Global & $\underbrace{\text{Local}}$ only in 2d

$\phi, \bar\phi$ → only infinitessimal

Consequences of conf. sym in $\underline{d}$ on correlation functions

* field : $A_i(x) \to \partial_x A_i \to \cdots \equiv \{A_i\}$

* "quasi" **(Global)** primary field : $\phi_j(x) \to \left|\dfrac{\partial x'}{\partial x}\right|^{\Delta_j/d} \phi_j(x')$

like " polynomial"

$$\Rightarrow \langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x_1}^{\frac{\Delta_1}{d}}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x_n}^{\frac{\Delta_n}{d}}\langle\phi_1(x_1')\cdots\phi_n(x_n')\rangle$$

* vacuum $|0\rangle$: invariant under global.

$\langle \cdots \rangle$:

- function of $(x_i - x_j)$ → $d \times (n-1)$ translation
- rotation → $\underbrace{|x_i - x_j|}$ ⇒ $\dfrac{n(n-1)}{2}$
- scale → $\dfrac{|x_{ij}|}{|x_{kl}|}$
- special → $|x_{12}'|^2 = \dfrac{(x_{12})^2}{(1+2b\cdot x_1+b^2x_1^2)(1+2b\cdot x_2+b^2x_2^2)}$

$$\Rightarrow \frac{|x_{ij}||x_{kl}|}{|x_{ik}||x_{jl}|}$$

# of indep. cross ratios

N-pt Corr.

$\Rightarrow \boxed{\dfrac{N(N-3)}{2}}$ variables

$N=3 \to$ fixed. (no ratios)

$N=4 \to 2 \quad \left(\to \dfrac{12|34}{13|24},\ \dfrac{14|23}{12|34}\right)$

2pt. $\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x_2}^{\Delta_2/d} \langle \phi_1(x_1')\,\phi_2(x_2')\rangle$

Rot. trns. $\overset{\shortparallel}{f(|x_1-x_2|)}$ under $x \xrightarrow[\text{dil}]{} \lambda x \overset{\shortparallel}{=} x'$ $\Rightarrow \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}$

$$\frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} = \lambda^{\Delta_1+\Delta_2}\frac{C_{12}}{|x_1'-x_2'|^{\Delta_1+\Delta_2}}$$

Special

$$= (1+2b\cdot x_1+\cancel{b^2 x_1^2})^{-\Delta_1}(\;''\;\cancel{x_2})^{-\Delta_2}\frac{C_{12}}{|x_1'-x_2'|^{\Delta_1+\Delta_2}} \rightarrow \frac{C_{12}\cancel{(x_1)^{\frac{\Delta_1+\Delta_2}{2}}}\cancel{(x_2)^{\frac{\Delta_1+\Delta_2}{2}}}}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$

$$|x_1'-x_2'| = \frac{|x_1-x_2|}{\sqrt{(x_1)(x_2)}}$$

only when $\Delta_1=\Delta_2$

3pt.

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = \sum_{a,b,c}\frac{C_{abc}}{r_{12}^a\, r_{23}^b\, r_{13}^c}$$

← cannot be fixed by G & Local conf. sym

$$\Rightarrow (x_1)^{-\Delta_1}(x_2)^{-\Delta_2}(x_3)^{-\Delta_3} \quad (x_1)^{\frac{a}{2}+\frac{c}{2}}(x_2)^{\frac{a}{2}+\frac{b}{2}}(x_3)^{\frac{b}{2}+\frac{c}{2}}$$

$a+c=2\Delta_1 \quad a+b=2\Delta_2 \quad b+c=2\Delta_3 \rightarrow a+b+c=\Delta_1+\Delta_2+\Delta_3$

$b=\Delta_2+\Delta_3-\Delta_1 \quad a=\Delta_1+\Delta_2-\Delta_3 \quad c=\Delta_1+\Delta_3-\Delta_2$

$\Delta=\Delta_1+\Delta_2+\Delta_3+\Delta_4$

4 (or higher) pt: $G^{(4)} = \underbrace{F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}},\frac{r_{12}r_{34}}{r_{23}r_{14}}\right)}_{\text{can not be fixed by global Conf.}}\prod_{i<j} r_{ij}^{-(\Delta_i+\Delta_j)+\frac{\Delta}{3}}$

2. CFT_2

$ds^2 = dz d\bar{z} \longrightarrow dw d\bar{w} \qquad w = w(z), \bar{w}(\bar{z})$

"Special field" [Primary field] satisfying for any w

$$\Phi(z,\bar{z}) = \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(w,\bar{w}) \qquad : h, \bar{h} : \text{real}$$

secondary could be quasi-primary

let $w = z + \epsilon$; $\delta\Phi = \left[(h\partial\epsilon + \epsilon\partial) + (\bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial})\right]\Phi$

$$\delta_{\epsilon,\bar{\epsilon}}\langle\Phi_1\Phi_2\rangle = \langle\delta\Phi_1\Phi_2\rangle + \langle\Phi_1\delta\Phi_2\rangle = 0$$

$$\left[(\epsilon_1\partial_1 + h_1\partial\epsilon_1) + (\epsilon_2\partial_2 + h_2\partial\epsilon_2) + \cdots\right]\langle\Phi_1\Phi_2\rangle = 0$$

$\epsilon = 1 \qquad (\partial_1 + \partial_2)\langle\Phi_1\Phi_2\rangle = 0 \longrightarrow \langle\Phi_1\Phi_2\rangle = f(z_{12})\bar{f}(\bar{z}_{12})$

$\epsilon = z \qquad \left[(z_1\partial_1 + h_1) + (z_2\partial_2 + h_2) + \cdots\right]\langle\Phi_1\Phi_2\rangle = 0$

$$\langle\Phi_1\Phi_2\rangle = \frac{C_{12}}{z_{12}^{h_1+h_2}\bar{z}_{12}^{\bar{h}_1+\bar{h}_2}} = \frac{C_{12}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}} = \underbrace{\frac{C_{12}}{|z_{12}|^{2\Delta}}}_{\text{normalization}}$$

