infinitessimal trasf for T

$$\delta_\epsilon T = \epsilon \partial T + \underset{\substack{\uparrow \\ h}}{2}\, \partial\epsilon\, T + \underbrace{\frac{c}{12}\partial^3\epsilon} \xrightarrow{\text{finite}} \boxed{(\partial f)^2 T(f) + \frac{c}{12} S}$$

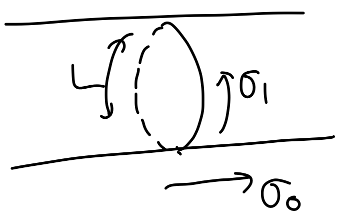
$f = z + \epsilon(z)$

$$S(f, z) = \frac{\partial f\, \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2} = 0 \quad \text{for } SL(2,\mathbb{R})$$

$$(\epsilon = 1,\ z,\ z^2 \longrightarrow \partial^3\epsilon = 0)$$

Relation to cylinder

$z = e^{\frac{2\pi}{L} w}$



let $\sigma_1 \to \sigma_1 + L$ $\quad w = \frac{L}{2\pi} \ln z = \sigma_0 + i\sigma_1$

$$T_{cyl}(w) = \underset{\left(\frac{2\pi}{L}\right)^2 z^2}{\left(\frac{\partial z}{\partial w}\right)^2} T_{pl}(z) + \frac{c}{12} \underbrace{S(z, w)}_{\left(\frac{2\pi}{L}\right)^2\left(1 - \frac{3}{2}\right)}$$

$$\boxed{T_{cyl}(w) = \left(\frac{2\pi}{L}\right)^2 \left(z^2 T_{pl}(z) - \frac{c}{24}\right)}$$

$$\left(\frac{2\pi}{L}\right)^2 \oint \frac{dz}{z} \left(z^2 T_{pl}(z) - \frac{c}{24}\right) = \left(L_0 - \frac{c}{24}\right)\left(\frac{2\pi}{L}\right)^2$$

$$= \oint \underbrace{\frac{dz}{z}}\ T_{cyl}(w) = \frac{2\pi}{L} \underbrace{\oint dw\, T_{cyl}(w)}$$

$$dw = \frac{L}{2\pi} \frac{dz}{z} \qquad \left(\underset{\substack{\sigma_0 \to \sigma_0 + \varepsilon \\ \varepsilon = \text{const}}}{\equiv} L_0^{cyl}\right)$$

$$\boxed{L_0^{cyl} = \frac{2\pi}{L}\left(L_0 - \frac{c}{24}\right),\ \bar{L}_0^{cyl} = \frac{2\pi}{L}\left(\bar{L}_0 - \frac{c}{24}\right)}$$

$$H^{cyl.} = L_0^{cyl} + \bar{L}_0^{cyl} = \frac{2\pi}{L}\left(\underbrace{L_0 + \bar{L}_0}_{\text{dilation in plane.}} - \frac{c}{24}\right)$$

$\sigma_0 \to \sigma_0 + \varepsilon$

"time translation"

Finite-size effect (Casimir effect)

$$T_{cyl}(w) = \left(\frac{2\pi}{L}\right)^2 \left(z^2 T_{pl}(z) - \frac{c}{24} \right)$$

SL(2,C) vac

$$\therefore \langle T_{cyl}(w) \rangle = -\frac{c\pi^2}{6L^2} \neq 0 \text{ "Casimir" Energy}$$

Free energy (of cylinder) : $Z = \int [D\varphi]\, e^{-S[\varphi]}$ $\left(S = \int \sqrt{g}\, \mathcal{L}\, d^2w \right) \rightarrow F = -\ln Z$

variation : $$\delta F = -\frac{1}{Z}\int [D\varphi] \left[\int \frac{\sqrt{g}}{2}\, \delta g_{\mu\nu} T^{\mu\nu}_{cyl}\, d^2w \right] e^{-S}$$

$$\equiv -\frac{1}{2}\int d^2w \sqrt{g}\, \delta g_{\mu\nu} \langle T^{\mu\nu}_{cyl} \rangle$$

time dilatation in cylinder : $\sigma^0 \rightarrow \lambda\sigma^0 = (1+\varepsilon)\sigma^0$

$$\delta g_{\mu\nu} = -2\varepsilon\, \delta_{\mu 0}\delta_{\nu 0} \qquad L \longrightarrow L + \underbrace{\varepsilon L}_{\delta L} \qquad \varepsilon = \frac{\delta L}{L}$$

$$\delta F_L = \frac{\delta L}{L} \int d^2w \sqrt{g} \left(\langle T^{00}_{cyl} \rangle + f_0 \right)$$

$\hookrightarrow$ vac. contr.

$$\left(\text{norm. } 2\pi T^{00} = -(T + \bar{T}) \right)$$

$$= \frac{\delta L}{L}(RL)\left[\frac{\langle T_{cyl} \rangle + \langle \bar{T}_{cyl} \rangle}{-(2\pi)} + f_0 \right] \qquad \int d^2w \sqrt{g} \equiv RL$$

$$\delta f_L = \left(f_0 + \frac{\pi c}{6L^2} \right) \delta L$$

$$\boxed{f_L = f_0 L - \frac{\pi c}{6L}}$$

finite Temperature $\longrightarrow$ specific heat $\gamma = -\frac{\partial f}{\partial T} = \frac{\pi c}{6}$

$$e^{-\frac{H}{T}} \longleftrightarrow e^{itH} \Rightarrow \boxed{\frac{1}{L} = T}$$

* Trace anomaly : $\langle T^{\mu}{}_{\mu} \rangle_g = \frac{c}{24\pi} R$

$g \in$ curved 2d manifold

$$\frac{c}{2} = \langle 0 | [L_2, L_{-2}] | 0 \rangle = \langle 0 | L_2 \underbrace{L_2^\dagger}_{L_{-2}} | 0 \rangle \geq 0$$

$\therefore$ $c \geq 0$ for positive Hilbertspace

3.5. HWS

define $|h\rangle = \phi(0)|0\rangle$

$$[L_n, \phi(w)] = \oint \frac{dz}{(2\pi i)} z^{n+1} \underbrace{T(z)\phi(w)}$$

$$= h(n+1) w^n \phi(w) + w^{n+1} \partial_w \phi \quad \Big\} \quad \frac{h}{(z-w)^2}\phi(w) + \frac{\partial_w \phi}{z-w}$$

as $\underbrace{w \to 0}$ with $n > 0$

$[L_n, \phi(0)] = 0 \to L_n|h\rangle = 0 \quad n \geq 1$

$n=0 \quad [L_0, \phi(0)] = h\,\phi(0) \Rightarrow L_0|h\rangle = h|h\rangle$

$n=-1 \quad [L_{-1}, \phi(0)] = \partial_w \phi(0) \Rightarrow L_{-1}|h\rangle = \partial\phi(0)|h\rangle$

$$\boxed{L_n|h\rangle = 0 \quad (n \geq 1), \quad L_0|h\rangle = h|h\rangle} \quad \text{hws}$$

$|h\rangle \to L_{-n_1} \cdots L_{-n_k}|h\rangle, \ (n_i \geq 1)$

"descendants"

$\Big(\langle h| L_0 = h\langle h|, \quad \langle h| L_n = 0 \quad n \leq -1\Big)$

$\langle h| L_{n_1} \cdots L_{n_k} \quad (n_i \geq 1)$ desc.

$$\Rightarrow \langle h| L_{-n}^\dagger L_{-n}|h\rangle = \langle h| L_n L_{-n}|h\rangle$$

$$= \langle h|[L_n, L_{-n}] + \cancel{L_{-n}L_n}|h\rangle = 2n\langle h|L_0|h\rangle + \frac{c}{12}(n^3-n)\langle h|h\rangle$$

$$2nh + \frac{c}{12}(n^3 - n) \geqq 0 \quad \text{for any } n$$

$$\Rightarrow \quad c, h \geqq 0 \quad \Rightarrow \quad \underset{(n=1)}{h=0} \to L_{-1}|h\rangle = 0$$

$\therefore$ $|0\rangle$ is only vac.

$c=0$

$L_{-n}|0\rangle$: zero norm $\equiv 0$

$$\phi(z) = \sum_{n \in \mathbb{Z}-h} \phi_n z^{-n-h} \to \phi_n = \oint \frac{dz}{2\pi i} z^{h+n-1} \phi(z)$$

$$\phi(z)|0\rangle \xrightarrow[z=0]{} \phi_n|0\rangle = 0 \quad n \geqq 1-h$$

$$\phi_{-h}|0\rangle = |h\rangle$$

$$[L_n, \phi_m] = \oint \frac{dw}{2\pi i} \underbrace{w^{h+m-1}(h(n+1)w^n \phi + w^{n+1}\partial\phi)}_{w^{h+m+n-1}(h(n+1) - (h+m+n))\phi}$$

$$= (n(h-1) - m)\phi_{m+n}$$

$$[L_0, \phi_m] = -m\phi_m \to L_0|h\rangle = L_0 \phi_{-h}|0\rangle$$
$$= h|h\rangle$$

$SL(2, C)$

$$\langle 0|\phi_1 \cdots \phi_n|0\rangle = \langle 0|\underbrace{U^{-1}\phi_1 U}_{\phi_1 + \epsilon[L_k, \phi_1]} \cdots U^{-1}\phi_n U|0\rangle \quad k = 0, \pm 1,$$

$$\Rightarrow 0 = \langle 0|[L_k, \phi_1] \cdots\rangle + \cdots + \langle 0|\phi_1 \cdots [L_k, \phi_n]|0\rangle$$
$$k = \pm 1, 0$$

$n=-1 \quad \sum_i \partial_i \langle 0 | \phi_1(z_1) \cdots \phi_n(z_n) | 0 \rangle = 0$

$n=0 \quad \sum_i (h_i + z_i \partial_{z_i}) \langle \quad " \quad \rangle = 0$

$n=1 \quad \sum_i (2h_i z_i + z_i^2 \partial_{z_i}) \langle \quad " \quad \rangle = 0$

also applies to $\phi_i = T$ "quasi-primary"

3.6. Descendant

$[\phi_n]$: "$\hat{L}_{-n}$"* $\phi(w) \equiv \oint \frac{dz}{2\pi i} \frac{1}{(z-w)^{n-1}} T(z) \phi(w)$

$ \shortparallel$

$\phi^{(-n)} : (h+n, \bar{h})$

$$L_{-2}\mathbb{1}(w) = \oint \frac{T(z)}{(z-w)} = T(w) = \mathbb{1}^{(-2)}$$

T: descendant
quasi-primary

$n>0$: $L_n \phi = 0$

$[\phi_n]$ n	Δ	field
0	h	ϕ
1	$h+1$	$L_{-1}\phi$
2	$h+2$	$L_{-2}\phi, L_{-1}^2\phi$
$\vdots$	$\vdots$	
N	$h+N$	$L_{-N}^{n_N} \cdots L_{-1}^{n_1}\phi$

