Reentrant melting of soliton lattice phase in a bilayer quantum Hall system

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At large parallel magnetic field $B_\parallel$, the ground state of the bilayer quantum Hall system forms a uniform soliton lattice phase. The soliton lattice will melt due to the proliferation of unbound dislocations at certain finite temperature leading to the Kosterlitz-Thouless (KT) melting. We calculate the KT phase boundary by numerically solving the newly developed set of Bethe ansatz equations, which fully take into account the thermal fluctuations of soliton walls. We predict that within certain ranges of $B_\parallel$, the soliton lattice will melt at T_{KT}. Interestingly enough, as temperature decreases, it melts at certain temperature lower than T_{KT} exhibiting the reentrant behavior of the soliton liquid phase.

When the interlayer spacing d is comparable to the mean particle distance, the bilayer quantum Hall system at total filling factor $\nu_{tot}=1$ can exhibit the quantum Hall effect due to the strong interlayer correlations.[1–3] In the presence of large parallel magnetic field $B_\parallel$, the ground state of the system is known to form a uniform soliton lattice (SL) phase made of the periodic array of the phase solitons of the field variable $\theta(\mathbf{r})$, which represents the relative phase difference between electrons in two layers.[3] Using the isospin language,[3] the effective energy functional for the bilayer system can be written by

$$E(\theta) = \int d^2r \left\{ \frac{1}{2}\rho_s |\nabla\theta|^2 - \frac{t}{2\pi l^2}\cos[\theta(\mathbf{r})+Qx] \right\}, \quad (1)$$

where ρ_s is the isospin stiffness, $t=t_0 e^{-Q^2 l^2/4}$ the interlayer tunneling amplitude, $l=(\hbar c/|e|B_\perp)^{1/2}$, and $Q=2\pi/L_\parallel$ with $L_\parallel=\Phi_0/B_\parallel d$, which defines the length associated with one flux quantum Φ_0 enclosed between two layers. We set the magnetic length $l=1$. This continuum model is valid, when the field $\theta(\mathbf{r})$ varies smoothly over the lengths $l, L_\parallel$.[4] With the increase of $B_\parallel$, the system exhibits a quantum phase transition from commensurate (C) to incommensurate (I) phase at $Q_c=(2/\pi)(2t/\pi\rho_s)^{1/2}$ at $T=0$.[3,5] The incommensurate phase at $Q>Q_c$ describes the uniform SL phase at zero temperature.

In the paper, we study the thermodynamics of the SL phase through a mapping of the 2D statistical mechanical model in Eq. (1) to the 1D quantum sine-Gordon (QSG) Hamiltonian. When one neglects the compactness of the field variable θ, which will be reasonably valid for $T<(\pi/2)\rho_s$, the 1D QSG model is exactly soluble by the Bethe ansatz method. The Bethe ansatz solution provides the thermodynamic CI phase boundary $Q_c(T)$. Depending on the values of $C=t/(32\pi\rho_s)$, the critical value $Q_c(T)$ will increase (decrease) with temperature for $C>1(C<1)$. Within the incommensurate phase, the ground state of the system is the uniform SL phase at $T=0$. As T increases to about $(\pi/2)\rho_s$,

the compactness of the angle variable θ will become important, which introduces dislocations to the system leading to the KT melting of the SL phase. Recently Hanna *et al.* have studied the melting of the SL phase by analyzing the elastic moduli based on the rigidity of the zero temperature ground state.[6,7] However, the renormalizaion of the elastic moduli due to thermal fluctuations of soliton walls has not been taken into account. Furthermore, as one approaches to the CI phase boundary, the soliton density will vanish. In order to understand the melting of the SL phase near the CI phase boundary, it is crucial to include the thermal fluctuations and the soliton density variations as well.

