# SPECTRA OF STRINGS ON NONSIMPLY-CONNECTED GROUP MANIFOLDS ☆

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We use the orbifold approach to confirm the formula of Felder, Gawedzki and Kupiainen for the one-loop partition function of strings on nonsimply-connected group manifolds.

Wess–Zumino–Witten models [1] are prototypical rational conformal field theories. The classification of rational conformal field theories has been the focus of much recent attention. But even the small subclass consisting of Wess–Zumino–Witten models is not well understood; all consistent models are not known.

One powerful restriction is modular invariance. For example, the states of the theory must be such that the one-loop partition function is modular invariant. A list of possible modular invariant partition functions has been compiled and proven complete only for the simplest case, that of SU(2) [2].

Remarkably, the SU(2) partition functions may be labelled by the simply-laced Lie algebras, i.e. those of the A, D and E types. There are the trivial diagonal modular invariants (A type) and also exceptional ones (E type) occurring for isolated values of Kač–Moody central charge $k$. The remaining modular invariants (D type) are the partition functions for strings propagating on the group manifold SO(3) [3,4]. So besides the trivial and exceptional, all SU(2) modular invariants are partition functions for strings on nonsimply-connected group manifolds. If this pattern continues for other Lie groups [#1], strings on nonsimply-connected group manifolds are certainly important.

Felder, Gawedzki and Kupiainen [6] have studied the canonical quantization of Wess–Zumino–Witten models. Using the geometry of line bundles over the loop groups of G, they derive consistent spectra for arbitrary nonsimply-connected groups $G=\tilde{G}/B$, where $\tilde{G}$ is the covering group, and B a subgroup of its centre $B(\tilde{G})$. In this letter we use the orbifold [7] approach advocated in ref. [3] to construct the partition functions, thus providing a simple confirmation of their results.

The crucial mathematical relation we use is the isomorphism between the outer automorphism group $O(\hat{g})$ of the (untwisted) Kač–Moody algebra g and the centre $B(\tilde{G})$, and its relation to the modular transformations of the torus. Bernard [8] has shown that in the space of characters of highest weight representations of $\hat{g}$, it is the modular transformation $S(\tau\to -1/\tau)$ that transforms an element $A\in O(\hat{g})$ into an element $\alpha\in B(\tilde{G})$, and vice versa. He and others [9] use this fact to derive many modular invariants [#2]. These are now understood to be some, but not all, of the partition functions for strings on nonsimply-connected group manifolds.

A Wess–Zumino–Witten model one-loop partition function is a sesquilinear combination of specialised characters

$$\chi_\lambda(\tau)=\mathrm{tr}_\lambda \exp[2\pi i(L_0-\tfrac{1}{24}c)\tau] \qquad (1)$$

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[#1] This was conjectured for SU(3) in ref. [5].
[#2] Many of these may also be derived from the branching rules for the conformal embedding $\widehat{su}(p)^q\otimes\widehat{su}(q)^p\subset\widehat{su}(pq)^1$ [10].

of highest weight representations of a Kač–Moody algebra $\hat{g}$. Here $\lambda=\lambda_\mu\omega^\mu$ is the highest weight of the corresponding representation, $\lambda_\mu\in\mathbb{Z}$, and $\omega^\mu$ are the fundamental affine weights. For a unitary representation, we must have $\lambda_\mu k^{\vee\mu}=k$, where $k$ is the Kač–Moody central charge and $k^{\vee\mu}$ are the dual Kač labels. $c$ denotes the Virasoro central charge.

