various semi-classical string calculations [27] as has been done for the AdS_5/CFT_4 in [28] and with perturbative CS computations along the line of [20]. The S -matrix can also be used to determine the bound state spectrum, which should be related to the multiplet states studied in [10].

Our result shows that the $SU(2|2)$ -invariant S -matrix appears in both AdS_5/CFT_4 and AdS_4/CFT_3 dualities. It would be interesting to understand this common integrability property at a more fundamental level.

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