$\sum_{j=1}^{N} n_j j = N \quad n_j \geq 0$

$$\boxed{\begin{array}{l} * \begin{cases} \hat{L}_n : \phi(z) \to [\phi](z) \ \text{field} \\ L_n : L_n|h\rangle : \text{state} \\ \mathcal{L}_{-n} : \text{differential op. for correl. fncs} \end{cases} \end{array}}$$

Consider $\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \prod_{n=1}^{\infty} \sum_{k_n=0}^{\infty} (q^n)^{k_n} = \sum_{N=0}^{\infty} P(N) q^N$

$\equiv q^{\frac{-1}{24}} \eta(q) \leftarrow$ Dedekind η

$P(N)$: # of partitions of N

4. Kac determinant and unitarity

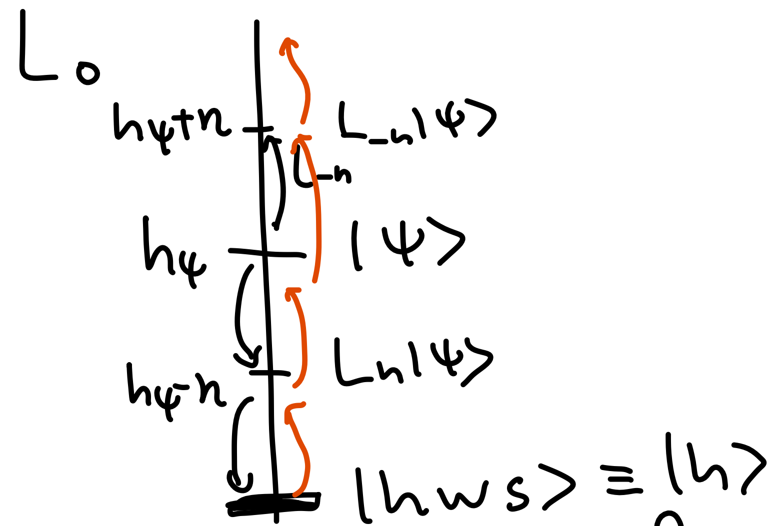
$|h\rangle = \phi(0)|0\rangle$ satisfy $L_0|h\rangle = h|h\rangle$

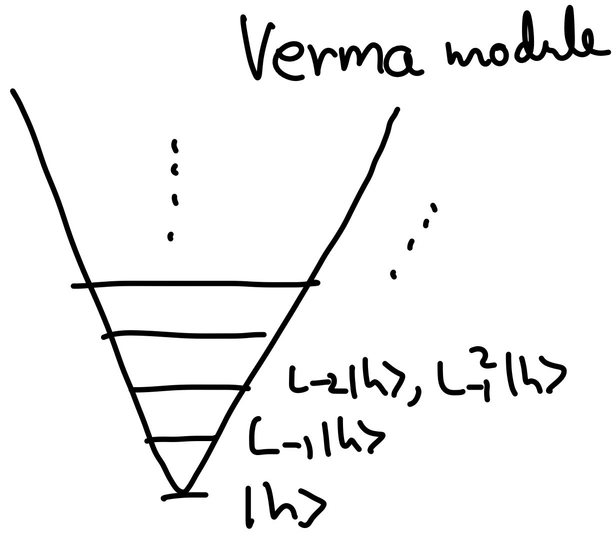
$L_n|0\rangle = 0$ for $n \geq -1$

we want to find "scaling operators" which is eigenstate of L_0.

$L_0|\psi\rangle = h_\psi|\psi\rangle$

$L_0 L_n|\psi\rangle = (h_\psi - n)L_n|\psi\rangle$, $[L_0, L_n] = -nL_n$





$E = L_0 + \bar{L}_0$: bounded from below →

some state (h.w.s) should be stopped: $L_n|h\rangle = 0$ for all $n>0$

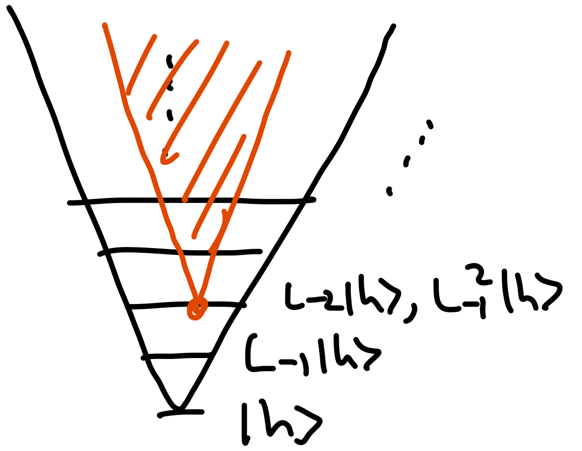
all states in V.M. are NOT linearly indep.

⇒ ∃ some $|\psi\rangle = a_i \mathcal{L}_i |h\rangle \equiv 0$ "null states"

- level 1 ($n=1$) $L_{-1}|h\rangle = 0 \to h=0$ only
- level 2 ($n=2$) $|\psi\rangle \equiv L_{-2}|h\rangle + aL_{-1}^2|h\rangle = 0$

i.e. $L_m|\psi\rangle = 0$ for all m

Verma module



$$L_{-1}\overbrace{[L_1,L_{-1}] + [L_1,L_{-1}]L_{-1}}^{2L_0}$$

$$[L_1,L_{-2}]|h\rangle + a[L_1,L_{-1}^2]|h\rangle = 3L_{-1}|h\rangle + 2a(L_{-1}L_0+L_0L_{-1})|h\rangle$$

$$= [3+2a(h+(h+1))]L_{-1}|h\rangle = 0$$

$$a = -\frac{3}{2(2h+1)}$$

$$\left(L_{-1}\overbrace{[L_2,L_{-1}]}^{\cancel{3L_1}} + 3\overbrace{L_1L_{-1}}^{2L_0}\right)|h\rangle$$

$$[L_2,L_{-2}]|h\rangle + a[L_2,L_{-1}^2]|h\rangle = \left(4h+\frac{c}{2}+6ah\right)|h\rangle$$

$$= 0 \longrightarrow c = -4h(2+3a) = 2h\frac{(5-8h)}{2h+1}$$

$$\therefore \left(\hat{L}_{-2} - \underbrace{\frac{3}{2(2h+1)}}_{a}\hat{L}_{-1}^2\right)\phi = 0 \quad \text{for } c.$$

$[\phi]=h$

is valid for any correlation fnch containing ϕ

or $\hat{L}_{-2}\phi = -a\hat{L}_{-1}^2\phi = -a\partial_z^2\phi$

from $T(z)\phi(w) \equiv \sum_{n\geq 0}(z-w)^{n-2}\hat{L}_{-n}\phi(w)$

$$= \frac{1}{(z-w)^2}\underbrace{\hat{L}_0\phi}_{h\phi} + \frac{1}{z-w}\partial_w\phi + \hat{L}_{-2}\phi + (z-w)\hat{L}_{-3}\phi + \cdots$$

$$\therefore \hat{L}_{-2}\phi = -\frac{h}{(z-w)^2}\phi - \frac{1}{z-w}\partial_w\phi + T(z)\phi(w)$$

$+$ reg.

$$\frac{3}{2(2h_1+1)}\frac{\partial^2}{\partial w_1^2}\langle \phi_1(w_1)\cdots\phi_n(w_n)\rangle$$

$$=\left\langle \left[\frac{h_1}{(z-w_1)^2}\phi_1 - \frac{1}{z-w_1}\partial_{w_1}\phi + \underbrace{T(z)\phi_1(w_1)\Big]\phi_2(w_2)\cdots\phi_n(w_n)}_{\frac{h_i\phi_i}{(z-w_i)^2}+\frac{\partial_{w_i}\phi_i}{z-w_i}}\right\rangle$$

as $z\to w_1$

$$=\sum_{j\neq 1}\left(\frac{h_j}{(w_1-w_j)^2}+\frac{1}{w_1-w_j}\frac{\partial}{\partial w_j}\right)\langle\phi_1(w_1)\cdots\phi_n(w_n)\rangle$$

2nd order DE for any n-pt fct.

$n=4$-pt fct: conf. blocks in terms of hypergeometric functions

level 3. (H.W.): $\overline{(L_{-3}+aL_{-1}L_{-2}+bL_{-1}^3)|h\rangle} \equiv |\psi_3\rangle$

impose condit. $L_n|\psi_3\rangle=0$ $n\geq 1$.

$$\left(L_{-3}-\frac{2}{h+1}L_{-1}L_{-2}+\frac{1}{(h+1)(h+2)}L_{-1}^3\right)|h\rangle = |\psi_3\rangle$$

$$\hat{L}_{-n}\phi = \oint\frac{dz}{2\pi i}(z-w)^{-n+1}T(z)\phi(w)$$

$$\langle \hat{L}_{-n}\phi_1\,\phi_2\cdots\rangle$$

$$=\underbrace{\sum_{j\neq 1}\left(\frac{(n-1)h_j}{(w_j-w_1)^n}-\frac{1}{(w_j-w_1)^{n-1}}\frac{\partial}{\partial w_j}\right)}_{\mathcal{L}_{-n}}\langle\phi_1(w_1)\cdots\phi_n(w_n)\rangle$$

$$\left[\frac{1}{h_1+1}\frac{\partial^3}{\partial w_1^3}-\sum_{j\neq 1}\frac{2h_1h_j}{(w_1-w_j)^3}+\frac{h_1}{(w_1-w_j)^2}\frac{\partial}{\partial w_j}+\frac{2h_j}{(w_1-w_j)^2}\frac{\partial}{\partial w_1}+\frac{2}{w_1-w_j}\frac{\partial^2}{\partial w_1\partial w_j}\right]$$

$$\langle\phi_1(w_1)\cdots\phi_n(w_n)\rangle = 0$$

4.2. Kac determinant

matrix A $(n \times n)$; $A|n\rangle = a_n|n\rangle$

if $\det A = 0$ some of $a_n = 0$ & $\{|n\rangle\}$ is not

$$A = \left(\begin{array}{c|ccc} 0 & & & \\ \hline & a_1 & & \\ & & \ddots & \\ & & & a_{n-1} \end{array}\right) \xrightarrow{\text{truncate this space}} \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{n-1} \end{pmatrix} \text{ all indep}$$

$n=2$
$$\begin{pmatrix} \langle h|L_2L_{-2}|h\rangle & \langle h|L_1^2L_{-2}|h\rangle \\ \langle h|L_2L_{-1}^2|h\rangle & \langle h|L_1^2L_{-1}^2|h\rangle \end{pmatrix} = \begin{pmatrix} 4h+\frac{c}{2} & 6h \\ 6h & 4h(1+2h) \end{pmatrix}$$

$$\det(\ ) = 2(16h^3 - 10h^2 + 2h^2c + hc) = 32(h-h_{1,1})(h-h_{1,2})(h-h_{2,1})$$

where $h_{1,1}=0$, $h_{1,2}, h_{2,1} = \frac{1}{16}(5-c) \mp \frac{1}{16}\sqrt{(1-c)(25-c)}$