$\epsilon = z^2 \qquad h_1 = h_2, \ \bar{h}_1 = \bar{h}_2$

$\left(\begin{array}{l}\Delta = h + \bar{h} \\ \text{if } S = h - \bar{h} = 0\end{array}\right)$

3-pt.

$$\frac{C_{123} \ (\text{structure const})}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \times [x \to \bar{x}]$$

most difficult (local eq. is NOT available)

Global

conf. transf

Local

4-pt. $\quad G^{(4)} = \underbrace{f(x,\bar{x})}_{\text{fixed by D.E. (Local conf)}} \prod_{i<j} z_{ij}^{-(h_i+h_j)+\frac{h}{3}} \cdot [x \to \bar{x}]$

$$\underbrace{x = \frac{z_{12}z_{34}}{z_{13}z_{24}}}_{\text{only one cross ratio, independent}} \qquad 1 - x = \frac{z_{13}z_{24} - z_{12}z_{34}}{z_{13}z_{24}} = \frac{\overbrace{12 + 34 - \cancel{23} - \cancel{14} + \cancel{23} + \cancel{14}}^{-13-24}}{\frac{z_{14}z_{23}}{z_{13}z_{24}}}$$

Global : can fix $\quad z_1 = \infty, z_2 = 1, z_4 = 0 \rightarrow \underbrace{z_3 \equiv x}_{\text{unfixed}}$

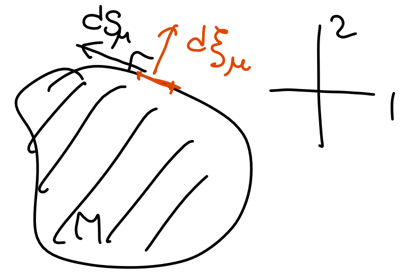
2d Ward Id.

① $\left\langle \left(\int d^2x\, \partial_\mu J^\mu_\varepsilon\right) X \right\rangle = \langle \delta X \rangle$

or

② $\partial_\mu \langle J^\mu_\varepsilon(x)\, X \rangle = \sum_i \delta(x-x_i)\, \delta_i \langle X \rangle$

① 2d Gauss theorem



$$\boxed{\int_M d^2x\, \partial_\mu J^\mu = \int_{\partial M} d\xi_\mu J^\mu = \int_{\partial M} \epsilon_{\mu\rho} J^\mu ds^\rho}$$

$d\xi_\mu = \epsilon_{\mu\rho} ds^\rho$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$dx_1 dx_2 = dz\, d\bar z\, \frac{\partial(x_1, x_2)}{\partial(z, \bar z)} = dz\, d\bar z \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{vmatrix} = dz\, d\bar z \left(\frac{i}{2}\right)$$

$$\partial_\mu J^\mu = \partial_1 J^1 + \partial_2 J^2 = 2(\partial_z J_{\bar z} + \partial_{\bar z} J_z) \leftarrow \partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$$

$\bar J/\pi \equiv J^1 + iJ^2$ $\qquad$ $J^1 - iJ^2 \equiv J/\pi$

$\epsilon_{zz} = 0$

$$\epsilon_{z\bar z} = \frac{\partial x^i}{\partial \bar z}\frac{\partial x^j}{\partial z}\epsilon_{ij} = \left(\frac{1}{2}\frac{(-i)}{2} - \frac{i}{2}\frac{1}{2}\right) = \frac{1}{2i}$$

$$\left(\begin{aligned} x_2 &= \frac{z - \bar z}{2i} \\ x_1 &= \frac{z + \bar z}{2} \end{aligned} \right)$$

$$\epsilon_{\mu\rho} J^\mu ds^\rho = \underbrace{\epsilon_{z\bar z}}_{-\frac{i}{2}} \underset{\substack{\uparrow \\ J^1 - iJ^2 \\ \equiv J/\pi}}{J^{\bar z}} dz + \underbrace{\epsilon_{\bar z z}}_{\frac{i}{2}} \underset{\substack{\uparrow \\ J^1 + iJ^2 \equiv \bar J/\pi}}{J^z} d\bar z$$

$$\therefore \int_M dz\, d\bar z\, (\partial \bar J + \bar\partial J) = \oint_{\partial M} (\bar J d\bar z - J dz)$$

$$\boxed{\frac{1}{2\pi i}\oint_{\partial M} \left\{ dz \langle J X \rangle - d\bar z \langle \bar J X \rangle \right\} = \langle \delta X \rangle}$$

$$\Rightarrow \quad -\oint \frac{dz}{2\pi i} \epsilon(z) \langle T X \rangle + \oint \frac{d\bar z}{2\pi i} \bar\epsilon(\bar z) \langle \bar T X \rangle = \langle \delta_{\epsilon\bar\epsilon} X \rangle$$

② 2d Ward Id.

$$\partial_\mu \langle T^\mu{}_\nu X \rangle \doteq -\sum_i \delta^{(2)}(x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle$$

$$\varepsilon_{\mu\nu} \langle T^{\mu\nu} X \rangle = -i \sum_{i=1}^{n} s_i \delta^{(2)}(x - x_i) \langle X \rangle$$

$$\langle T^\mu{}_\mu X \rangle = -\sum_i \delta^{(2)}(x - x_i) \Delta_i \langle X \rangle$$