In the paper, we develop a set of Bethe ansatz equations, which fully take into account both the thermal fluctuations of the soliton walls and the density variations. By numerically solving the Bethe ansatz equations, we have calculated both the compression modulus κ_{xx} and the shear modulus κ_{yy} of the SL phase. By making an asymptotic expansion near the CI phase boundary, we have shown that κ_{xx} goes as $(2\tau/M)^{1/2}[Q-Q_c(T)]^{1/2}\rho_s$ and $\kappa_{yy}\cong(8M\tau^3)^{1/2}[Q-Q_c(T)]^{-1/2}\rho_s$, where M is the soliton mass which varies with τ and $\tau=T/(8\pi\rho_s)$. This asymptotic behavior explicitly confirms the predictions made by Coppersmith *et al.*[7] Based on the elastic moduli thus obtained, T_{KT} is calculated and compared with the zero-temperature estimates.[6,8] We predict the following reentrant behavior of the soliton liquid phase. At certain values of Q slightly below $Q_c(0)$, the system initially stays at the C phase. As T increases, it makes a transition to the soliton liquid phase. With the further increase of T, the soliton liquid solidifies to the SL phase due to the rapid increase of the soliton density, which subsequently melts reentering the soliton liquid phase.

Following the effective energy functional of Eq. (1), the low-temperature thermodynamics of the system can be defined by the statistical partition function $\mathcal{Z}=\int[\mathcal{D}\theta]e^{-E(\theta)/T}$. One can map the statistical ensemble summation into quantum transfer matrix by identifying $\mathcal{Z}=\lim_{R\to\infty}\text{Tr}[e^{-R\hat{H}}]$,

$$\hat{H} = \int dx \left[\frac{T}{2\rho_s} \Pi^2 + \frac{\rho_s}{2T}(\partial_x \theta)^2 - \frac{t}{2\pi T} \cos(\theta + Qx) \right]. \tag{2}$$

If one neglects the compactness of the angle variable $\theta(\mathbf{r})$, the canonical conjugate momentum $\Pi(\mathbf{r})$ can be easily integrated out leading to the corresponding classical partition function. We make the following change of variables $\phi(\mathbf{r}) = \sqrt{\rho_s/T}[\theta(\mathbf{r}) + Qx]$, which leads to the quantum Hamiltonian of the 1D sine-Gordon model with external U(1) coupling

$$\hat{H} = \int dx \left[\frac{1}{2} \Pi^2 + \frac{1}{2}(\partial_x \phi)^2 - A \frac{\beta}{2\pi} \partial_x \phi + 2\mu \cos(\beta \phi) \right.$$
$$\left. + \frac{1}{2T} \rho_s Q^2 \right]. \tag{3}$$

Here $\mu = t/(4\pi T)$, $\beta = \sqrt{T/\rho_s}$, and $A = 2\pi Q \rho_s /T$. The compactness of $\theta(\mathbf{r})$ would restrict $\Pi(\mathbf{r})$ to the integer values. Hence the ground state energy $\mathcal{E}$ of the 1D QSG model corresponds to $\mathcal{F}/T$, where $\mathcal{F}$ is the free energy of the 2D statistical mechanical system.

The ground state of the 1D QSG model is exactly soluble by the Bethe ansatz method. There is a competition between the finite soliton mass M which prefers the commensurate phase and the external field A coupled to the topological charge which favors the incommensurate phase. This is a "quantum" version of the CI transition,[9] where the soliton mass includes full "quantum" fluctuation effects. With external U(1) field, a soliton can be created with energy cost $\Delta\mathcal{E} = M \cosh\Theta - A$. The soliton mass M is given by

$$M = \frac{2\Gamma\left(\frac{p}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{p+1}{2}\right)} \left[\pi\mu \frac{\Gamma\left(\frac{1}{p+1}\right)}{\Gamma\left(\frac{p}{p+1}\right)} \right]^{1/2(p+1)}, \tag{4}$$

where $p = T/(8\pi\rho_s - T)$ and Θ represents the rapidity of the soliton.[10]