$\downarrow$

null state at level 1 ; $L_{-1}|0\rangle = 0 \to L_{-1}(L_{-1}|0\rangle) = 0$

__Kac__ computed $\det M_N(c,h) = \alpha_N \prod\limits_{\substack{pq \le N \\ N \ge p,q \ge 1}} (h - h_{p,q}(c))^{P(N-pq)}$

where $\alpha_N = \text{const}$, indep of c, h

$$h_{p,q}(c) = \frac{1-c}{96}\left[\left((p+q)+(p-q)\sqrt{\frac{25-c}{1-c}}\right)^2 - 4\right]$$

$$= \frac{((m+1)p - mq)^2 - 1}{4m(m+1)} \quad \text{with} \quad m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}$$

$$\text{or } c = 1 - \frac{6}{m(m+1)}$$

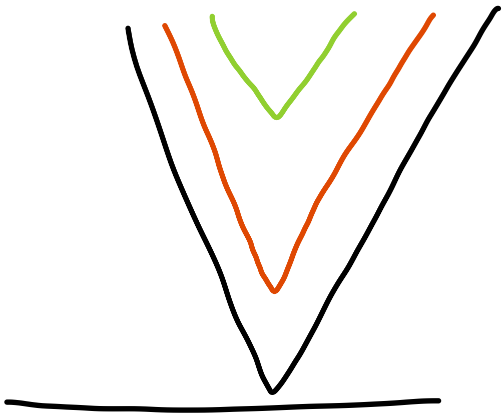
$$= \frac{1}{4}\left[(\underbrace{p\alpha_+ + q\alpha_-}_{\alpha})^2 - (\alpha_+ + \alpha_-)^2\right] \qquad \alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$$

descendants of null states at level n

$|h+n\rangle = 0 \longrightarrow$ at level N, $P(N-n)$ null states

which are $L_{-n_1} \cdots L_{-n_k} |h+n\rangle = 0$

$$\left(\sum_{j=1}^{k} n_j = N-n \right)$$



at level N; $\sum_{\{n_i\}} a_{n_1 \cdots n_k} L_{-n_1} \cdots L_{-n_k} |h\rangle$

$\Downarrow$

$\sum_i n_i = N \rightarrow P(N)$ $\qquad$ $P(N) \times P(N)$ matrix $M_N^{(c,h)}$

$$\det M_N \begin{cases} > 0 & \longrightarrow \text{OK} \\ = 0 & \longrightarrow \text{zero norm, OK (Reducible)} \\ < 0 & \longrightarrow \text{odd \# of neg. norm} \rightarrow \text{NOT OK} \end{cases}$$

[Consequencies]

$25 > c > 1,\ h \geq 0$; m is not real $\to h_{p,q} < 0$ $(p=q)$ or imaginary

$c \geq 25$; $-1 < m < 0 \to h_{p,q} < 0$

$$\mathrm{Det} = \prod_{p,q}^{p} (h - h_{p,q}) \neq 0 \quad \text{for } h > 0$$

$$= \prod_i M_i \quad \Rightarrow \quad \underbrace{M_i > 0}$$

$\therefore\ c > 1,\ h \geq 0$; no null states.

$c = 1$, $h_{p,q} = \frac{(p-q)^2}{4}$

$$M = \prod_{p,q} \left(h - \frac{(p-q)^2}{4}\right)\left(h - \frac{(q-p)^2}{4}\right) = \prod_{p,q} \left(h - \frac{(p-q)^2}{4}\right)^2 \geq 0$$

(if $h \neq \frac{n^2}{4} \Rightarrow M > 0 \to$ no null state)

$\therefore\ c \geq 1,\ h \geq 0 \Rightarrow$ unitary rep of Virasoro Alg.

$0 < c < 1,\ h > 0$

$$h_{p,q}(c) = \frac{1-c}{96}\left[\left((p+q) \pm (p-q)\sqrt{\frac{25-c}{1-c}}\right)^2 - 4\right] \qquad (p \leftrightarrow q)$$

if $c = 1 - 6\epsilon$ $\quad \left(\epsilon \equiv \frac{1}{m(m+1)} \ll 1,\ m \gg 1\right)$

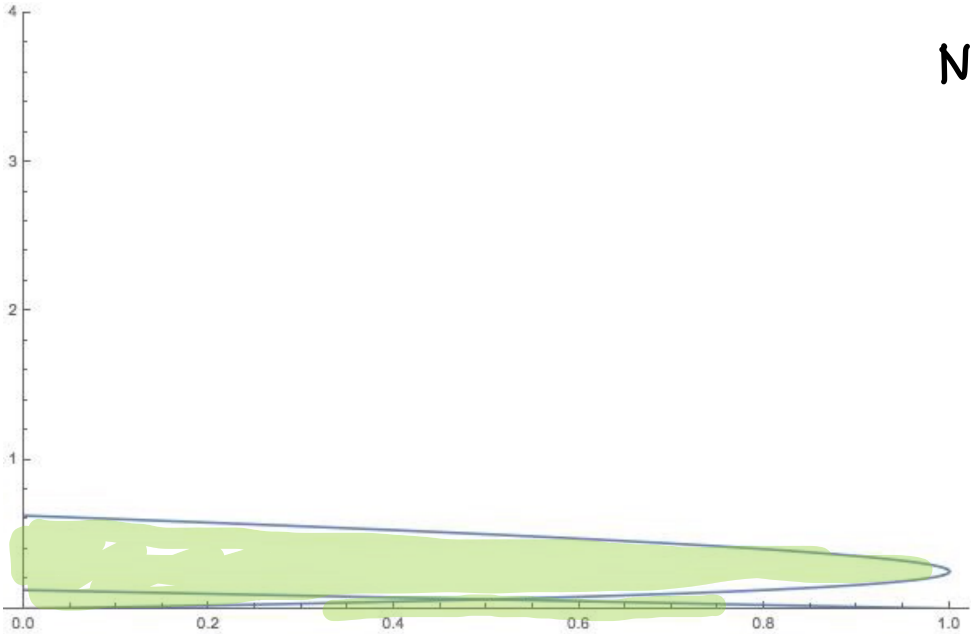
$$h_{p,q} = \frac{1}{96}\left[\left((p+q)\sqrt{6\epsilon} \pm (p-q)\sqrt{24+6\epsilon}\right)^2 - 24\epsilon\right]$$

$$= \frac{1}{4}\left[\left(\frac{(p+q)^2}{4} - 1\right)\epsilon + (p-q)^2\left(1 + \frac{\epsilon}{4}\right) + (p^2 - q^2)\sqrt{\epsilon}\right]$$

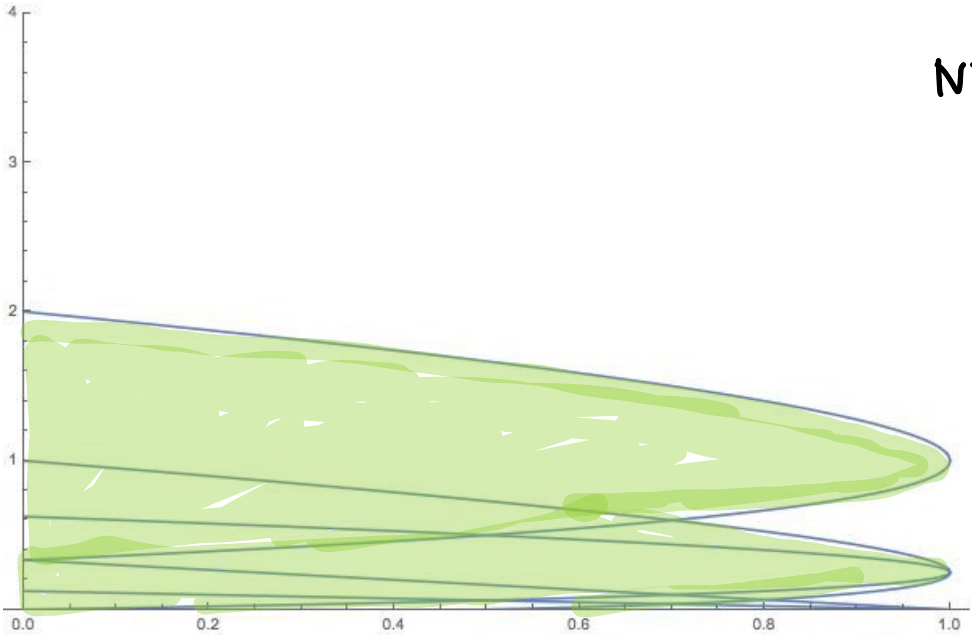
$$\approx \frac{1}{4}(p-q)^2 + \frac{1}{4}(p^2 - q^2)\sqrt{\epsilon} \qquad (p \neq q)$$