Using $\boxed{\delta^{(2)}(x) = \frac{1}{\pi} \bar\partial \frac{1}{z}}$

$$\int_M d^2x \, \underbrace{\delta^{(2)}(x)}_{\equiv \partial_{\bar z} g(z)} f(z) = f(0) = \int_M d^2x \, \partial_{\bar z}(g f) = \frac{i}{2} \int dz d\bar z \, \partial_{\bar z} \underbrace{(g f)}_{J}$$

$$\int_M dz d\bar z \, (\partial \bar J + \bar\partial J) = \oint_{\partial M} (\bar J d\bar z - J dz) \xrightarrow{\bar J = 0} -\frac{i}{2} \oint g f \, dz$$

$$f(0) = \oint g(z) f(z) \frac{dz}{2i} \Rightarrow g(z) = \frac{1}{\pi z}$$

$$2\pi \partial_z \langle T_{\bar z z} X \rangle + 2\pi \partial_{\bar z} \langle T_{zz} X \rangle = -\sum_{i=1}^{n} \partial_{\bar z} \frac{1}{z - z_i} \partial_{z_i} \langle X \rangle$$

$$2\pi \partial_z \langle T_{\bar z \bar z} X \rangle + 2\pi \partial_{\bar z} \langle T_{z \bar z} X \rangle = -\sum_{i=1} \partial_z \frac{1}{\bar z - \bar z_i} \partial_{\bar z_i} \langle X \rangle$$

$$2 \langle T_{z \bar z} X \rangle + 2 \langle T_{\bar z z} X \rangle = -\sum \delta^{(2)}(x - x_i) \Delta_i \langle X \rangle$$

$$-2 \langle T_{z \bar z} X \rangle + 2 \langle T_{\bar z z} X \rangle = -\sum \delta^{(2)}(x - x_i) s_i \langle X \rangle$$

define: $T = -2\pi T_{zz}$, $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$, $\Theta = -2\pi T_{z\bar{z}}$
$\bar{\Theta} = -2\pi T_{\bar{z}z}$

$$\partial_z \langle \Theta X \rangle + \partial_{\bar{z}} \langle T X \rangle = \sum_{i=1}^{n} \partial_{\bar{z}} \frac{1}{z - z_i} \partial_{z_i} \langle X \rangle$$

$$\partial_z \langle \bar{T} X \rangle + \partial_{\bar{z}} \langle T_{z\bar{z}} X \rangle = \sum_{i=1}^{n} \partial_{z} \frac{1}{\bar{z} - \bar{z}_i} \partial_{\bar{z}_i} \langle X \rangle$$

$$\left\{ \begin{aligned} \langle \Theta X \rangle + \langle \bar{\Theta} X \rangle &= \pi \sum \delta^{(2)}(x - x_i) \Delta_i \langle X \rangle \\ \langle \Theta X \rangle - \langle \bar{\Theta} X \rangle &= \pi \sum \delta^{(2)}(x - x_i) s_i \langle X \rangle \end{aligned} \right.$$

$$\left\{ \begin{aligned} \langle \Theta X \rangle &= \sum_i \partial_{\bar{z}} \frac{h_i}{z - z_i} \langle X \rangle \\ \langle \bar{\Theta} X \rangle &= \sum_i \partial_{z} \frac{\bar{h}_i}{\bar{z} - \bar{z}_i} \langle X \rangle \end{aligned} \right. \qquad \left\{ \begin{aligned} \Delta_i &= h_i + \bar{h}_i \\ s_i &= h_i - \bar{h}_i \end{aligned} \right.$$

Insert

$$\partial_{\bar{z}} \left\{ \langle T X \rangle - \sum_{i=1}^{n} \left[\frac{1}{z - z_i} \partial_{z_i} \langle X \rangle + \frac{h_i}{(z - z_i)^2} \langle X \rangle \right] \right\} = 0$$

$$\partial_{z} \left\{ \langle \bar{T} X \rangle - \sum_{i=1}^{n} \left[\frac{1}{\bar{z} - \bar{z}_i} \partial_{\bar{z}_i} \langle X \rangle + \frac{\bar{h}_i}{(\bar{z} - \bar{z}_i)^2} \langle X \rangle \right] \right\} = 0$$

$$\langle T X \rangle = \sum_{i=1}^{n} \left[\frac{1}{z - z_i} \partial_{z_i} \langle X \rangle + \frac{h_i}{(z - z_i)^2} \langle X \rangle \right] + \text{regular}$$

$$\langle \bar{T} X \rangle = \sum_{i=1}^{n} \left[\frac{1}{\bar{z} - \bar{z}_i} \partial_{\bar{z}_i} \langle X \rangle + \frac{\bar{h}_i}{(\bar{z} - \bar{z}_i)^2} \langle X \rangle \right] + \text{reg.}$$

Consistent with

$$-\oint \frac{dz}{2\pi i} \epsilon(z) \langle T X \rangle + \oint \frac{d\bar{z}}{2\pi i} \bar{\epsilon}(\bar{z}) \langle \bar{T} X \rangle = \langle \delta_{\epsilon\bar{\epsilon}} X \rangle$$

[pf]

$$z \to w = z + \epsilon(z)\,; \qquad \phi(z,\bar z) \to \phi'(w,\bar w) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{\partial \bar w}{\partial \bar z}\right)^{-\bar h}\phi(z,\bar z)$$

$$\langle X\rangle = \langle \phi_1(z_1,\bar z_1)\cdots\phi_n(z_n,\bar z_n)\rangle$$

$$\Rightarrow \prod_j \left(\frac{\partial w_j}{\partial z_j}\right)^{h_j}\left(\frac{\partial \bar w_j}{\partial \bar z_j}\right)^{\bar h_j}\langle \phi_1(w_1,\bar w_1)\cdots\phi_n(w_n,\bar w_n)\rangle$$

$$\langle \delta X\rangle = \sum_i \langle \phi_1(z_1,\bar z_1)\cdots \underbrace{\Delta\phi_i}\cdots\phi_n(z_n,\bar z_n)\rangle$$

$$\Delta\phi_i = \phi_i'(z,\bar z) - \phi_i(z,\bar z) = \left[-\left(\epsilon\partial_{z_i} + h_i\partial_{z_i}\epsilon\right) - \left(\bar\epsilon\partial_{\bar z_i} + \bar h_i\partial_{\bar z_i}\bar\epsilon\right)\right]\phi_i$$

holomorphic part:

$$-\oint\frac{dz}{2\pi i}\langle \epsilon T(z)\,\phi_1\cdots\phi_n\rangle = -\sum_j\langle\phi_1\cdots\left(\epsilon\partial_{z_i} + h_i\partial_{z_i}\epsilon\right)\phi_i\cdots\phi_n\rangle$$

$$\therefore\ T\phi = \frac{h\,\phi(w)}{(z-w)^2} + \frac{1}{z-w}\partial_w\phi + \text{reg.}$$

$$T(z)T(w) = \underbrace{\frac{c/2}{(z-w)^4}} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T + \text{reg.}$$