In particular, the partition function for strings propagating on the nonsimply-connected group manifold $G=\tilde{G}/B$, $B\subset B(\tilde{G})$, is of the form

$$Z(\tilde{G}/B)=\sum_{\lambda',\lambda}\chi^*_{\lambda'}Z_{\lambda'\lambda}\chi_\lambda. \tag{2}$$

It can also be written as an orbifold [7] partition function [3]. If $\sigma_1+\sigma_2\tau$ are the coordinates of the torus and $\tau$ its modulus, we let $(\alpha_1, \alpha_2)$ denote the contribution to the partition function from fields obeying the twisted boundary conditions

$$\phi(\sigma_1+2\pi, \sigma_2)=\alpha_1\phi(\sigma_1, \sigma_2), \quad \phi(\sigma_1, \sigma_2+2\pi)=\alpha_2\phi(\sigma_1, \sigma_2). \tag{3}$$

Then the partition function can be written as

$$Z(\tilde{G}/B)=\frac{1}{|B|}\sum_{\substack{\alpha_1,\alpha_2\in B\\ [\alpha_1,\alpha_2]=0}}(\alpha_1, \alpha_2), \tag{4}$$

where $|B|$ is the order of B. The modular invariance of this expression is guaranteed, since under any transformation $\tau\to(a\tau+b)/(c\tau+d)$ $(ad-bc=1; a, b, c, d\in\mathbb{Z})$, $(\alpha_1, \alpha_2)$ transforms to $(\alpha_1^d\alpha_2^c, \alpha_1^b\alpha_2^a)$ [7,11]. If $B=\mathbb{Z}_N$, (4) reduces to

$$Z(\tilde{G}/\mathbb{Z}_N)=\frac{1}{N}\sum_{m,n=0}^{N-1}(\alpha_1^m, \alpha_2^n). \tag{5}$$

The trivial example is the partition function on the simply-connected group manifold $\tilde{G}$ [3]:

$$Z(\tilde{G})_{\lambda'\lambda}=(1, 1)_{\lambda'\lambda}=\delta_{\lambda'\lambda}. \tag{6}$$

Untwisted fields are those obeying (3) with $\alpha_1=1$. The contribution to (5) from the untwisted sector is denoted $Z_1$:

$$Z_1(\tilde{G}/\mathbb{Z}_N)=\frac{1}{N}\sum_{n=0}^{N-1}(1, \alpha^n). \tag{7}$$

Using these last two objects and the generators $S$ $(\tau\to-1/\tau)$ and $T$ $(\tau\to\tau+1)$ of the modular group, it is in principle possible to obtain the full partition function $Z(\tilde{G}/\mathbb{Z}_N)$ [3]. The following formula is valid for $N$ prime:

$$Z(\tilde{G}/\mathbb{Z}_N)=\left(1+\sum_{v=0}^{N-1}T^vS\right)Z_1(\tilde{G}/\mathbb{Z}_N)-Z(\tilde{G}). \tag{8}$$

For $N$ not prime, the situation is more complicated. For example, one can verify

$$Z(\tilde{G}/\mathbb{Z}_4)=\left(1+\sum_{v=0}^{3}T^vS\right)Z_1(\tilde{G}/\mathbb{Z}_N)-Z(\tilde{G})-\tfrac{1}{2}Z(\tilde{G}/\mathbb{Z}_2). \tag{9}$$

The $\mathbb{Z}_2$ group of the last term is generated by $\alpha^2$ if $\alpha$ generates $\mathbb{Z}_4$. For general $N$ not prime, we expect subtraction of terms proportional to $Z(\tilde{G}/\mathbb{Z}_p)$ for $p|N$ would be necessary. For simplicity, we therefore restrict to $N$ prime, and use (8) [#3].

[#3] This is a significant restriction only for $g=A$ (g is the Lie algebra of G); $\mathbb{Z}_N$ with $N$ prime covers all other cases, except for half the possibilities with $g=D$.