For $A < M$, the ground state will be the vacuum: the commensurate phase. However, for $A > M$, a nontrivial vacuum will arise, since the energy cost to create a soliton can be negative. In this case, the ground state will be described by the soliton condensations: The SL phase. The CI transition will occur at $A = M$. By equating the two quantities, one can obtain the exact CI phase boundary $Q_c(T)$

$$Q_c(T) = \frac{8}{\sqrt{\pi}} \frac{\tau\Gamma\left(\frac{\tau}{2(1-\tau)}\right)}{\Gamma\left(\frac{1}{2(1-\tau)}\right)} \left[C\frac{\Gamma(1-\tau)}{\Gamma(1+\tau)} \right]^{1/2(1-\tau)}, \tag{5}$$

where we have neglected the Q dependence of t, since the continuum model is only valid for $Q \lesssim 1$. The parameter C is given by $t/(4T_0)$. The reduced temperature variable τ is between 0 and 1. As $\tau \to 0$, one can reproduce the classical limit $Q_c(0) = 16\sqrt{C}/\pi$.

Now we want to study the melting of the SL phase by numerically solving the Bethe ansatz equations. Using the fact that the exact scattering matrix is known for the 1D QSG model, one can calculate the ground state energy and the soliton density $\bar{n}_s(Q,T)$ as a function of Q and T.[10] The Bethe ansatz equation is given by

$$2\pi\rho(\Theta) = M\cosh\Theta + \int_{-B}^{B} d\Theta' \varphi(\Theta - \Theta')\rho(\Theta'), \tag{6}$$

where $\rho(\Theta)$ is the density of solitons between Θ and $\Theta + d\Theta$ per unit length and the integral kernel $\varphi(\Theta)$ is given by

$$\varphi(\Theta) = \int_{-\infty}^{\infty} d\omega e^{i\omega\Theta} \frac{\sinh\left(\frac{\pi(p-1)\omega}{2}\right)}{2\cosh\left(\frac{\pi\omega}{2}\right)\sinh\left(\frac{\pi p\omega}{2}\right)}. \tag{7}$$

In terms of the density function, one can calculate the ground state energy as follows:

$$\Delta\mathcal{E} = \int_{-B}^{B} d\Theta (M\cosh\Theta - A)\rho(\Theta), \tag{8}$$

where the ground state is determined by imposing the condition $\partial\Delta\mathcal{E}/\partial B = 0$. By making an asymptotic expansion of Eq. (6) near the CI phase boundary, one can confirm that the soliton density exhibits the power-law behavior $\bar{n}_s \cong (2M/\tau)^{1/2}[Q - Q_c(T)]^{1/2}/(2\pi)$.[5]

As T increases to about $(\pi/2)\rho_s$, the compactness of the variable $\theta(\mathbf{r})$ becomes important, which introduces topological defects (dislocations) into the system leading to the KT melting of the SL phase at temperatures much below the CI transition temperature. Although the 1D QSG Hamiltonian does not involve dislocation excitations, one can get a reasonable estimate for T_{KT} by analyzing the elastic moduli of the SL at finite temperature. The elastic moduli κ_{ij} are defined as follows:[6]

$$\kappa_{ij} = \frac{\partial^2}{\partial Q_s^i \partial Q_s^j}\mathcal{F}, \tag{9}$$

where Q_s^x is given by $2\pi/L_s$, where L_s is the lattice constant of the SL phase along the $\hat{x}$ direction, $Q_s^y = \tan\phi Q_s^x$ with ϕ the tilt angle of the soliton walls, and $i,j = x,y$. In order to obtain the elastic moduli of the system, one needs to calculate the energy of the system by varying the soliton density from the ground state value and/or tilting the solitons from the vertical orientations. We first calculate the compression modulus κ_{xx}. By shifting the Fermi momentum from B to $B+\epsilon$, one can increase (decrease) the soliton density for positive (negative) values of ϵ. The soliton density $\rho(\Theta)$ is a function of both ϵ and Θ. For small ϵ, one can expand the soliton density up to quadratic order in ϵ