$$h_{p,p} = \frac{1}{4}(p^2 - 1)\,\epsilon$$

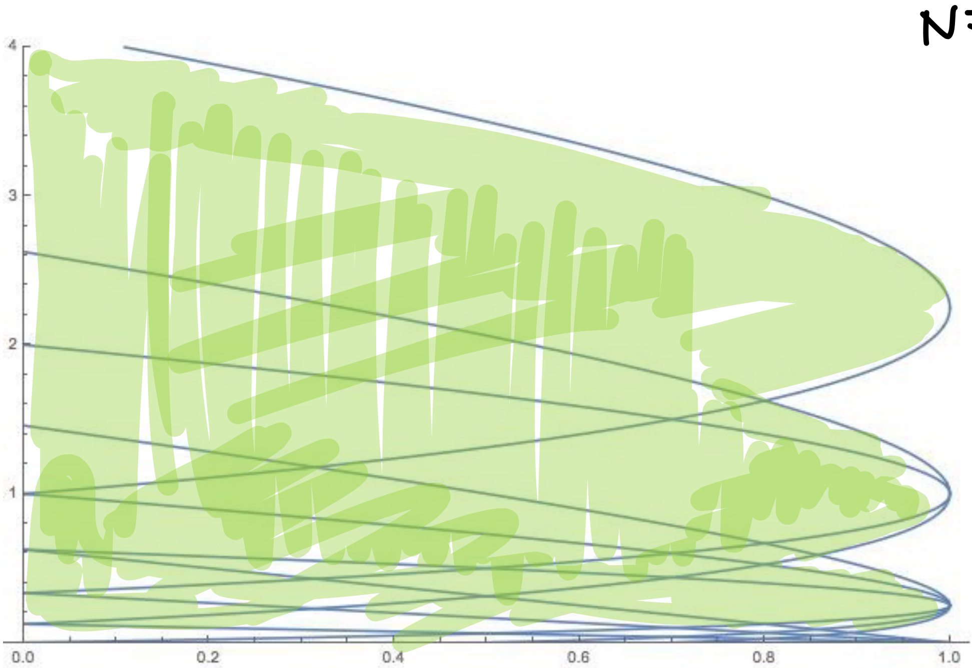
N=2

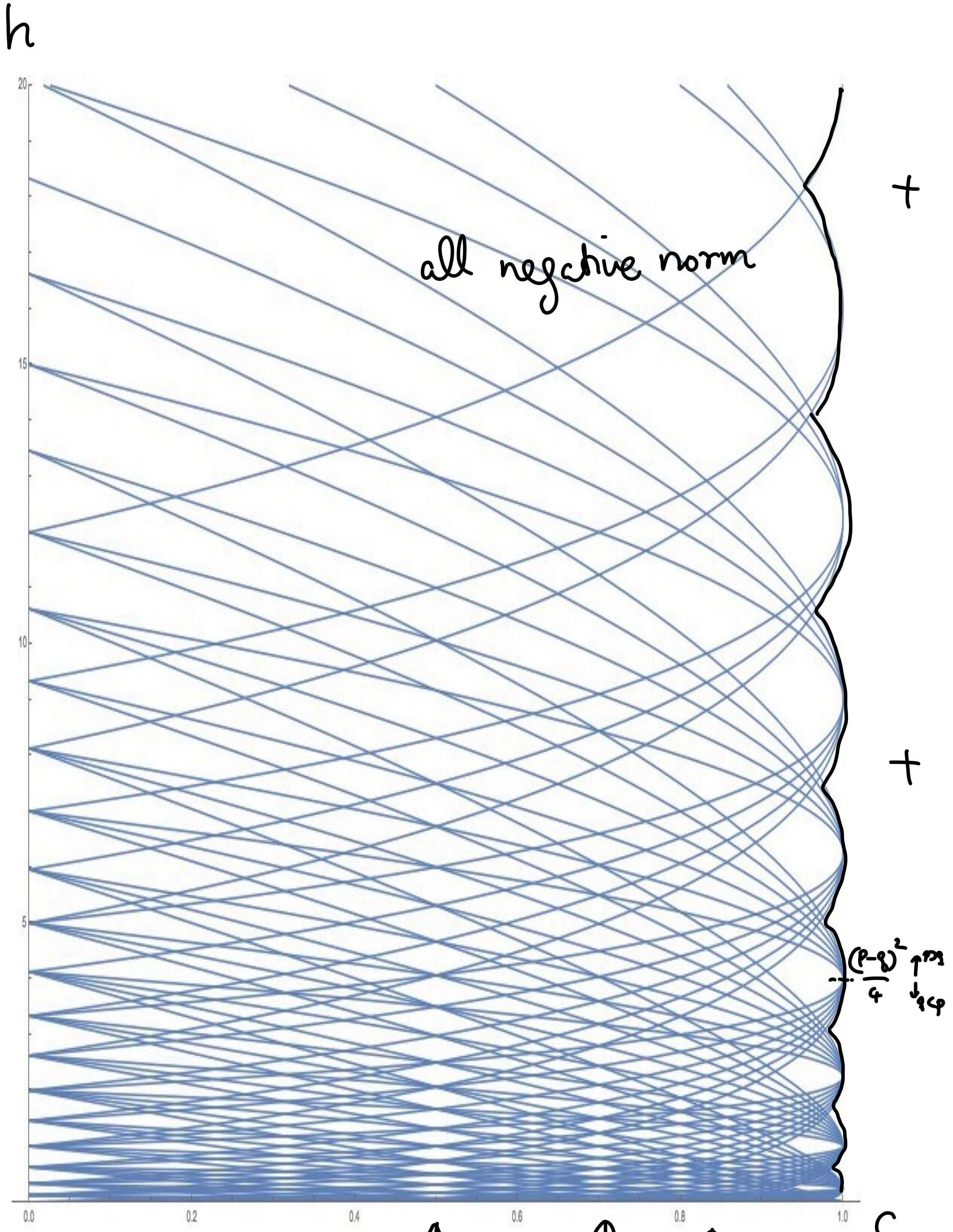


N=3



N=4





cross points are OK.
("reducible")

$\frac{1}{2}$ $\quad \frac{7}{10}$ $\quad \frac{4}{5}$ ...

at $c = 1 - \frac{6}{m(m+1)}$

with m = integer!

$$h_{rs}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$$

(show later)

$m \in \mathbb{Z} \Rightarrow 1 \le r \le m-1,\ 1 \le s \le r$

bounded

i.e. finite # of h's

"rational CFT"

called Minimal model

if we allow $c, h < 0$ "non-unitary"

$\Rightarrow m \in \frac{\mathbb{N}}{\mathbb{Z}}$ (rational)

$$C = 1 - \frac{6}{m(m+1)} \qquad m = \frac{p'}{p-p'} \qquad m+1 = \frac{p}{p-p'} \quad (p > p') \text{ coprime}$$

$$= 1 - \frac{6(p-p')^2}{pp'}, \quad h_{rs} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'}$$

$$\begin{pmatrix} 1 \le r \le p'-1 \\ 1 \le s \le p-1 \end{pmatrix}$$

$$\boxed{r \to m-r,\ s \to m+1-s}$$

$$h_{m-r,\, m+1-s} = h_{rs}$$

one can extend: $1 \le s \le m$

$1 \le r \le m-1$

(ex) $m = 3$

s \ r	1	2
3	$\frac{1}{2}$	0
2	$\frac{1}{16}$	$\frac{1}{16}$
1	0	$\frac{1}{2}$

Overview of MM

$|\chi\rangle$: null state at level 2 ; $h = h_{1,2}$ or $h_{2,1}$

$$= \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right)$$

$\uparrow$

$|\phi\rangle_h \qquad \chi = \hat{L}_{-2}\phi - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\phi \equiv 0$

$$\left\{\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2\right\}\langle\phi(z)X\rangle = 0 \qquad \phi_1(z_1)\cdots\phi_N(z_N)$$

$$\left[\sum_{i=1}^{N}\left[\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{h_i}{(z-z_i)^2}\right] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right]\langle\phi(z)X\rangle = 0$$

① $X = \phi(w) \quad \left[\frac{1}{z-w}\frac{\partial}{\partial w} + \frac{h}{(z-w)^2} - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right]\langle\phi(z)\phi(w)\rangle = 0$

$\langle\phi(z)\phi(w)\rangle = (z-w)^{-2h}$

② $X = \phi_1(z_1)\phi_2(z_2) \rightarrow \langle\phi\phi_1\phi_2\rangle = \frac{C_{hh_1h_2}}{(z-z_1)^{h_2-h-h_1}\cdots}$

$\Rightarrow$ a relation between h_1, h_2, h

$$2(2h+1)(h+2h_2-h_1) = 3(h-h_1+h_2)(h-h_1+h_2+1)$$

$$h_2 = \frac{1}{6} + \frac{h}{3} + h_1 \pm \frac{2}{3}\sqrt{h^2+3hh_1 - \frac{h}{2} + \frac{3h_1}{2} + \frac{1}{16}} \qquad h(\alpha) = h_0 + \frac{1}{4}\alpha^2$$

$\Rightarrow \boxed{\alpha_2 = \alpha_1 \pm \alpha_+}$

with $h = h_{12} \rightarrow \boxed{\alpha_2 = \alpha_1 \pm \alpha_-}$

$\phi_{21}\times\phi_{rs} = \phi_{r-1\,s} + \phi_{r+1,s}$

$\phi_{12}\times\phi_{rs} = \phi_{rs-1} + \phi_{rs+1}$

fusion rule : $\phi_{21}\times\phi_\alpha = \phi_{\alpha-\alpha_+} + \phi_{\alpha+\alpha_+}$

$\phi_{12}\times\phi_\alpha = \phi_{\alpha-\alpha_-} + \phi_{\alpha+\alpha_-}$

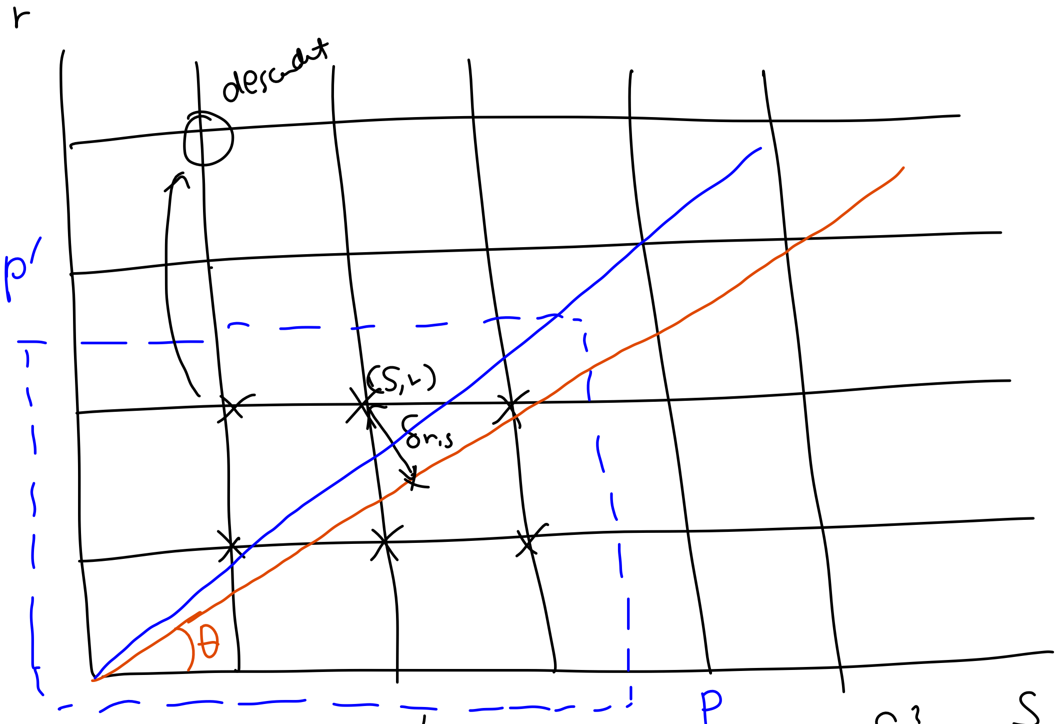
associativity $\qquad \phi_{12}\phi_{21} = \phi_{22}$

$$\phi_{22}\times\phi_{rs} = \phi_{12}\phi_{r-1s} + \phi_{12}\phi_{r+1\,s} = \phi_{r-1\,s-1} + \phi_{r-1\,s+1} + \phi_{r+1,s-1} + \phi_{r+1,s+1}$$

etc

$$\Rightarrow \phi_{(r_1s_1)}\times\phi_{(r_2s_2)} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=\text{odd}}}^{r_1+r_2-1}\ \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2=\text{odd}}}^{s_1+s_2-1}\phi_{(k,l)}$$

so far, no restrictions on $(r,s)\in\mathbb{Z}$.