← central charge

T is NOT primary

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4} \qquad \therefore\ c \geq 0 \text{ for positive def. H.}$$

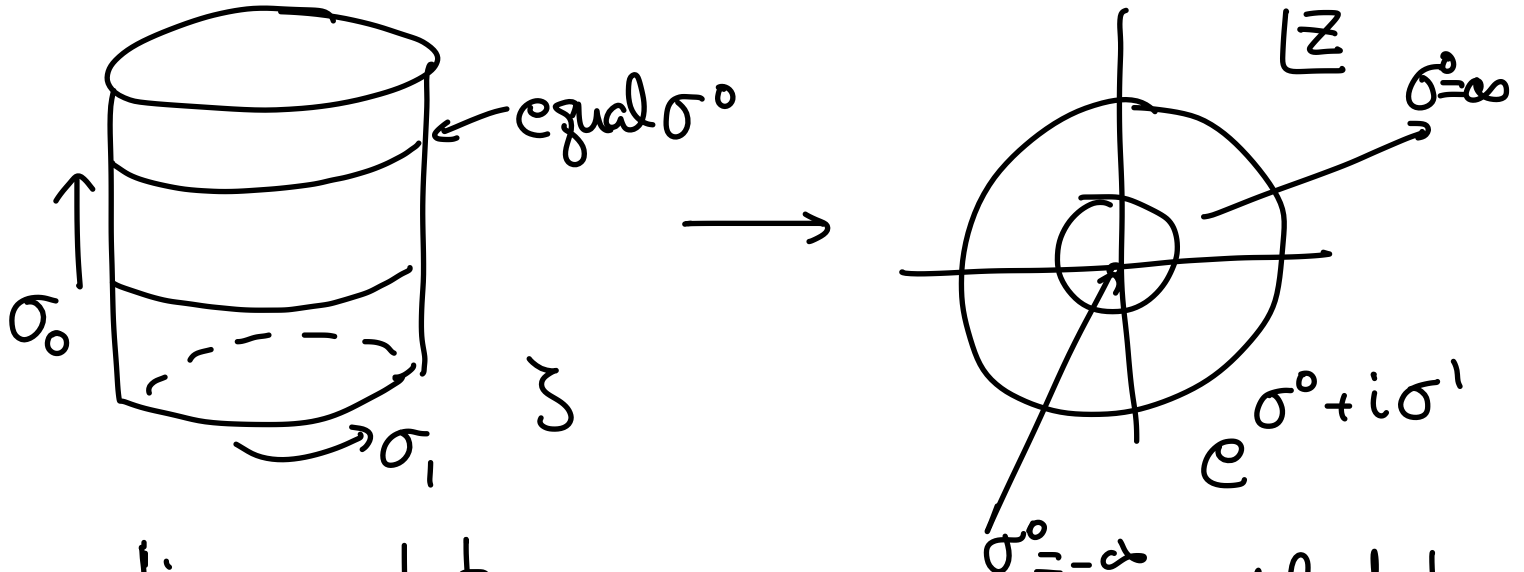
$$\bar T(\bar z)\bar T(\bar w) = \frac{c/2}{(\bar z-\bar w)^4} + \cdots$$

Radial

$\sigma^0, \sigma^1 \to \zeta = \sigma^0 + i\underset{\sim}{\sigma^1}$ Compact on C

$\sigma^1 \equiv \sigma^1 + 2\pi$

conf. map: $\zeta \to \underline{z = e^{\zeta}}$



time evolution

$\sigma^0 \to \sigma^0 + a$

Hamil.

dilatation

$z \to \lambda z$ $\quad L_0$

"radial quantization"

$(d+1)$ dim

$$\partial_\mu j^\mu = 0 \to Q = \int d^d x \, j_0 \Leftarrow t \text{ fixed}$$

$\uparrow$ conserved charges

$$\delta_\epsilon A = \epsilon [Q, A]$$

$x \cdot y = x_\mu y^\mu = x_1 y_1 + x_2 y_2$ $\quad x_1 = x^1$ $\quad z = x_1 + i x_2$

$= g_{\mu\nu} x^\mu x^\nu$ $\quad g_{11} = g_{22} = 1$ $\quad y = y_1 + i y_2$

$= \frac{1}{2}(\bar{z} y + z \bar{y}) = \underbrace{g_{z\bar{y}}}_{\frac{1}{2}} \bar{z} y + \underbrace{g_{y\bar{z}}}_{\frac{1}{2}} \bar{y} z$

$\therefore g_{zz} = g_{\bar{z}\bar{z}} = 0$ $\quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$ $\quad (g^{z\bar{z}} = g^{\bar{z}z} = 2)$ $\quad \to x^z = g^{z\bar{z}} x_{\bar{z}} = \frac{1}{2} x_{\bar{z}}$

OPE: $A(z)B(w)$ $\quad |z|>|w|$

Path Int.