$$\rho(\Theta) \cong \bar{\rho}(\Theta) + \epsilon\rho_b'(\Theta) + \frac{\epsilon^2}{2}\rho_b''(\Theta), \tag{10}$$

where $\bar{\rho}(\Theta)$ is the soliton density profile of the ground state, $\rho_b'=\partial\rho/\partial\epsilon|_{\epsilon=0}$, and $\rho_b''=\partial^2\rho/\partial\epsilon^2|_{\epsilon=0}$. At the lowest order of ϵ, we obtain the usual Bethe ansatz equation for the ground state as shown in Eq. (6) with the condition $\partial\Delta\mathcal{E}/\partial\epsilon=0$. Up to the order of ϵ^2, we obtain the following two equations:

$$2\pi\rho_b'(\Theta)=[\varphi(\Theta-B)+\varphi(\Theta+B)]\bar{\rho}(B)$$
$$+\int_{-B}^{B}d\Theta'\varphi(\Theta-\Theta')\rho_b'(\Theta'), \qquad (11)$$

$$2\pi\rho_b''(\Theta)=\frac{\partial}{\partial\Theta}[\varphi(\Theta+B)-\varphi(\Theta-B)]\bar{\rho}(B)+[\varphi(\Theta-B)$$
$$+\varphi(\Theta+B)]\left[\frac{\partial\bar{\rho}(B)}{\partial B}+2\rho_b'(B)\right]$$
$$+\int_{-B}^{B}d\Theta'\varphi(\Theta-\Theta')\rho_b''(\Theta'). \qquad (12)$$

The energy of the system is given as follows: $\Delta\mathcal{E}=\Delta\bar{\mathcal{E}}+(\epsilon^2/2)\chi_b$, where $\Delta\bar{\mathcal{E}}$ is the ground state energy and χ_b is given by

$$\chi_b=2M\sinh B\bar{\rho}(B)+2(M\cosh B-A)\left[\frac{\partial\bar{\rho}(B)}{\partial B}+2\rho_b'(B)\right]$$
$$+\int_{-B}^{B}d\Theta(M\cosh\Theta-A)\rho_b''(\Theta). \qquad (13)$$

The Q_s^x can be calculated as follows: $Q_s^x=2\pi\int_{-B+\epsilon}^{B+\epsilon}d\Theta\rho(\Theta)\cong\bar{Q}_s^x+\alpha_b\epsilon+\mathcal{O}(\epsilon^2)$, where $\bar{Q}_s^x=2\pi\int_{-B}^{B}d\Theta\bar{\rho}(\Theta)$ is the ground state value and $\alpha_b=4\pi\bar{\rho}(B)+2\pi\int_{-B}^{B}d\Theta\rho_b'(\Theta)$. The compression modulus κ_{xx} is given as follows: $\partial^2\mathcal{F}/\partial Q_s^{x2}=T\chi_b/\alpha_b^2$.

Now we calculate the shear modulus κ_{yy}. To obtain the shear modulus, one needs to fix Q_s^x and vary Q_s^y alone. Global shift of the rapidity will not only tilt the soliton walls but also change the soliton density along the $\hat{x}$ direction. In order to vary Q_s^y alone, one needs to change the fermi momentum B to $B+\epsilon$ and rapidity by η simultaneously along the following trajectory: $\epsilon=-\alpha_T\eta^2$ for some constant α_T.[11] By making a perturbative expansion in η, we obtain the following two Bethe ansatz equations up to η^2:

$$2\pi\rho_s'(\Theta)=[\varphi(\Theta-B)-\varphi(\Theta+B)]\bar{\rho}(B)$$
$$+\int_{-B}^{B}d\Theta'\varphi(\Theta-\Theta')\rho_s'(\Theta'), \qquad (14)$$

