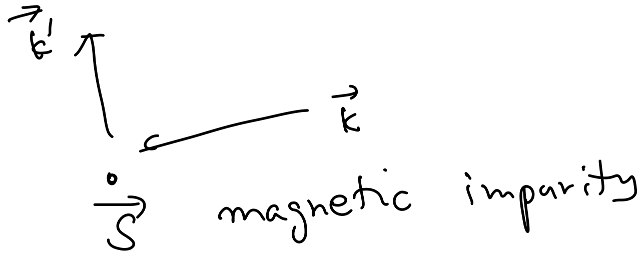
Kondo Effect (Affleck & Ludwig)

$$H = \sum_{\vec{k},\alpha} \psi^{\dagger\alpha}_{\vec{k}} \psi_{\vec{k}\alpha} \epsilon(\vec{k}) + \lambda \vec{S} \cdot \sum_{\vec{k},\vec{k}'} \psi^{\dagger}_{\vec{k}} \frac{\vec{\sigma}}{2} \psi_{\vec{k}'}$$



Spherical symmetry: $\epsilon(\vec{k}) = \epsilon(k) \approx v_F (k - k_F)$

$$\psi(\vec{k}) = \frac{1}{\sqrt{4\pi}\, k} \psi_0(k) + \cdots$$

$\uparrow$ $l=0$ "S-wave"

$$H = \underbrace{\int dk\, \psi^{\dagger}_{0k} \psi_{0k} \epsilon(k)}_{\text{excitation near Fermi surface}} + \lambda v_F \int dk\, dk'\, \psi^{\dagger}_{0k} \frac{\vec{\sigma}}{2} \psi_{0k'} \cdot \vec{S}$$

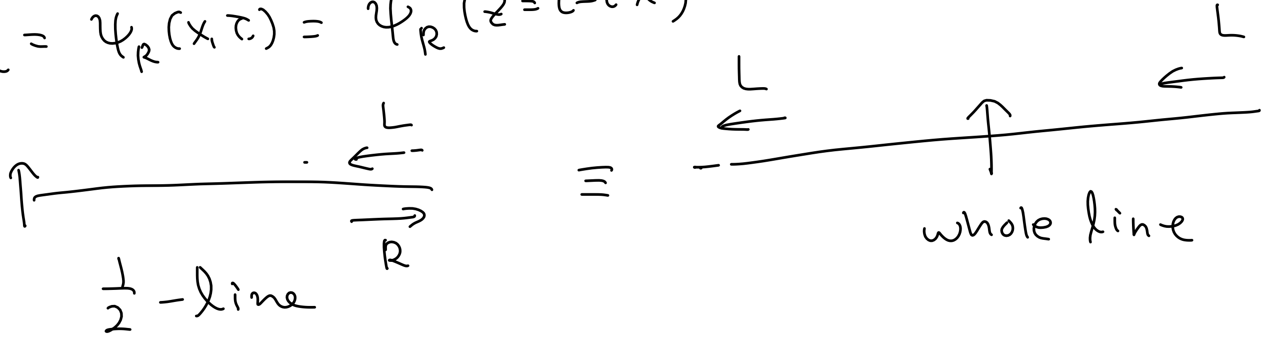
F.T. back to real space

$$= \frac{v_F}{2\pi} \int_0^\infty dr \left(\psi^{\dagger}_L i \frac{d}{dr} \psi_L - \psi^{\dagger}_R i \frac{d}{dr} \psi_R \right) + \lambda v_F \vec{S} \cdot \psi^{\dagger}_L(0) \frac{\vec{\sigma}}{2} \psi_L(0)$$

$(\psi_L(0) = \psi_R(0))$

$$\psi_L = \psi_L(x,\tau) = \psi_L(z = \tau + ix) \qquad (v_F \equiv 1)$$

$$\psi_R = \psi_R(x,\tau) = \psi_R(\bar{z} = \tau - ix)$$



$$= \frac{v_F}{2\pi} \int_{-\infty}^{\infty} dx\, \psi^{\alpha\dagger}_L i \frac{d}{dx} \psi_{L\alpha} + \cdots$$

$$\mathcal{H} = \psi^{\alpha\dagger}_L i \frac{d}{dx} \psi_{L\alpha} + \lambda \psi^{\dagger\alpha}_L \frac{\vec{\sigma}_\alpha^{\ \beta}}{2} \psi_{L\beta} \cdot \vec{S}\, \delta(x)$$

spin-charge separation

$J = :\psi^{\alpha\dagger}\psi_\alpha:$ $U(1)$ charge $\qquad \vec{J} = \psi^{\dagger\alpha} \frac{\vec{\sigma}_\alpha^{\ \beta}}{2} \psi_\beta$ $SU(2)$

using $\vec{\sigma}_\alpha^{\ \beta} \cdot \vec{\sigma}_\gamma^{\ \delta} = 2 \delta^\beta_\gamma \delta^\delta_\alpha - \delta^\alpha_\beta \delta^\gamma_\delta$

$$\vec{J}^2 = -\frac{3}{4} :\psi^{\dagger\alpha}\psi_\alpha \psi^{\dagger\beta}\psi_\beta: + \frac{3}{2} i \psi^{\alpha\dagger} \frac{d}{dx} \psi_\alpha$$

$$J^2 = :\psi^{\dagger\alpha}\psi_\alpha \psi^{\dagger\beta}\psi_\beta: + 2i\, \psi^{\alpha\dagger} \frac{d}{dx} \psi_\alpha$$

$$\mathcal{H} = \frac{1}{8\pi} J^2 + \frac{1}{6\pi}\vec{J}^2 + \lambda \vec{J}\cdot\vec{S}\,\delta(x)$$