$$\langle\Omega|T[\phi_1\cdots\phi_n]|\Omega\rangle = \frac{\int \varphi_1\cdots\varphi_n e^{-S[\varphi]}\mathcal{D}\varphi}{\int e^{-S[\varphi]}\mathcal{D}\varphi}$$

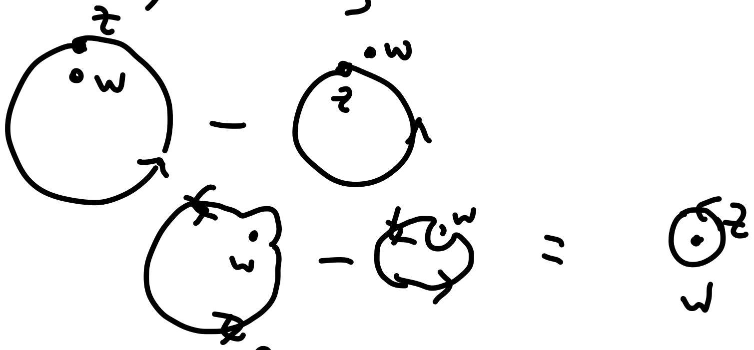
time ordered result

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z|>|w| \\ B(w)A(z) & |z|<|w| \end{cases}$$

Commutator $\longleftrightarrow$ Contour int.

$$\Big[\underbrace{\int dx\, B}, A\Big]_{E.T.} \longrightarrow \oint_w dz\, R(B(z)A(w))$$

$$\Big[\oint dz\, B(z), A(w)\Big]$$



$$\delta_{\epsilon,\bar\epsilon}\Phi = \frac{1}{2\pi i}\oint\Big(dz\,\epsilon(z)R(T(z)\Phi(w)) + c.c.\Big)$$

$$= [(h\partial\epsilon + \epsilon\partial) + c.c.]\,\Phi$$

$$\Rightarrow R(T(z)\Phi(w)) = \frac{h}{(z-w)^2}\Phi + \frac{1}{z-w}\partial\Phi + \cdots$$

$\qquad\qquad\qquad\qquad \rotatebox{90}{$\equiv$}$

OPE (rad. order) $T(z)\Phi(w,\bar w)$

OPE $\longleftrightarrow$ 3-pt

Mode exp.

$$T = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad \text{or} \quad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$$

$$[L_{-n}] = n$$

$$[L_n, L_m] = \left(\oint dz \oint dw - \oint dw \oint dz\right) z^{n+1} w^{m+1} \underbrace{T(z)T(w)}$$

$$= \oint_{w^{m+1}} dw \oint_w dz \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T + \frac{\partial_w}{z-w} T\right) z^{n+1}$$

$$z^{n+1} = w^{n+1} + (n+1)(z-w) w^n + \frac{n(n+1)}{2}(z-w)^2 w^{n-1} + \frac{n(n+1)(n-1)}{6}(z-w)^3 w^{n-2} + \cdots$$

$$= \oint dw\, w^{m+1} \oint \left(\frac{c/2}{z-w} \frac{n(n^2-1)}{6} w^{n-2} + \frac{2(n+1)}{(z-w)} w^n T(w) + \frac{\partial_w T}{z-w} w^{n+1} \right) dz$$

$$= \frac{c}{12} n(n^2-1) \delta_{n+m,0} + \underbrace{2(n+1) \underbrace{\oint dw\, w^{n+m+1} T(w)}_{L_{n+m}}} + \underbrace{\oint w^{n+m+2} \partial_w T\, dw}_{\underbrace{\partial_w(w^{\cdots} T)}_{0} - T(n+m+2) w^{n+m+1}}$$

$$+ (n-m) L_{n+m} \qquad -(n+m+2) L_{n+m}$$

In-out states

$$t \to -t \; ; \; z = e^{t+i\sigma} \to e^{-t+i\sigma} = \frac{1}{z^*} = \frac{1}{\bar{z}}$$

$$A(z,\bar{z})^\dagger = A\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \frac{1}{\bar{z}^{2h}} \frac{1}{z^{2\bar{h}}}$$

$$|A_{in}\rangle = \lim_{\sigma=0,\, t\to -\infty} A|0\rangle = \lim_{z,\bar{z}\to 0} A(z,\bar{z})|0\rangle$$

$$\langle A_{out}| = \lim_{w,\bar{w}\to 0} \langle 0| \underbrace{A(w,\bar{w})^\dagger}_{A(w,\bar{w}) w^{2h} \bar{w}^{2\bar{h}}} = (|A_{in}\rangle)^\dagger$$

$$T(z) = \sum \frac{L_n}{z^{n+2}} \longrightarrow T^\dagger = \sum \frac{L_n^\dagger}{\bar{z}^{n+2}}$$

$$T\left(\frac{1}{\bar{z}}\right) \frac{1}{\bar{z}^4} = \sum L_n \bar{z}^{n-2} = \sum L_{-n} \bar{z}^{-n-2}$$

equal

$$\Rightarrow L_n^\dagger = L_{-n} \quad \& \quad \bar{L}_n^\dagger = \bar{L}_{-n}$$

Regularity: $T(z)|0\rangle = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}}|0\rangle$

should be regular as $z\to 0$

$\therefore\ L_m|0\rangle = 0$ for $m \geq -1$

$m=\pm 1, 0 \Rightarrow$ global conf. $SL(2,\mathbb{R})$

$\theta\ L_m^{+}|0\rangle = 0\ \ m\leq 1 \Longrightarrow \langle 0|L_m = 0$

$\Big\}$ $|0\rangle$ $SL(2,\mathbb{R})$ invariant vacuum

$\langle 0|L_{\pm 1,0} = L_{\pm 1,0}|0\rangle = 0$

ex:

$(L_{-2}|0\rangle \neq 0)$ * $\langle 0|L_m|0\rangle = 0 \to \langle 0|T|0\rangle = 0$