$$\alpha^2 = (\alpha_+ + \alpha_-)^2 - 2\alpha_+\alpha_-$$

$$\alpha_+ + \alpha_- = 2\alpha_0$$

$$\alpha_+\alpha_- = -1$$

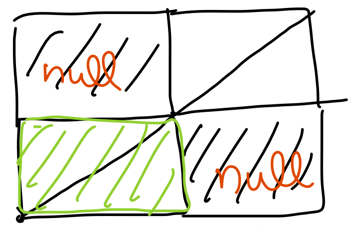
$$\tan\theta = -\frac{\alpha_+}{\alpha_-} \qquad h_{rs} = h_0 + \frac{\delta_{rs}^2}{4}(\alpha_+^2 + \alpha_-^2)$$

$$y = -\frac{\alpha_+}{\alpha_-}x \longrightarrow y = \frac{\alpha_-}{\alpha_+}(x - s) + r \Rightarrow \left(\frac{\alpha_+}{\alpha_-} + \frac{\alpha_-}{\alpha_+}\right)x = \frac{\alpha_-}{\alpha_+}s - r$$

$$x = \frac{r - \frac{\alpha_-}{\alpha_+}s}{\alpha_+^2 + \alpha_-^2} \to y = \frac{s - \frac{\alpha_+}{\alpha_-}r}{\alpha_+^2 + \alpha_-^2} \qquad \delta^2 = \left(\frac{r - \frac{\alpha_-}{\alpha_+}s}{\alpha^2} - s\right)^2 + \left(\frac{s - \frac{\alpha_+}{\alpha_-}r}{\alpha^2} - r\right)^2$$

if $-\frac{\alpha_+}{\alpha_-} = \text{irrational} \neq \frac{p}{p'}$ $\to$ inf. many $h_{r,s}$ (all different)
$\Rightarrow$ "irrational" CFT.

if $-\frac{\alpha_+}{\alpha_-} = \text{rational} = \frac{p}{p'}$ $\longrightarrow$
coprime



$$p\alpha_- + p'\alpha_+ = 0 \Rightarrow \boxed{h_{r,s} = h_{r+p', s+p}} \quad (1)$$

$$c = 1 - 6\underbrace{\frac{(p-p')^2}{pp'}}_{p \leftrightarrow p' \text{ (choose } p > p')} \qquad h_{rs} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'}$$

$$: \quad h_{rs} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'}$$

$$\|$$

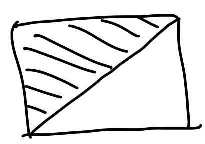
also notice

$$\boxed{h_{p'-r, p-s} = h_{r,s}} \quad (3)$$

(2) $h_{rs} + rs = h_{p'+r, p-s} = h_{p'-r, p+s}$; null states at rs

$$h_{rs} + (p'-r)(p-s) = h_{r, 2p-s} = h_{2p'-r, s}$$

$$\overset{(1)+(2)}{\Longrightarrow} 1 \leq r < p', \; 1 \leq s < p \xrightarrow{(3)\ 1/2}$$



$$\phi_{(r_1 s_1)} \times \phi_{(r_2 s_2)} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=\text{odd}}}^{k_{max}} \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2=\text{odd}}}^{l_{max}} \phi_{(k,l)}$$

$(r_1+r_2-1) \equiv r$

$$k_{max} = \min(\underbrace{r_1+r_2-1}_{r}, \underbrace{2p'-1-r_1-r_2}_{2p'-2-r})$$

$$l_{max} = \min(s_1+s_2-1, 2p-1-s_1-s_2)$$

$r < 2p'-2-r \rightarrow r < \underline{p'-1}$

$\underbrace{2p'-2-r}$ $< r \rightarrow r > p'-1$

$\therefore 2p'-2-r < \underline{p'-1}$

$\boxed{\text{unitarity}: \ h > 0}$ to have reasonable 2-pt fct. ($r \to \infty$ $G \to 0$)

But it is not absolute condition ; (ex) Yang-Lee edge sing.

(IM with imag. mag. field)

$h_{r,s} \to (pr - p's)^2 \Rightarrow \exists (r_0, s_0) \ni : pr_0 - p's_0 = 1$

$$\underbrace{h_{r_0,s_0} = \frac{1-(p-p')^2}{4pp'}}_{\text{lowest dimension}} < 0 \quad \text{except } \underbrace{(p-p') = 1 \ (p=m+1, p'=m)}$$

" unitary min. model

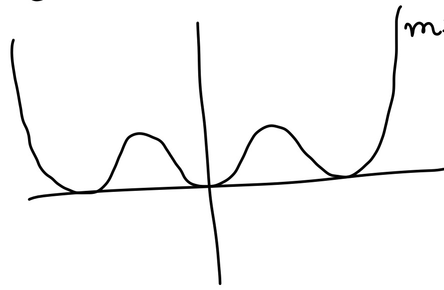
$r_0 = s_0 = 1 \rightarrow h_{1,1} = 0$

"vac."

---

* Landau-Ginzburg model for minimal model [Zamol. paper]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \underbrace{V(\varphi)}_{\sum_{n=1}^{(m-1)} g_n :\varphi^{2n}:} \xrightarrow[\text{RG Fixed pt.}]{} \mathcal{L}_* = \frac{1}{2}(\partial_\mu \varphi)^2 - \sum_n^{m-1} g_n^* :\varphi^{2n}:$$



$m=4$

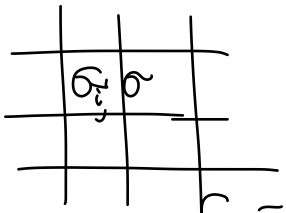
E. of. M : $\partial_z \partial_{\bar z} \varphi \cong :\phi^{2m-3}:$

$$\underset{\phi_{22}}{\varphi} \times \underset{\phi_{22}}{\varphi} = \underset{\phi_{11}}{\mathbb{1}} + \underset{\phi_{33}}{\varphi^2} \quad \longrightarrow \underline{:\varphi^k: = \phi_{(k+1,k+1)}}$$

"

Critical stat Mech model

$m=3$: 2D IM



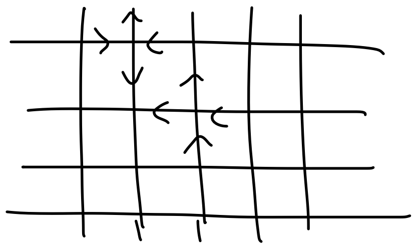
$\sigma_{\vec{i}} = \pm 1$

$E = -J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$ $c = \frac{1}{2}$

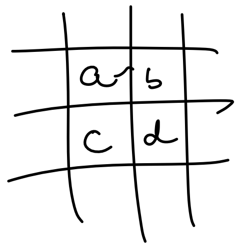
$m=4$: tricritical IM $E = -\sum (k + \delta_{\sigma_i \sigma_j}) t_i t_j - \mu \sum t_i$ $t_i = \sigma_i^2$

$m=5$: 3 states Potts $E = -\sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j}$ $\sigma_i = \pm 1, 0$

6 vertex model



$\equiv$



$b = a+1$

$|a-b| = 1$

$\equiv$

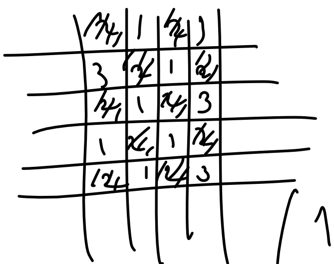
SOS (IRF) model $c=1$ $1 \leq a, b, \cdots \leq \infty$

(ABF)

$\downarrow$

Restricted SOS (RSOS) $1 \leq a, b, \cdots \leq m$

$m=3$:



$c = 1 - \frac{6}{m(m+1)}$

$\begin{pmatrix} 1 \equiv +1 \\ 3 \equiv -1 \end{pmatrix}$

$m=3$ $\left(L_{-2} - \frac{3}{\underbrace{2(2h_{21}+1)}_{\frac{3}{4}}} L_{-1}^2 \right) |\tfrac{1}{2}\rangle = 0$

$s \uparrow$		
	1/2	0
	1/16	1/16
	0	1/2
	$\rightarrow r$	

$\langle \sigma_{\vec{n}} \sigma_{\vec{0}} \rangle \sim \frac{1}{|\vec{n}|^{\eta = \frac{1}{4}}} = \frac{1}{r^{2\Delta_\sigma}}$ $\Delta_\sigma = h_\sigma + \bar{h}_\sigma = \frac{1}{8}$

$\langle \varepsilon_{\vec{n}} \varepsilon_{\vec{0}} \rangle \sim \frac{1}{|\vec{n}|^{2(2-1)}} = \frac{1}{r^{2\Delta_\varepsilon}}$ $\Delta_\varepsilon = \frac{1}{2} + \frac{1}{2} = 1$

$\varepsilon_{\vec{n}} \equiv \sigma_{\vec{n}} \sigma_{\vec{n}+1}$

$\langle \mathbb{1} \; \mathbb{1} \rangle \sim 1 = \frac{1}{r^{2 \cdot 0}}$ $h_{\mathbb{1}} = 0$

Minimal CFT

$C_{pp'}$, $h_{r,s}$, 2-pt. are known.

But 3-pt. needs $C^{(r_3 s_3)}_{(r_1 s_1)(r_2 s_2)}$

Also 4-pt. needs $F(x)$ conformal block.

$G^{(4)}$ can be determined by DE or integration. (Coulomb gas)

from $G^{(4)}$ and conf. block $\Rightarrow C_{ijk}$

---

① Special operator $\phi_{(21)} \to$ null state at level 2.
$\Rightarrow$ D.E.

let $\phi_1 = \phi_{(21)}$ with $h_1 = h_{2,1}$ $\quad x = \frac{z_{12} z_{34}}{z_{13} z_{24}}$

$$G^{(4)} = \langle \phi_1(z_1) \cdots \phi_4(z_4) \rangle = \left[\prod_{i<j} z_{ij}^{\mu_{ij}} \right] H(x) \equiv A \cdot H$$

$\mu_{ij} \equiv \frac{1}{3} \sum_k h_k - h_i - h_j$, $A = z_{12}^{\mu_{12}} z_{13}^{\mu_{13}} z_{14}^{\mu_{14}} z_{23}^{\mu_{23}} z_{24}^{\mu_{24}} z_{34}^{\mu_{34}}$

null-state condition gives

$$\frac{1}{t} \frac{\partial^2}{\partial z_1^2} [AH(x)] = \sum_{j=2}^{4} \left(\frac{h_j}{(z_1 - z_j)^2} + \frac{1}{z_1 - z_j} \frac{\partial}{\partial z_j} \right) [AH]$$

after computation, set $z_1 = x$, $z_2 = 0$, $z_4 = 1$, $z_3 = \infty$

$$\partial_{z_1}(AH) = (\partial_{z_1} A) H + A \partial_{z_1} H = \left(\frac{\mu_{12}}{z_{12}} + \frac{\mu_{13}}{z_{13}} + \frac{\mu_{14}}{z_{14}} \right) A (H + \partial_{z_1} H)$$

$$\partial_{z_1}^2 (AH) = -\left(\frac{\mu_{12}}{z_{12}^2} + \frac{\mu_{13}}{z_{13}^2} + \frac{\mu_{14}}{z_{14}^2} \right) A (H + \partial_{z_1} H)$$

$$+ \left(\frac{\mu_{12}}{z_{12}} + \frac{\mu_{13}}{z_{13}} + \frac{\mu_{14}}{z_{14}} \right)^2 A (H + \partial_{z_1} H) + \left(\frac{\mu_{12}}{z_{12}} + \frac{\mu_{13}}{z_{13}} + \frac{\mu_{14}}{z_{14}} \right) A \partial_{z_1} (H + \partial_{z_1} H)$$