$$2\pi\rho_s''(\Theta)=\frac{\partial}{\partial\Theta}[\varphi(\Theta+B)-\varphi(\Theta-B)]\bar{\rho}(B)+[\varphi(\Theta-B)$$
$$+\varphi(\Theta+B)]\left[\frac{\partial\bar{\rho}(B)}{\partial B}+2\rho_s'(B)\right]$$
$$+\int_{-B}^{B}d\Theta'\varphi(\Theta-\Theta')\rho_s''(\Theta'), \qquad (15)$$

where one can easily notice that $\rho_s'(\Theta)$ is an odd function of Θ and $\rho_s''(\Theta)$ an even function. The energy of the system along the trajectory is given by

$$\Delta\mathcal{E}=\Delta\bar{\mathcal{E}}+\frac{\eta^2}{2}\chi_s, \qquad (16)$$

where χ_s is given by

$$\chi_s=2M\sinh B\bar{\rho}(B)+2(M\cosh B-A)\left[\frac{\partial\bar{\rho}(B)}{\partial B}+2\rho_s'(B)\right]$$
$$+\int_{-B}^{B}d\Theta(M\cosh\Theta-A)\rho_s''(\Theta). \qquad (17)$$

Since $Q_s^y=\tan\phi\bar{Q}_s^x=\eta\bar{Q}_s^x$, the shear modulus κ_{yy} is given by the following relation: $\partial^2\mathcal{F}/\partial Q_s^{y2}=T\chi_s/(\bar{Q}_s^x)^2$. The off-diagonal elements of κ_{ij} are zero. Using the asymptotic expansion, one can show that near the CI phase boundary, κ_{xx} goes as $(2\tau/M)^{1/2}[Q-Q_c(T)]^{1/2}\rho_s$, and $\kappa_{yy}\cong(8M\tau^3)^{1/2}[Q-Q_c(T)]^{-1/2}\rho_s$.[11]