$\langle 0|T(z)T(w)|0\rangle = \langle 0|\frac{c/2}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial_w T}{(z-w)} + \dots |0\rangle$

$\sum_{n\geq 2} L_n z^{-n-2} \underbrace{\sum_{m\leq -2} L_m w^{-m-2}}$

$[L_n, L_m] = (n-m)L_{n+m} + C_n\delta_{n+m}$, $\quad C_n = \frac{c}{12}n(n^2-1)$

$L_nL_m = L_mL_n + \quad ''$

$\langle 0|\ \downarrow\ |0\rangle = 0 + (n-m)\langle 0|\cancel{L_{n+m}}|0\rangle + \sum_{n,m} C_n\delta_{n+m} z^{-n-2}w^{-m-2}$

$\sum_{n,m\geq 2} C_n\delta_{n-m}(zw)^{-2}\,(z^{-n}w^{n})$

$= \underbrace{\frac{c}{12w^4}\sum_{n\geq 2} n(n^2-1)\left(\frac{w}{z}\right)^{n+2}} = \frac{c/2}{(z-w)^4}$

$= \frac{6\left(\frac{w}{z}\right)^4}{\left(1-\frac{w}{z}\right)^4} = \sum_{n\geq 2}\frac{C_n}{w^4}\left(\frac{w}{z}\right)^{n+2}$

$\frac{1}{1-\frac{w}{z}} = \sum_{n\geq -1}\underbrace{\left(\frac{w}{z}\right)}_{\alpha}^{n+1}$, $\quad \alpha < 1\ (w<z)$

$\frac{\partial}{\partial\alpha}\sum_{n}(n+1) \qquad \frac{1}{(1-\alpha)^2}$

$\partial_\alpha^2\sum_{n\geq -1} n \qquad \frac{2}{(1-\alpha)^3}$

$\partial_\alpha^3\sum_{n\geq 2}(n-1) \qquad \frac{6}{(1-\alpha)^4}$

$\frac{6}{\left(1-\frac{w}{z}\right)^4} = \sum_{n\geq 2} n(n^2-1)\left(\frac{w}{z}\right)^{n-2}$

infinitessimal trasf for T

finite

$$\delta_\epsilon T = \epsilon \partial T + \underset{\substack{\uparrow \\ h}}{2} \partial\epsilon T + \underbrace{\frac{c}{12}\partial^3\epsilon} \longrightarrow \boxed{(\partial f)^2 T(f) + \frac{c}{12} S}$$

$$f = z + \epsilon(z)$$

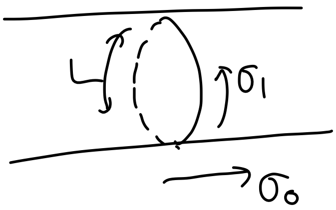
$$S(f, z) = \frac{\partial f \, \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2} = 0 \quad \text{for } SL(2,\mathbb{R})$$

$$(\epsilon = 1, z, z^2 \longrightarrow \partial^3\epsilon = 0)$$

Relation to cylinder

$$z = e^{\frac{2\pi}{L} w}$$

let $\sigma_1 \to \sigma_1 + L$ $\quad w = \frac{L}{2\pi} \ln z = \sigma_0 + i\sigma_1$



$$T_{cyl}(w) = \underset{\substack{\shortparallel \\ \left(\frac{2\pi}{L}\right)^2 z^2}}{\left(\frac{\partial z}{\partial w}\right)^2} T_{pl}(z) + \frac{c}{12} \underbrace{S(z, w)}_{\left(\frac{2\pi}{L}\right)^2\left(1 - \frac{3}{2}\right)}$$

$$\boxed{T_{cyl}(w) = \left(\frac{2\pi}{L}\right)^2 \left(z^2 T_{pl}(z) - \frac{c}{24}\right)}$$

$$\left(\frac{2\pi}{L}\right)^2 \oint \frac{dz}{z} \left(z^2 T_{pl}(z) - \frac{c}{24}\right) = \left(L_0 - \frac{c}{24}\right)\left(\frac{2\pi}{L}\right)^2$$

$$= \oint \underbrace{\frac{dz}{z}} T_{cyl}(w) = \frac{2\pi}{L} \underbrace{\oint dw \, T_{cyl}(w)}$$

$$dw = \frac{L}{2\pi} \frac{dz}{z} \qquad \left(\underset{\substack{\sigma_0 \to \sigma_0 + \varepsilon \\ \varepsilon = \text{const}}}{\equiv} L_0^{cyl}\right)$$

$$\boxed{L_0^{cyl} = \frac{2\pi}{L}\left(L_0 - \frac{c}{24}\right), \quad \bar{L}_0^{cyl} = \frac{2\pi}{L}\left(\bar{L}_0 - \frac{c}{24}\right)}$$

$$H^{cyl.} = L_0^{cyl} + \bar{L}_0^{cyl} = \frac{2\pi}{L}\left(\underbrace{L_0 + \bar{L}_0} - \frac{c}{24}\right)$$

$\sigma_0 \to \sigma_0 + \varepsilon$ $\qquad$ dilation in plane.