(Ex) Ising model

$$[\sigma][\sigma] = 1 + [\varepsilon] \qquad [\sigma][\varepsilon] = [\sigma] \qquad [\varepsilon][\varepsilon] = 1$$

$$G^{(4)} = \langle \sigma(1) \cdots \sigma(4) \rangle = (\quad)^{\frac{1}{8}} \overline{(\quad)}^{\frac{1}{8}} F(x, \bar{x})$$

$$x = \frac{z_{12} z_{34}}{z_{13} z_{24}}$$

$$\left[\frac{4}{3} \frac{\partial^2}{\partial z_i^2} - \sum_{j \neq i}^{4} \left(\frac{1/16}{(z_i - z_j)^2} + \frac{1}{z_i - z_j} \frac{\partial}{\partial z_j} \right) \right] G^{(4)} = 0$$

$$\hookrightarrow \left[x(1-x) \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2} - x\right) \frac{\partial}{\partial x} + \frac{1}{16} \right] F(x) = 0 \quad (\text{similarly for } \bar{x})$$

$\Rightarrow$ two solutions for $F(x) = f_i(x)$ $\quad i = 1, 2$

$$\underline{f_{1,2} \equiv \left(1 \pm \sqrt{1-x}\right)^{\frac{1}{2}}}$$

$$\therefore G^{(4)} = \left| \frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{41}} \right|^{\frac{1}{4}} \sum_{i,j=1}^{2} \underbrace{a_{ij}}_{\text{not fixed yet}} f_i(x) f_j(\bar{x})$$

$$a_{11} (1 + \sqrt{1-x})^{\frac{1}{2}} (1 + \sqrt{1-\bar{x}})^{\frac{1}{2}} + a_{22} (1 - \sqrt{1-x})^{\frac{1}{2}} (1 - \sqrt{1-\bar{x}})^{\frac{1}{2}}$$

$$+ a_{12} (1 + \sqrt{1-x})^{\frac{1}{2}} (1 - \sqrt{1-\bar{x}})^{\frac{1}{2}} + a_{21} (1 - \sqrt{1-x})^{\frac{1}{2}} (1 + \sqrt{1-\bar{x}})^{\frac{1}{2}} \Big] \rightarrow \text{not single-value.}$$

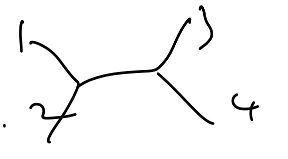
$$\therefore a_{11} = a_{22} \equiv a, \quad a_{12} = a_{21} = 0$$

$$\boxed{\frac{z_{12} z_{34} z_{23} z_{41}}{\underbrace{z_{13} z_{24}}_{x} \underbrace{z_{13} z_{24}}_{1-x}}}$$

$$\therefore G^{(4)} = a \overbrace{\left| \frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{41}} \right|^{\frac{1}{4}}}^{[z_{12} z_{34} (1-x)]^{-1}} \left[\left| 1 + \sqrt{1-x} \right| + \left| 1 - \sqrt{1-x} \right| \right]$$

Now, consider OPE

$$\sigma(1)\sigma(2) \sim \underbrace{\frac{1}{z_{12}^{1/8} \bar{z}_{12}^{1/8}}}_{|z_{12}|^{\frac{1}{4}}} + C_{\sigma\sigma\varepsilon} \overbrace{z_{12}^{3/8} \bar{z}_{12}^{3/8}}^{\frac{1}{2} - \frac{1}{16} - \frac{1}{16} \;\rightarrow\; |z_{12}|^{\frac{3}{4}}} \varepsilon(z_2, \bar{z}_2) + \cdots$$



(similarly for $\sigma(3)\sigma(4)$), $\langle \varepsilon \rangle = 0$

$$G^{(4)} \sim \frac{1}{|z_{12}|^{1/4} |z_{34}|^{\frac{1}{4}}} + C_{\sigma\sigma\varepsilon}^2 |z_{12}|^{\frac{3}{4}} |z_{34}|^{\frac{3}{4}} \underbrace{\langle \varepsilon(2) \varepsilon(4) \rangle}_{\frac{1}{|z_{24}|^2}} + \cdots$$

$$G^{(4)} = \frac{a}{|z_{12}|^{\frac14}|z_{34}|^{\frac14}} \left(\frac{\left|1+\sqrt{1-x}\right| + \left|1-\sqrt{1-x}\right|}{|1-x|^{\frac14}} \right) \approx \frac{2a}{\cdots}\left(1+\frac{|x|}{4}\right)$$

$$G^{(4)} \sim \frac{1}{|z_{12}|^{1/4}\,|z_{34}|^{\frac14}}\left(1 + C_{\sigma\sigma\varepsilon}^{2}\,\frac{|z_{12}|\,|z_{34}|}{|z_{24}|^{2}} + \cdots\right)$$

as $x \to 0$ ($|z_{13}| \gg$ others) $\qquad a = \frac{1}{2}$

$$\left(\frac{\left|1+\sqrt{1-x}\right|}{|1-x|^{\frac14}} \;\begin{array}{c}\to\\ \simeq\end{array}\; \frac{2-\frac{x}{2} = 2\left|1-\frac{x}{4}\right|}{\left|1-\frac{x}{4}\right|} \approx 2 + O(x^2) \right.$$

$$\left. \left| \frac{\frac{x}{2}}{1-\frac{x}{4}} \right| \approx \frac{1}{2}|x| \qquad x = \frac{z_{12}z_{34}}{z_{13}\,z_{24}} \qquad 1-x = \frac{z_{23}z_{41}}{z_{13}z_{24}} \right)$$

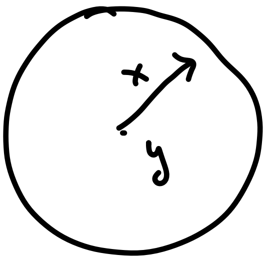
set $z_1 = 0$, $z_2 = x$, $z_3 = 1$, $z_4 = \infty$ $\Longrightarrow$ $C_{\sigma\sigma\varepsilon} = \frac{1}{2}$ //

free boson

$$S = \frac{g}{2}\int d^2x\left[(\partial_\mu\varphi)^2 + m^2\varphi^2\right] \qquad \overset{\text{propagator}}{K(x,y)} \equiv \langle\varphi(x)\,\varphi(y)\rangle$$

$$\equiv \frac{1}{2}\int d^2x\, d^2y\;\varphi(x)\,A(x,y)\,\varphi(y) \quad \text{w/}\quad A(x,y) = g\,\delta(x-y)(-\partial^2+m^2)$$

$$K = A^{-1} \quad \text{or} \quad g(-\partial_x^2 + m^2)\underbrace{K(x,y)}_{K(\underbrace{|x-y|}_{\rho})} = \delta^{(2)}(x-y)$$



$$1 = \int \underbrace{d^2x}_{2\pi\rho\, d\rho}\; g\,(-\underbrace{\partial_x^2}_{}+m^2)\,K(x,y) \qquad \nabla_2^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \cancel{\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}}$$

$$= 2\pi g\left\{-\int_0^r \frac{\partial}{\partial\rho}(\rho K')\,d\rho + \int m^2\rho K\,d\rho\right\} = \int_0^r 2\pi g\,\rho\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho K'(\rho)) + m^2 K\right)d\rho$$

$$= 2\pi g\left[-rK'(r) + m^2\int_0^r d\rho\,\rho\,K(\rho)\right] = 1$$

if $m=0$; $\quad K'(r) = -\frac{1}{2\pi g}\frac{1}{r} \longrightarrow K(r) = -\frac{1}{2\pi g}\log r$

$$\therefore\ \langle\varphi(x)\varphi(y)\rangle = -\frac{1}{2\pi g}\log|x-y| = -\frac{1}{4\pi g}\log(\vec{x}-\vec{y})^2$$

$$(\vec{x}-\vec{y})^2 \downarrow |z-w|^2$$

$$g = \frac{1}{4\pi} \longrightarrow \boxed{\langle\varphi(z)\varphi(w)\rangle = -\log(z-w)}$$

$$\left(\phi(x) \equiv \varphi(z) + \bar{\varphi}(\bar{z})\right)$$

($m \neq 0$; one-more derivative: $m^2K = \frac{1}{r}\frac{d}{dr}\left(r\frac{dK}{dr}\right)$

Bessel; $K(r) = \frac{1}{2\pi g}K_0(mr)$, $K_0(x) = \int_0^\infty dt\,\frac{\cos(xt)}{\sqrt{t^2+1}}$)

$$\langle \varphi(z)\varphi(w)\rangle = -\log(z-w)$$

$$\langle \partial\varphi(z)\partial\varphi(w)\rangle = -\partial_z\partial_w \log(z-w) = -\frac{1}{(z-w)^2}$$

OPE

or $$\partial\varphi(z)\,\partial\varphi(w) = -\frac{1}{(z-w)^2} + \cdots$$

$$T_{\mu\nu} = -2\,\partial_\mu\phi\,\partial_\nu\phi$$

$$\downarrow$$

$$T_{zz} = -\frac{1}{2}:\partial\varphi\,\partial\varphi: = -\frac{1}{2}\lim_{z\to w}\left[\partial\varphi(z)\partial\varphi(w) + \frac{1}{(z-w)^2}\right]$$

$$T(z)\,\partial\varphi(w) = -\frac{1}{2}:\partial\varphi(z)\partial\varphi(z):\partial\varphi(w)$$

$$= -\frac{1}{2}\cdot 2\cdot\partial\varphi(z)\langle\partial\varphi(z)\partial\varphi(w)\rangle + \cdots$$

$$= \frac{1}{(z-w)^2}\underbrace{\partial\varphi(z)}_{\partial\varphi(w) + (z-w)\partial^2\varphi(w) + \cdots} + \cdots$$

$$= \frac{1}{(z-w)^2}\partial\varphi(w) + \frac{1}{z-w}\partial(\partial\varphi)(w) + \cdots$$

①

$\therefore$ $\underline{\partial\varphi}$ is a $(1,0)$ primary.

$$\left[\oint\frac{dz}{2\pi i}T(z)\epsilon(z),\ \partial\varphi(w)\right] = \oint_w\frac{dz}{2\pi i}\epsilon(z)\left[\frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}\right]$$

$$\| \qquad\qquad = [\partial\epsilon\cdot\partial\varphi + \epsilon\,\partial^2\varphi](w)$$

$$\delta_\epsilon(\partial\varphi)(w)$$

$$\left(\partial\varphi(z+\epsilon) = \partial[\varphi + \epsilon\partial\varphi] = \partial\varphi + \partial\epsilon\,\partial\varphi + \epsilon\,\partial^2\varphi\right)$$