The $T$ transformation is of quite simple form:

$$T_{\lambda'\lambda}=\delta_{\lambda'\lambda}\exp\left(\frac{\pi i|\lambda+\rho|^2}{k+h^\vee}-\frac{\pi i|\rho|^2}{h^\vee}\right). \tag{10}$$

Here $\rho=\sum_\mu \omega^\mu$ and the dual Coxeter number is $h^\vee=\sum_\mu k^{\vee\mu}\omega^\mu$. But the dimension of the $S$-transformation matrix grows rapidly with $k$, and the expression for its elements involves a sum over the Weyl group of g. So explicitly constructing the $S$-transformation matrix is extremely tedious. This is the main obstruction to using formulae like (8) to derive orbifold partition functions.

However, identities proved by Bernard [8] allow us to bypass this difficulty. Consider an element $A$ of the outer automorphism group $O(\hat{g})$ of $\hat{g}$ acting on a highest weight $\lambda$ for a given value of the Kač–Moody central charge $k$ ($\lambda_\mu k^\mu=k$). [An example is the generator of $O(\widehat{su}(N))$, *which permutes the fundamental weights as follows*: $A\omega^\mu=\omega^{\mu+1}$; $\omega^N\equiv\omega^0$.] Restricting to the weight lattice of the finite Lie algebra g, one can write

$$\overline{A(\lambda+\rho)}=(k+h^\vee)\omega^{A(0)}+w_A(\overline{\lambda+\rho}). \tag{11}$$

Here $\bar{\kappa}=\sum_{i\neq 0}\kappa_i\omega^i$ is the restriction of an affine weight $\kappa$ to the g weight lattice, $\omega^{A(0)}=A\omega^0$, and $w_A$ is an element of the Weyl group of g acting in the following way:

$$w_A(\omega^i)=\omega^{A(i)}-k^{\vee i}\omega^{A(0)}. \tag{12}$$

Using (11) it is easy to show [8]

$$S_{A(\lambda)\lambda'}=S_{\lambda\lambda'}\epsilon(w_A)\exp[2\pi i(\omega^{A(0)}|\lambda'+\rho)]. \tag{13}$$

Here $\epsilon(w_A)$ is the signature of $w_A$; i.e. $\epsilon=+1$ $(-1)$ for the product of an even (odd) number of reflections. Now for all outer automorphisms $A$, we have [6]

$$\epsilon(w_A)=\exp[2\pi i(\omega^{A(0)}|\rho)]=\exp(\pi i h^\vee|\omega^{A(0)}|^2). \tag{14}$$

So (13) reduces to

$$S_{A(\lambda)\lambda'}=S_{\lambda\lambda'}\exp[2\pi i(\omega^{A(0)}|\lambda')]. \tag{15}$$

This last equation is the starting point. Considering it with $A$ replaced by $A^r$ yields

$$(\omega^{A^r(0)}|\lambda)=r(\omega^{A(0)}|\lambda)\bmod 1 \tag{16}$$

implying

$$N(\omega^{A(0)}|\lambda)=0\bmod 1 \tag{17}$$

if $A^N=1$. So we see that the phase on the right hand side of (15) is an $N$th root of unity. In fact, it is the eigenvalue of an element of $B(\tilde{G})$ of order $N$. So, as mentioned above, the modular transformation $S$ maps elements of $O(\hat{g})$ into elements of $B(\tilde{G})$.

Now the untwisted sector partition function $Z_1(\tilde{G}/\mathbb{Z}_N)$ is built from the diagonal partition function $Z(\tilde{G})$ by projecting onto $\mathbb{Z}_N$ invariant states [compare (7) and (5)]. So

$$Z_1(\tilde{G}/\mathbb{Z}_N)_{\lambda'\lambda}=\delta_{\lambda'\lambda}\frac{1}{N}\sum_{r=0}^{N-1}\exp[2\pi i r(\omega^{A(0)}|\lambda)]=\delta_{\lambda'\lambda}\delta_1\{(\omega^{A(0)}|\lambda)\} \tag{18}$$

where we have defined

$$\begin{aligned}\delta_1(x)&=1, \quad \text{if } x=0\bmod 1,\\ &=0 \quad \text{otherwise.}\end{aligned} \tag{19}$$