"time translation"

Finite-size effect (Casimir effect)

$$T_{cyl}(w) = \left(\frac{2\pi}{L}\right)^2 \left(z^2 T_{pl}(z) - \frac{c}{24}\right)$$

SL(2,C) vac

$$\therefore \langle T_{cyl}(w) \rangle = -\frac{c\pi^2}{6L^2} \neq 0$$ "Casimir" Energy

Free energy (of cylinder) ; $Z = \int [D\varphi] e^{-S[\varphi]}$ $\left(S = \int \sqrt{g}\,\mathcal{L}\, d^2w\right) \to F = -\ln Z$

variation : $$\delta F = -\frac{1}{Z}\int [D\varphi] \left[\int \frac{\sqrt{g}}{2}\, \delta g_{\mu\nu} T^{\mu\nu}_{cyl}\, d^2w\right] e^{-S}$$

$$\equiv -\frac{1}{2}\int d^2w \sqrt{g}\; \delta g_{\mu\nu} \langle T^{\mu\nu}_{cyl} \rangle$$

time dilatation in cylinder ; $\sigma^0 \to \lambda \sigma^0 = (1+\varepsilon)\sigma^0$

$$\delta g_{\mu\nu} = -2\varepsilon\, \delta_{\mu 0}\delta_{\nu 0} \qquad L \to L + \underbrace{\varepsilon L}_{\delta L} \qquad \varepsilon = \frac{\delta L}{L}$$

$$\delta F_L = \frac{\delta L}{L} \int d^2w \sqrt{g} \left(\langle T^{00}_{cyl} \rangle + f_0 \right)$$

f_0 ↳ vac. contr.

(norm. $2\pi T^{00} = -(T + \bar{T})$)

$$= \frac{\delta L}{L}(RL)\left[\frac{\langle T_{cyl}\rangle + \langle \bar{T}_{cyl}\rangle}{-(2\pi)} + f_0\right] \qquad \int d^2w \sqrt{g} \equiv RL$$

$$\delta f_L = \left(f_0 + \frac{\pi c}{6L^2}\right)\delta L$$

$$\boxed{f_L = f_0 L - \frac{\pi c}{6L}}$$

finite Temperature $\longrightarrow$ specific heat $\gamma = -\frac{\partial f}{\partial T} = \frac{\pi c}{6}$

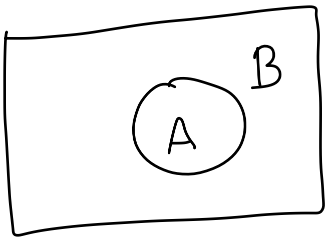
$$e^{-\frac{H}{T}} \longleftrightarrow e^{itH} \Rightarrow \boxed{\frac{1}{L} = T}$$

* Trace anomaly : $\langle T^{\mu}{}_{\mu} \rangle_g = \frac{c}{24\pi} R$

$g \in$ curved 2d manifold

Entanglement Entropy (Calabrese & Cardy)

$|\Psi\rangle$ quantum state. $\rho = |\Psi\rangle\langle\Psi|$ density matrix

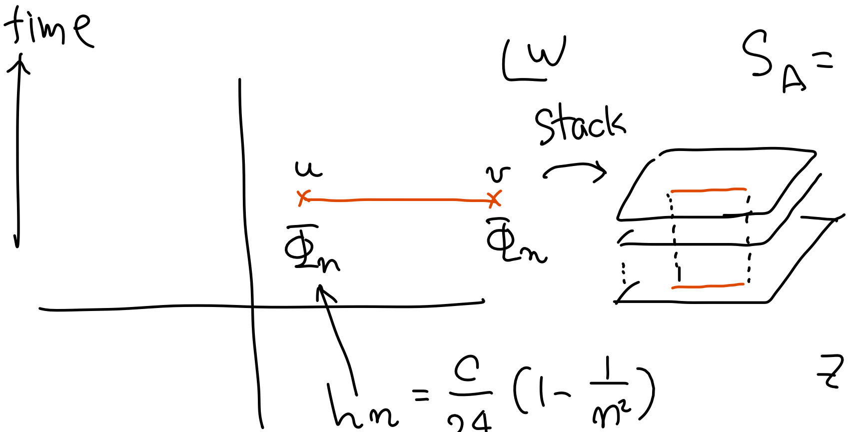


$\rho_A = \mathrm{Tr}_B\, \rho \longrightarrow$ entropy

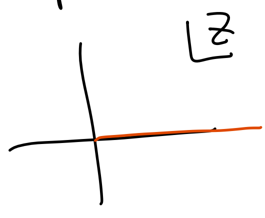
$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A$

2d CFT (1D spatial) $A = \overbrace{\qquad\qquad}^{l}$

$$S_A = -\lim_{n\to 1} \underbrace{\frac{\partial}{\partial n}}_{\text{replica}} \frac{\overbrace{Z_n(A)}^{\mathrm{Tr}\,\rho_A^n = \mathrm{Tr}\, e^{n \log \rho_A}}}{Z^n}$$



$$z = \left(\frac{w-u}{w-v}\right)^{\frac{1}{n}}$$



$$h_n = \frac{c}{24}\left(1 - \frac{1}{n^2}\right)$$

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12} S(z,w)$$

$$\langle T(w) \rangle_{\text{stack}} = \frac{c}{12} S(z,w) = \frac{c\left(1-\frac{1}{n^2}\right)}{24} \frac{(u-v)^2}{(w-u)^2 (w-v)^2} = \langle T(w)\, \Phi_n \Phi_n \rangle_{\mathbb{C}}$$

$$\Rightarrow h(\Phi_n) = \frac{c}{24}\left(1 - \frac{1}{n^2}\right)$$

normalization $\alpha = 1$

$$\frac{Z_n(A)}{Z^n} = \langle \Phi_n(u) \Phi_{-n}(v) \rangle^n = \left(\frac{\alpha}{\left(\frac{u-v}{a}\right)^{4h_n}} \right)^n$$

$$= e^{-n 4 h_n \log\left(\frac{u-v}{a}\right)}$$

$$\frac{\partial}{\partial n}(\,''\,) = -\frac{c}{6}\left(1 + \frac{1}{n^2}\right) \log\left(\frac{u-v}{a}\right) e^{-n 4 h_n \log\left(\frac{u-v}{a}\right)}$$

$n \to 1$, $u - v = l$ $\quad \boxed{S_A = \frac{c}{3} \log \frac{l}{a}}$.

$$\frac{c}{2} = \langle 0|[L_2, L_{-2}]|0\rangle = \langle 0|L_2 \underbrace{L_2^\dagger}_{L_{-2}}|0\rangle \geq 0$$