② $$T e^{\sqrt{2}i\alpha\varphi} = -\frac{1}{2}:\partial\varphi\,\partial\varphi:\,e^{i\sqrt{2}\alpha\varphi(w)}$$

$$= -\frac{1}{2}\Big(\underbrace{i\sqrt{2}\alpha\langle\partial\varphi(z)\varphi(w)\rangle}_{-\frac{\sqrt{2}i\alpha}{z-w}}\Big)^2 e^{i\alpha x(w)} - \frac{1}{2}\,2\,\underbrace{\partial\varphi}_{\partial\varphi(w)+\text{reg.}}\langle\partial\varphi\sqrt{2}i\alpha\varphi\rangle e^{i\sqrt{2}\alpha\varphi(w)}$$

$$= \frac{\alpha^2 \equiv h_\alpha}{(z-w)^2}e^{\sqrt{2}i\alpha x} + \frac{\sqrt{2}i\alpha\,\partial\varphi(z)}{z-w}\underbrace{e^{\sqrt{2}i\alpha\varphi(w)}}_{\partial_w e^{\sqrt{2}i\alpha\varphi}}$$

$$\frac{1}{z-w}\partial_w(e^{\sqrt{2}i\alpha\varphi(w)})$$

③

$$T = -\frac{1}{2} :\partial\varphi\partial\varphi:$$

$$T(z)T(w) = +\frac{1}{4}\Big\{ :\partial_z\varphi\,\partial_z\varphi: \; :\partial_w\varphi\,\partial_w\varphi: \; - \; 4\frac{1}{(z-w)^2}\underbrace{\partial_z\varphi\,\partial_w\varphi}$$

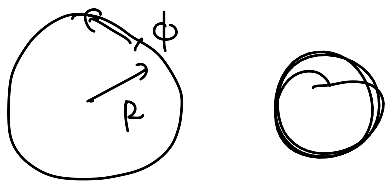
$$2\frac{1}{(z-w)^4} \qquad \partial_z\varphi\partial_w\varphi = (\partial_w\varphi)^2 + (z-w)\underbrace{\partial_w^2\varphi\,\partial_w\varphi}_{\frac{1}{2}\partial_w((\partial\varphi)^2)}$$

$$= \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2}\underbrace{\left(-\frac{1}{2}:\partial_w\varphi\partial_w\varphi:\right)}_{T(w)} + \frac{1}{z-w}\partial_w(T)$$

$$\boxed{T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T}$$

with $c=1$

Compactified boson.



let $\phi(x,t)$ satisfy $\phi(x+L,t) = \phi(x,t) + 2\pi R\, m$

$\uparrow$ winding #

$$\phi(x) \equiv \varphi(x-t) + \bar\varphi(x+t) \Leftarrow \partial_+\partial_-\varphi = 0$$

mode expan $\Rightarrow i\sum_{n\neq 0}\frac{1}{n}a_n \underbrace{e^{2\pi i n(x-t)/L}}_{z^{-n}}$ + zero-mode

$$\phi = \varphi_0 + \underbrace{\frac{4\pi n}{R}\frac{t}{L}}_{\text{momentum}} + 2\pi R\frac{x}{L} + i\sum_{n\neq 0}\left[\frac{1}{n}a_n e^{\frac{2\pi i n(x-t)}{L}} + c.c\right]$$

$$z \equiv e^{-2\pi i\left(\frac{x-t}{L}\right)}$$

$$\Rightarrow \varphi = \varphi_0 - i\left(\frac{n}{R} + \frac{mR}{2}\right)\ln z + i\sum_{n\neq 0}\frac{1}{n}a_n z^{-n}$$

$(\ln\bar z)$

$$\Rightarrow L_0 = \sum_{n>0} a_{-n}a_n + \frac{1}{2}\left(\frac{n}{R} + \frac{mR}{2}\right)^2$$

$$\bar L_0 = \sum_{n>0}\bar a_{-n}\bar a_n + \frac{1}{2}\left(\frac{n}{R} - \frac{mR}{2}\right)^2$$

Coulomb gas formalism
for correlation function

$\boxed{\text{free boson}}$

$$S = \frac{g}{2}\int d^2x\left[(\partial_\mu\phi)^2\right] \qquad g = \frac{1}{4\pi} \qquad \phi = \varphi + \bar{\varphi}$$

$$T = -\frac{1}{2}:\partial\varphi\,\partial\varphi: \;\rightarrow\; V = e^{i\alpha\varphi} \rightarrow h = \frac{\alpha^2}{2}$$

or $V_\alpha \equiv e^{\sqrt{2}\,i\alpha\varphi} \longrightarrow h_\alpha = \alpha^2$

$$\varphi(z) = \varphi_0 - i a_0 \ln z + i\sum_{n\neq 0}\frac{1}{n} a_n z^{-n}$$

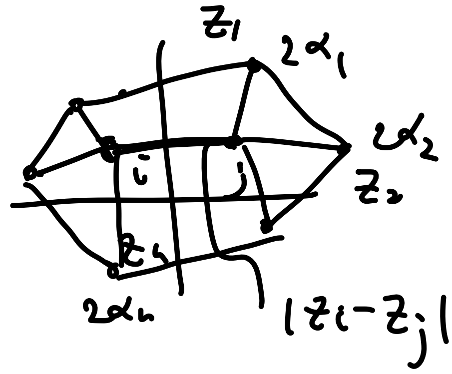
$[a_n, a_m] = n\,\delta_{n+m,0}$

$\mathcal{V}_\alpha(z,\bar{z}) = V_\alpha(z)\bar{V}_\alpha(\bar{z})$

* zero-mode $[\varphi_0, a_0] = i$

$$\boxed{\text{Correl. } \langle \mathcal{V}_{\alpha_1}\cdots\mathcal{V}_{\alpha_n}(z_n,\bar{z}_n)\rangle = \prod_{i<j}|z_i - z_j|^{4\alpha_i\alpha_j} \quad \text{if } \alpha_1 + \cdots + \alpha_n = 0}$$

$$= \exp\Big[\underbrace{\sum_{i<j} 4\alpha_i\alpha_j \ln|z_i - z_j|}_{\text{potential energy of } n \text{ charge coulomb interaction in 2d}}\Big]$$



Holomorphic only

$$\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\rangle = \prod_{i<j}(z_i - z_j)^{2\alpha_i\alpha_j}$$

$\boxed{\text{ex}}$ Consistent with 2, 3-pt.

$$\begin{cases} \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\rangle = (z_1 - z_2)^{-2\alpha_1^2} & (\alpha_1 = -\alpha_2) \\ \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)\rangle = (z_1 - z_2)^{2\alpha_1\alpha_2}(z_2 - z_3)^{2\alpha_2\alpha_3}(z_1 - z_3)^{2\alpha_1\alpha_3} \end{cases}$$

$$2\alpha_1\alpha_2 = (\alpha_1 + \alpha_2)^2 - \alpha_1^2 - \alpha_2^2 = \alpha_3^2 - \alpha_1^2 - \alpha_2^2 = h_3 - h_1 - h_2 \quad \text{etc}$$

$\rightarrow$ global conf. sym fixes.

$$V_\alpha(z) = :e^{i\sqrt{2}\alpha\,\varphi(z)}: = :e^{i\sqrt{2}\alpha\tilde{\varphi}}:\ \underbrace{:e^{-\sqrt{2}\alpha\sum_{n>0}\frac{1}{n}a_{-n}z^n}::e^{\sqrt{2}\alpha\sum_{n>0}a_n\frac{z^{-n}}{n}}:}_{V'_\alpha(z)}$$

$$\tilde{\varphi} = \varphi_0 - i a_0 \ln z$$

---

simpler case: $A_i = \alpha_i a + \beta_i a^\dagger$

$$:e^{A_i}: = e^{\beta_i a^\dagger} e^{\alpha_i a} \qquad :e^{A_1}::e^{A_2}:\cdots:e^{A_n}:$$

$$e^{\beta_i a^\dagger}\,\underbrace{e^{\alpha_i a}:e^{A_{i+1}}:\cdots:e^{A_n}:}_{}$$

$$e^{:A_{i+1}:}\cdots:e^{A_n}:\ e^{\alpha_i a}\cdot e^{\alpha_i\sum_{j=i+1}^{n}\beta_j}$$

$$e^B e^A = e^A e^B e^{\overbrace{[B,A]}^{C\#}}$$

$$\left(\prod_j e^{\beta_j\alpha_i\underbrace{[a,a^\dagger]}_{1}}\right)$$

$$\left.\begin{aligned} i&=1 \quad e^{\alpha_1\sum_{j=2}^{n}\beta_j} \\ i&=2 \quad e^{\alpha_2\sum_{j=3}^{n}\beta_j} \\ &\vdots \\ i&=n-1 \quad e^{\alpha_{n-1}\beta_n}\end{aligned}\right\}\ \underbrace{e^{\sum_{i<j}^{n}\alpha_i\beta_j}}_{e^{\sum_{i<j}^{n}\langle A_iA_j\rangle}}\cdot\underbrace{e^{(\beta_1+\cdots+\beta_n)a^\dagger}e^{(\alpha_1+\cdots+\alpha_n)a}}_{\vdots e^{A_1+\cdots+A_n}\vdots}$$

$$= \ :e^{A_1}:\cdots:e^{A_n}:$$

$$V'_{\alpha_i}(z_i) = [V_i]_-[V_i]_+ \qquad [V_i]_- \equiv e^{-\sqrt{2}\alpha_i\sum_n\frac{z_i^n}{n}a_{-n}}$$

$$[V_i]_+ \equiv e^{\sqrt{2}\alpha_i\sum_n\frac{z_i^{-n}}{n}a_n}$$

$$[V_i]_-[V_i]_+\,V_{i+1}\cdots V_n$$

$$\prod_{j=i+1}^{n} e^{\underbrace{[\sqrt{2}\alpha_i\sum_n\frac{z_i^{-n}}{n}a_n,\ -\sqrt{2}\alpha_j\sum_m\frac{z_j^m}{m}a_{-m}]}_{}}$$

$$-2\alpha_i\alpha_j\underbrace{\sum_{n,m}\frac{z_i^{-n}z_j^{m}}{nm}\underbrace{[a_n,a_{-m}]}_{n\delta_{nm}}}_{\sum_{n>0}\frac{1}{n}\left(\frac{z_j}{z_i}\right)^n = f(w)}$$

$$w = \frac{z_j}{z_i}$$

$$f(w) = \sum_{n>0}\frac{1}{n}w^n$$

$$f'(w) = \sum_{n>0}w^{n-1} = \frac{1}{1-w}$$

$$\therefore f(w) = -\ln(1-w) = -\ln\left(1-\frac{z_j}{z_i}\right)$$

$$\prod e^{2\alpha_i\alpha_j \ln\left(1-\frac{z_j}{z_i}\right)} = \prod_{j=i+1}^{n}\left(1-\frac{z_j}{z_i}\right)^{2\alpha_i\alpha_j}$$