In the experiment of Murphy *et al.,* for equally populated layers, the experimental values of the various parameters are given as follows: $\rho_s\cong0.35$ K, $t_0\cong1.2$ K, $l\cong126$ Å, and $d\cong200$ Å.[1] For the above sample, the parameter C can be estimated to be about 0.033. In the inset of Fig. 1, the open circles represent κ_{xx} and the open squares κ_{yy} at fixed value of $Q\cong0.919$, which obey the correct asymptotic behavior near the CI phase boundary. We have confirmed that both κ_{xx} and κ_{yy} approach to ρ_s at large Q.[11] It has been argued that at large Q, there can be an additional transition to canted incommensurate phase with $m_z\neq0$.[12,13] The KT transition temperature T_{KT} can be estimated by solving the following equation: $k_BT/(\pi/2)\rho_s=[\kappa_{xx}(T)\kappa_{yy}(T)]^{1/2}/\rho_s$.[8] In Fig. 1, the closed circles represent $[\kappa_{xx}(T)\kappa_{yy}(T)]^{1/2}/(16\rho_s)$ as a function of τ. The intersections with the solid line with slope 1 locate the positions of T_{KT}.[14] It can be shown that for $Q_{min}<Q<Q_c(0)$, the equations have two solutions, where $Q_{min}\cong0.9,Q_c(0)\cong0.927$. In Fig. 2, we plot the KT melting temperature as a function of Q. The closed circles represent the KT phase boundary from our Bethe ansatz calculation, the open squares from Hanna *et al.* based on the zero temperature value of the elastic moduli, and the solid curve the CI phase boundary. For $Q_{min}<Q<Q_c(0)$, the system initially stays at the C phase. As T increases, it makes a transition to the soliton liquid phase. With the further increase of T, the soliton liquid becomes the SL phase due to the rapid increase of the soliton density upon entering the CI phase boundary. Subsequently it melts exhibiting the reentrant behavior of the soliton liquid phase. We have confirmed that at
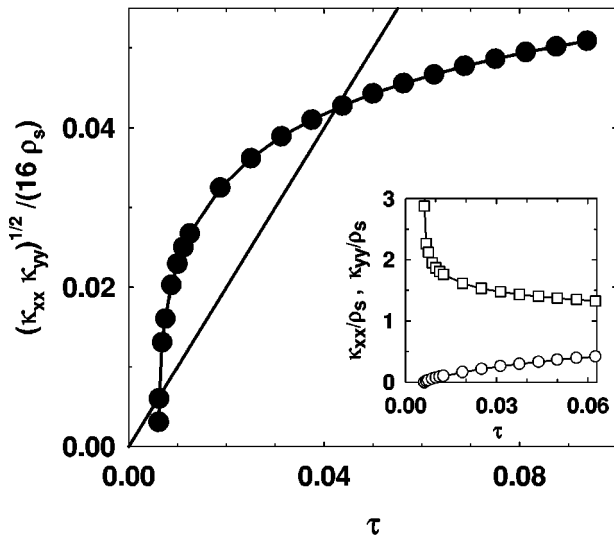
FIG. 1. Determination of the KT melting temperature. The closed circles represent $[\kappa_{xx}(T)\kappa_{yy}(T)]^{1/2}/(16\rho_s)$ as a function of τ. The intersections with the solid line with slope 1 locate the positions of T_{KT}. In the inset, the open circles represent the compression modulus κ_{xx} as a function of τ at fixed $Q=0.919$ and the open squares the shear modulus κ_{yy}.
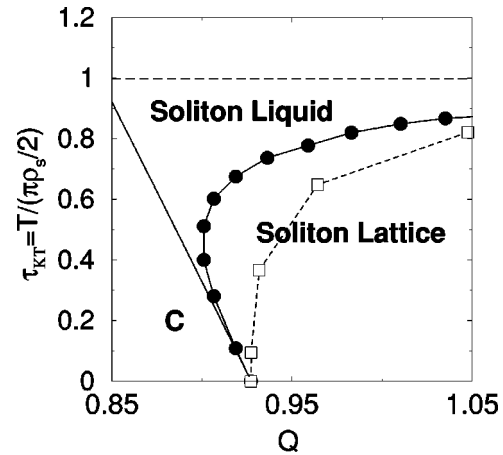

FIG. 2. The KT melting temperature as a function of Q. The KT phase boundary is shown and compared with the result of Hanna *et al.* (Ref. 6), where $\tau_{KT}=16\tau$. The closed circles are from our Bethe ansatz result and the open squares from Hanna *et al.* The solid line represents the CI phase boundary.

large Q, T_{KT} approaches $(\pi/2)\rho_s$.[11] Read argues that at $T=0$, quantum fluctuations are not important, since the domain "sheets" are marginally rough.[8,15] However at finite T, thermal length scale $\xi_\beta=(\hbar v/k_B T)$ becomes finite. Hence near the CI phase boundary and/or KT phase boundary, the large distance thermal fluctuations will be still important.

To summarize, we have studied the melting of the SL phase by numerically solving the newly developed set of Bethe ansatz equations, which fully take into account both the thermal fluctuations of the soliton walls and the density variations. Based on the elastic moduli thus obtained, T_{KT} is calculated. We predict that the system will exhibit the reentrant behavior of the soliton liquid phase within certain

ranges of Q. Our proposed KT phase boundary can be experimentally observed by measuring the longitudinal resistance of the high purity bilayer sample. Towards the completion of the paper, we became aware that Papa *et al.*[16] also studied the bilayer quantum Hall system based on the exact solution of the 1D sine-Gordon model. They studied the temperature dependence of the specific heat and the magnetization. In our paper, we mainly focus on the elastic moduli, the temperature-renormalized value of the KT temperature, and the reentrant behavior of the soliton liquid.

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