$\therefore$ $c \geq 0$ for positive Hilbertspace

3.5. HWS

define $|h\rangle = \phi(0)|0\rangle$

$$[L_n, \phi(w)] = \oint \frac{dz}{(2\pi i)} z^{n+1} \underbrace{T(z)\phi(w)}$$

$$= h(n+1) w^n \phi(w) + w^{n+1}\partial_w\phi \quad \Big\} \; \frac{h}{(z-w)^2}\phi(w) + \frac{\partial_w\phi}{z-w}$$

as $\underbrace{w \to 0}$ with $n > 0$

$[L_n, \phi(0)] = 0 \to L_n|h\rangle = 0 \quad n \geq 1$

$n=0 \quad [L_0, \phi(0)] = h\,\phi(0) \Rightarrow L_0|h\rangle = h|h\rangle$

$n=-1 \quad [L_{-1}, \phi(0)] = \partial_w\phi(0) \Rightarrow L_{-1}|h\rangle = \partial\phi(0)|h\rangle$

$$\boxed{L_n|h\rangle = 0 \quad (n \geq 1), \quad L_0|h\rangle = h|h\rangle} \quad \text{hws}$$

$|h\rangle \to L_{-n_1} \cdots L_{-n_k}|h\rangle, \; (n_i \geq 1)$

"descendants"

$\Big(\langle h|L_0 = h\langle h|, \quad \langle h|L_n = 0 \quad n \leq -1\Big)$

$\langle h|L_{n_1} \cdots L_{n_k} \quad (n_i \geq 1)$ desc.

$$\Rightarrow \langle h|L_{-n}^\dagger L_{-n}|h\rangle = \langle h|L_n L_{-n}|h\rangle$$

$$= \langle h|[L_n, L_{-n}] + \cancel{L_{-n}L_n}|h\rangle = 2n\langle h|L_0|h\rangle + \frac{c}{12}(n^3-n)\langle h|h\rangle$$

$$2nh + \frac{c}{12}(n^3 - n) \geqq 0 \quad \text{for any } n$$

$$\Rightarrow \quad c, h \geqq 0 \quad \Rightarrow \quad \underset{(n=1)}{h=0} \rightarrow L_{-1}|h\rangle = 0$$

$c=0$ $\quad \therefore |0\rangle$ is only vac.

$\underline{L_{-n}|0\rangle}$: zero norm $\cdot \equiv 0$

$$\phi(z) = \sum_{n \in \mathbb{Z}-h} \phi_n z^{-n-h} \rightarrow \phi_n = \oint \frac{dz}{2\pi i} z^{h+n-1} \phi(z)$$

$$\phi(z)|0\rangle \xrightarrow[z=0]{} \phi_n|0\rangle = 0 \quad n \geqq 1-h$$

$$\phi_{-h}|0\rangle = |h\rangle$$

$$[L_n, \phi_m] = \oint \frac{dw}{2\pi i} \underbrace{w^{h+m-1}(h(n+1)w^n \phi + w^{n+1}\partial\phi)}_{w^{h+m+n-1}(h(n+1) - (h+m+n))\phi}$$

$$= (n(h-1) - m)\phi_{m+n}$$

$$[L_0, \phi_m] = -m\phi_m \rightarrow L_0|h\rangle = L_0 \phi_{-h}|0\rangle$$
$$= h|h\rangle$$

$SL(2, C)$

$$\langle 0|\phi_1 \cdots \phi_n|0\rangle = \langle 0|\underbrace{U^{-1}\phi_1 U}_{\phi_1 + \epsilon[L_k, \phi_1] \quad k=0,\pm1} \cdots U^{-1}\phi_n U|0\rangle$$

$$\Rightarrow \underset{k=\pm1,0}{0} = \langle 0|[L_k, \phi_1] \cdots\rangle + \cdots + \langle 0|\phi_1 \cdots [L_k, \phi_n]|0\rangle$$

$n=-1 \quad \sum_i \partial_i \langle 0|\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle = 0$

$n=0 \quad \sum_i (h_i + z_i \partial_{z_i}) \langle \quad " \quad \rangle = 0$

$n=1 \quad \sum_i (2h_i z_i + z_i^2 \partial_{z_i}) \langle \quad " \quad \rangle = 0$

also applies to $\phi_i = T$ "quasi-primary"

3.6. Descendant

$[\phi_n] \; : \; {}^{\text{``}}\hat{L}_{-n}{}^{\text{''}*}\phi(w) \equiv \oint \frac{dz}{2\pi i} \frac{1}{(z-w)^{n-1}} T(z)\phi(w)$

$\quad \phi^{(-n)} : (h+n, \bar{h})$

$L_{-2}\mathbb{1}(w) = \oint \frac{T(z)}{(z-w)} = T(w) = \mathbb{1}^{(-2)}$

T: descendant
quasi-primary

$n>0 : \quad L_n\phi = 0$

$[\phi_n]$ n	Δ	field
0	h	ϕ
1	$h+1$	$L_{-1}\phi$
2	$h+2$	$L_{-2}\phi, L_{-1}^2\phi$
$\vdots$	$\vdots$	
N	$h+N$	$L_{-N}^{n_N}\cdots L_{-1}^{n_1}\phi$

$\sum_{j=1}^{N} n_j j = N \quad n_j \geq 0$

* $\hat{L}_n : \phi(z) \to [\phi](z)$ field

$L_n : L_n|h\rangle$: state

$\mathcal{L}_{-n}$: differential op. for correl. funcs

Consider $\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \prod_{n=1}^{\infty} \sum_{k_n=0}^{\infty} (q^n)^{k_n} = \sum_{N=0}^{\infty} P(N) q^N$

$\equiv q^{\frac{-1}{24}}\eta(q)$ ← Dedekind η ; $P(N)$: # of partitions of N