$i=1,\dots,n-1$

$$\prod_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(1-\frac{z_j}{z_i}\right)^{2\alpha_i\alpha_j} = \prod_{i<j}(z_i-z_j)^{2\alpha_i\alpha_j}\, z_i^{-2\alpha_i\alpha_j}$$

$$\left\langle :e^{i\sqrt{2}\alpha_1\tilde{\varphi}_1}: \cdots :e^{i\sqrt{2}\alpha_n\tilde{\varphi}_n}: \right\rangle = \prod_{i<j} z_i^{2\alpha_i\alpha_j}$$

$\tilde{\varphi} = \varphi_0 - i a_0 \ln z_1 \qquad [\varphi_0, a_0] = i \quad \to 2\alpha_i\alpha_j \ln z_i$

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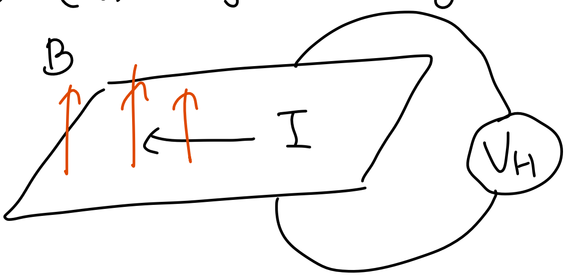
neutrality condition:

symmetry: $\phi \to \phi + a \qquad e^{ia\sqrt{2}(\alpha_1+\dots+\alpha_n)} = 1$

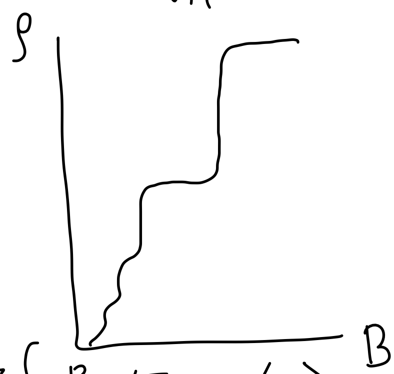
[Application] Laughlin wave function for <u>fractional QHE</u>



$$\sigma = \frac{I}{V_H} = \nu\frac{e^2}{h}$$

$\nu =$ fractional number

$\nu = \frac{1}{q}$ (Laughlin)



<u>Read & Moore</u>

$$\left\langle \prod_{i=1}^{N} e^{i\sqrt{q}\,\phi(z_i)}\, e^{-i\sqrt{q}\int_D d^2z'\,\bar{\rho}\,\phi(z')} \right\rangle$$

$\bar{\rho} = \frac{1}{2\pi q}$ on $D =$ Disk, Area $= 2\pi q N$

$$= \underbrace{\prod_{i<j}(z_i-z_j)^q\, e^{-\frac{1}{2\pi}\sum_i \int d^2z' \log(z_i-z')}}_{e^{-\frac{1}{4}\sum_i |z_i|^2}}$$

real part $\to \frac{\pi}{2}|z_i|^2$

Laughlin wave function for lowest Landau level for $\nu = \frac{1}{q}$

Excited states (quasi-holes)

$$\left\langle e^{\frac{i}{\sqrt{q}}\phi(z)}\, e^{\frac{i}{\sqrt{q}}\phi(w)} \prod_{i=1}^{N} e^{i\sqrt{q}\,\phi(z_i)}\; e^{-i\sqrt{q}\int d^2z'\,\bar{\rho}\,\phi(z')} \right\rangle$$

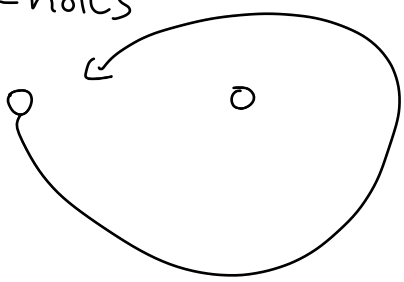
$$= (z-w)^{1/q} \prod_k (z-z_k)(w-z_k) \prod_{i<j} (z_i-z_j)^q\; e^{-\frac{1}{4}\left(\sum_i |z_i|^2+|z|^2+|w|^2\right)}$$

" wave function " for the excited states.

as z goes around w

i.e. $\quad z-w \to e^{2\pi i}(z-w)$

quasi-holes



$$\Psi \to \underbrace{e^{\frac{2\pi i}{q}}}_{\text{anyon statistics}} \Psi$$

Background charge (imaginary $\rightarrow$ non unitary) for generic α_0

b.c.

$$S=\frac{1}{8\pi}\int d^2x\sqrt{g}\left(\partial_\mu\varphi\partial^\mu\varphi+2i\sqrt{2}\alpha_0\varphi R\right)$$

no more sym : $\varphi\rightarrow\varphi+a$

$$\delta S=\frac{i\sqrt{2}\alpha_0 a}{4\pi}\underbrace{\int d^2x\sqrt{g}R}_{8\pi(1-h)}=i2a\sqrt{2}\alpha_0$$

$h=0$ for sphere

$$i\sqrt{2}\langle X\rangle\sum_k\alpha_k=\frac{i}{4\pi}\oint dz\underbrace{\langle\partial\phi X\rangle}_{-i\sqrt{2}\sum\frac{\alpha_k}{z-z_k}\langle X\rangle}\cancel{-\frac{i}{4\pi}\oint d\bar{z}\underbrace{\langle\bar\partial\phi X\rangle}_{-i\sqrt{2}\sum\frac{\alpha_k}{\bar z-\bar z_k}\langle X\rangle}}+2i\sqrt{2}\alpha_0\langle X\rangle$$

$$\therefore\ \boxed{\sum_k\alpha_k=2\alpha_0}$$

$$T_{\mu\nu}=\underset{c=1}{T^{(0)}_{\mu\nu}}-\frac{i\sqrt{2}\alpha_0}{2\pi}\left(\partial_\mu\partial_\nu\varphi-\frac{1}{2}\eta_{\mu\nu}\partial^\sigma\partial_\sigma\varphi\right)$$

or $T(z)=-\frac{1}{2}:\partial\varphi\partial\varphi:+i\sqrt{2}\alpha_0\partial^2\varphi$ (Feigin-Fuchs)

$$-2\alpha_0^2\ \partial_z^2\varphi\ \partial_w^2\varphi=2\alpha_0^2\partial_z^2\partial_w\frac{1}{(z-w)^2}$$

$$=2\alpha_0^2(2)(-3)\frac{1}{(z-w)^4}=\frac{-24\alpha_0^2/2}{(z-w)^4}$$

$$\underline{c=1-24\alpha_0^2}$$

for general $\alpha_0\rightarrow$ nonunit.
some $\alpha_0\rightarrow$ unitary

$$T(z)\partial\varphi(w)=\underline{\underline{\frac{2\sqrt{2}i\alpha_0}{(z-w)^3}}}+\frac{\partial_w\varphi}{(z-w)^2}+\frac{\partial_w^2\varphi}{z-w}+\text{reg.}$$

$T+i\sqrt{2}\alpha_0\partial^2\varphi(z)$

$\therefore\ \partial_w\varphi$ is NOT primary

but

$$\partial^2\varphi V_\alpha(w)\sim\frac{i\sqrt{2}\alpha}{(z-w)^2}V_\alpha(w)\qquad\therefore\ h_\alpha=\alpha^2-2\alpha\alpha_0=\alpha(\alpha-2\alpha_0)$$

$\alpha\longleftrightarrow 2\alpha_0-\alpha$

i.e. $V_\alpha=V_{2\alpha_0-\alpha}$

if ψ has $h=1$, $\oint dz\, \psi(z) \equiv A$ is $h=0$ and invariant under conf. transf. $z \to w : A \to A$ i.e.

$$[L_n, A] = \oint dz\, [L_n, \psi] = \oint dz \{\underbrace{h}_{=1}(n+1)z^n \psi + z^{n+1}\partial\psi\} = \oint dz\, \partial(z^{n+1}\psi) = 0$$

two α's satisfying $h_\alpha = 1 \to \alpha = \alpha_0 \pm \sqrt{\alpha_0^2+1} \equiv \alpha_\pm$, $\alpha_+ + \alpha_- = 2\alpha_0$, $\alpha_+\alpha_- = -1$

$Q_\pm \equiv \oint dz\, V_\pm(z) = \oint dz\, e^{i\sqrt{2}\alpha_\pm \varphi}$ are "screening operator" which do not affect "conformal properties" in a correl. function.

But can change total charge & correlation.

$\langle V_\alpha(z) V_\alpha(w) \rangle = 0$, $\langle V_\alpha(z) V_\alpha(w) Q_-^m Q_+^n \rangle = \langle V_\alpha V_{2\alpha_0 - \alpha} \rangle \neq 0$

if $2\alpha + m\alpha_- + n\alpha_+ = 2\alpha_0 = \alpha_+ + \alpha_-$

$\Rightarrow$ neutrality condition for arbitrary correl. fct. leads to

$$\alpha = \{ \underline{\alpha_{r,s} = \tfrac{1}{2}(1-r)\alpha_+ + \tfrac{1}{2}(1-s)\alpha_-}, \ r,s \in \mathbb{Z} \}$$

$V_{\alpha_{r,s}} \equiv V_{r,s} \to h_{r,s} = \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 - \alpha_0^2$ ← consist. with Kac's formula (no restriction on r,s)

Rational (# is finite) is possible if $\frac{\alpha_+}{\alpha_-}$ = rational

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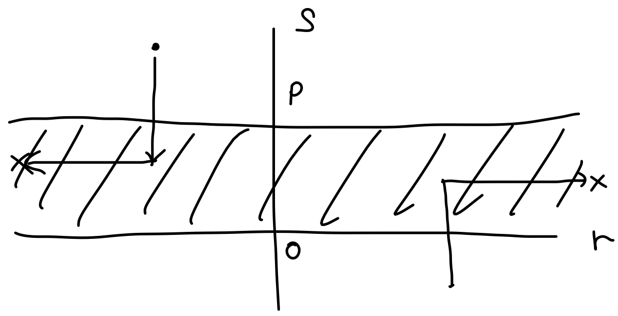
$\exists\ p, p'\ (p > p')$ coprime $\ni$: $p'\alpha_+ + p\alpha_- = 0$

$r\alpha_+ + s\alpha_- \equiv (r+p')\alpha_+ + (s+p)\alpha_- \Rightarrow V_{r+p', s+p} \equiv V_{r,s}$



still need to restrict r

$\alpha_+ = \sqrt{\frac{p}{p'}}$, $\alpha_- = -\sqrt{\frac{p'}{p}}$, $2\alpha_0 = \frac{p-p'}{\sqrt{pp'}}$ $\longrightarrow c = 1 - 24\alpha_0^2 = 1 - \frac{6(p-p')^2}{pp'}$

$$h_{r,s} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'}$$

Now consider 3-pt

$\langle \phi_{r_1 s_1} \phi_{r_2 s_2} \phi_{r_3 s_3} \rangle = \langle V_{r_1 s_1} V_{r_2 s_2} V_{r_3 s_3} Q_+^r Q_-^s \rangle$ → can not fix C_{123}

will be determined by 4-pt.