Using (15) it is then easy to show

$$SZ_1(\tilde{G}/\mathbb{Z}_N)_{\lambda'\lambda}=\frac{1}{N}\sum_{r=0}^{N-1}\delta_{\lambda' A^r(\lambda)}. \tag{20}$$

Applying successive $T$-transformations yields

$$T^vSZ_1(\tilde{G}/\mathbb{Z}_N)_{\lambda'\lambda}=\frac{1}{N}\sum_{r=0}^{N-1}\delta_{\lambda' A^r(\lambda)}\exp\left[-2\pi iv\left(\frac{|A^r(\lambda+\rho)|^2-|\lambda+\rho|^2}{2(k+h^\vee)}\right)\right]. \tag{21}$$

Since $k^\vee A^{(0)}=1$,

$$(\omega^{A(0)}|w(\nu))=(\omega^{A(0)}|\nu)\ \mathrm{mod}\ 1 \tag{22}$$

for any element $w$ of the Weyl group of g, and any integral weight $\nu$. This and eqs. (11) and (14) simplify (21) to

$$T^vSZ_1(\tilde{G}/\mathbb{Z}_N)_{\lambda'\lambda}=\frac{1}{N}\sum_{r=0}^{N-1}\delta_{\lambda' A^r(\lambda)}\exp\{-2\pi iv[(\omega^{A^r(0)}|\lambda)+\tfrac{1}{2}k|\omega^{A^r(0)}|^2]\}. \tag{23}$$

Twisting a string by $\alpha^N=1$ must make no difference. Replacing $v$ with $v+N$ in (23) therefore demands

$$\tfrac{1}{2}Nk|\omega^{A^r(0)}|^2=0\ \mathrm{mod}\ 1 \tag{24}$$

for all $r$. This can be simplified, however, since

$$\tfrac{1}{2}Nk|\omega^{A(0)}|^2=0\ \mathrm{mod}\ 1 \tag{25}$$

is sufficient to ensure (24) and furthermore that

$$\tfrac{1}{2}k(\omega^{A^r(0)}|\omega^{A^s(0)}+\omega^{A^t(0)})=\tfrac{1}{2}k(\omega^{A^r(0)}|\omega^{A^{s+t}(0)})\ \mathrm{mod}\ 1. \tag{26}$$

Eq. (25) disallows certain integer values of Kač–Moody central charge $k$. It was derived in ref. [6] by requiring consistency of the Wess–Zumino term on a torus, with one of its cycles mapped into a nontrivial closed path in G. Thus it is a consequence of the nontrivial fundamental group $\pi_1(G=\tilde{G}/\mathbb{Z}_N)=\mathbb{Z}_N$.

Substituting (23) into the general formula (8), and using (26), we finally obtain

$$Z(\tilde{G}/\mathbb{Z}_N)_{\lambda'\lambda}=Z_1(\tilde{G}/\mathbb{Z}_N)_{\lambda'\lambda}+\sum_{r=1}^{N-1}\delta_{\lambda' A^r(\lambda)}\frac{1}{N}\sum_{v=0}^{N-1}\exp\{-2\pi ir[(\omega^{A^v(0)}|\lambda+\tfrac{1}{2}k\omega^{A^r(0)})]\}. \tag{27}$$

Since $N$ is prime the factor $r$ outside the square brackets may be dropped, and we can write

$$Z(\tilde{G}/\mathbb{Z}_N)_{\lambda'\lambda}=\frac{1}{N}\sum_{m,n=0}^{N-1}\delta_{\lambda' A^m(\lambda)}\exp[-2\pi i(\omega^{A^n(0)}|\lambda+\tfrac{1}{2}k\omega^{A^m(0)})]. \tag{28}$$

This is exactly the form found by Felder, Gawedzki and Kupiainen [6] [#4]. Furthermore, it is easy to convince oneself that

$$(\alpha^m,\alpha^n)_{\lambda'\lambda}=\delta_{\lambda' A^m(\lambda)}\exp[-2\pi i(\omega^{A^n(0)}|\lambda+\tfrac{1}{2}k\omega^{A^m(0)})]. \tag{29}$$

The condition (25) guarantees the integrality of the elements of the matrix $Z(\tilde{G}/\mathbb{Z}_N)$. This must be, since these quantities count the numbers of primary fields. We may rewrite the final result in a way that manifests this property:

$$Z(\tilde{G}/\mathbb{Z}_N)_{\lambda'\lambda}=\sum_{m=0}^{N-1}\delta_{\lambda' A^m(\lambda)}\delta_1\{(\omega^{A(0)}|\lambda+\tfrac{1}{2}km\omega^{A(0)})\}. \tag{30}$$

[#4] The authors of ref. [6] also considered the unique semi-simple possibility for $\tilde{G}$ simple: $B=\mathbb{Z}_2\times\mathbb{Z}_2$ for $g=D_l$, $l$ even.

We should emphasise here that (28) is a modular invariant for all cases, whether or not it is integer valued. But it reduces to the physically sensible partition function (30) (with integral values) only when the Kač–Moody central charge $k$ obeys (25).

We will now write the partition function in a more compact notation, and use it to verify modular invariance. Considering an outer automorphism $A$ as acting on the space of highest weights of unitary representations, we have

$$A_{\lambda'\lambda}=\delta_{\lambda' A(\lambda)}. \tag{31}$$

Then (15) becomes in matrix notation

$$AS=S\alpha, \tag{32}$$

where $\alpha\in B(\tilde{G})$ is of course diagonal

$$\alpha_{\lambda'\lambda}=\delta_{\lambda'\lambda}\exp[2\pi i(\omega^{A(0)}|\lambda)]. \tag{33}$$

Thus the modular transformation $S$ diagonalises the outer automorphisms of $\hat{g}$.

If we have another related pair $A'\in O(\hat{g})$, $\alpha'\in B(\tilde{G})$, i.e. $A'S=S\alpha'$, we define

$$A'\circ\alpha=A'\alpha\exp[+\pi ik(\omega^{A'(0)}|\omega^{A(0)})],\quad \alpha\circ A'=\alpha A'\exp[-\pi ik(\omega^{A(0)}|\omega^{A'(0)})], \tag{34}$$

so that

$$A'\circ\alpha=\alpha\circ A'. \tag{35}$$

Then the partition function may be written simply as

$$Z(\tilde{G}/\mathbb{Z}_N)=\frac{1}{N}\sum_{m,n=0}^{N-1}A^m\circ\alpha^n. \tag{36}$$

If $C$ is the charge conjugation matrix, we have

$$S^2=C,\quad CA=A^{-1}C, \tag{37}$$

so that (32) also implies

$$S^\dagger\alpha S=A^{-1}. \tag{38}$$

Therefore

$$SZ(\tilde{G}/\mathbb{Z}_N)=\frac{1}{N}\sum_{m,n}\alpha^m\circ A^{-n}=Z(\tilde{G}/\mathbb{Z}_N). \tag{39}$$

Finally, it is straightforward to prove

$$T^\dagger(A\circ\alpha')T=A\circ\alpha\alpha', \tag{40}$$

establishing the $T$-invariance of $Z(\tilde{G}/\mathbb{Z}_N)$.

The simple structure just discussed may exist in a more general class of rational conformal field theories, perhaps those obtained by the coset construction [12]. Then partition functions for these theories of the type (36) could be easily obtained. We hope to report on this possibility in the future.

In conclusion, let us emphasise that the advantage of the orbifold approach used here is simplicity, and consequent generality. It remains to be seen if the elegant methods of ref. [6] are applicable to more general rational conformal field theories.

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