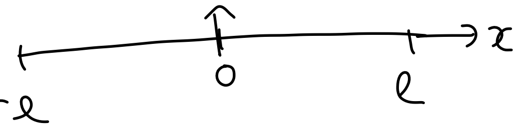
$\frac{1}{8\pi}$ ↳ decouples

$\vec{J}$ ↳ $\vec{J}(z)$ satisfy $SU(2)_1$ Kac-Moudy algebra (later)

$$\vec{J}(x) = \frac{1}{2\pi}\int_{-l}^{l} dx\, e^{\frac{in\pi}{l}x}\vec{J}_n \quad \leadsto \quad [J_n^a, J_m^b] = i\epsilon^{abc} J^c_{n+m} + \frac{nk}{2}\delta^{ab}\delta_{n,-m}$$

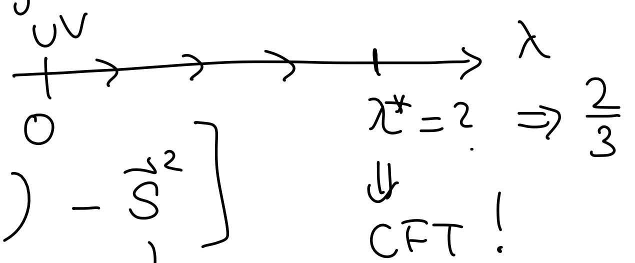
$$\mathcal{H}_S = \frac{1}{6\pi}\vec{J}^2 + \lambda\vec{J}\cdot\vec{S}\,\delta(x)$$

$$H = \int_{-\infty}^{\infty} \mathcal{H}_S\, dx = \frac{\pi}{l}\left(\frac{1}{3}\sum_n \vec{J}_{-n}\cdot\vec{J}_n + \lambda\vec{J}_n\cdot\vec{S}\right)$$



RG flow of λ : $\frac{d\lambda}{d\log\mu} = -\lambda^2 + \cdots < 0$

① nontrivial fixed point



if $\lambda = \frac{2}{3}$ one can square

$$= \frac{\pi}{3l}\cdot\sum_n\left[(\vec{J}_{-n}+\vec{S})(\vec{J}_n+\vec{S}) - \vec{S}^2\right]$$

$\vec{S}^2$ ↓ $\frac{3}{4}$ if $S=\frac{1}{2}$

still conformal. since $\vec{\mathcal{J}}_n \equiv \vec{J}_n + \vec{S}$ also satisfy Kac-Moudy algebra

② near IR. $\vec{\mathcal{J}}(x) = \vec{J}(x) + 2\pi\delta(x)\vec{S}$

Physical quantities near IR fixed point.

$$\mathcal{H}_S = \frac{1}{6\pi}\left(\vec{\mathcal{J}}(x) - 2\pi\delta(x)\vec{S}\right)^2 + \lambda\left(\vec{\mathcal{J}} - 2\pi\delta(x)\vec{S}\right)\cdot\vec{S}\,\delta(x)$$

$$= \frac{1}{6\pi}\left(\vec{\mathcal{J}}(x)\right)^2 + (\vec{S}\text{ dependence disappears})$$

$$+ \lambda_1 \vec{\mathcal{J}}(0)^2\,\delta(x)$$

$\lambda_1 \vec{\mathcal{J}}(0)^2\delta(x)$: leading dim=2 operator near IR FP.

Susceptibility

$$\chi = \frac{1}{3T}\left\langle \int dx \left(\vec{\mathcal{J}}(x)\right)^2 \right\rangle_{\lambda_1} = \chi_0 - \frac{\lambda_1}{3T^2}\left\langle \int dx \left(\vec{\mathcal{J}}(x)\right)^2 \vec{\mathcal{J}}(0)^2 \right\rangle$$

...

Extended conformal symmetries

① WZW model

nonlinear σ-model on group manifold

describe string moving in curved background

$$L = G_{\mu\nu}(X)\,\partial_a X^\mu \partial_b X^\nu g^{ab} + \dots$$

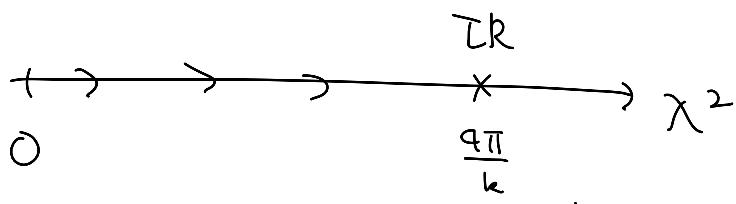
if curved is group manifold:

$$S = \frac{1}{4\lambda^2}\int \mathrm{Tr}\left[\partial_a g^{-1}\partial^a g\right] d^2\sigma + k\,\Gamma_{WZ}$$

$$\Gamma_{WZ} = \frac{1}{24\pi}\int_M d^3\xi\, \epsilon^{\alpha\beta\gamma}\,\mathrm{Tr}\left[(g^{-1}\partial_\alpha g)(g^{-1}\partial_\beta g)(g^{-1}\partial_\gamma g)\right]$$

∂M = 2d Eucl.

- $k=0$: ordinary σ-model; asymp free & massive
- $\lambda^2 = \frac{4\pi}{k}$ → conformal.



b, c

$$\therefore\ J^a(z) J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \sum_c \frac{i f_{abc} J^c(w)}{z-w} + \cdots$$

mode expansion $J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n$ $\quad J^a_n = \oint z^n J^a(z)$

$$[J^a_n, J^b_m] = \sum_c i f_{abc} J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}$$

"affine Lie algebra" (Kac-Moody) (also for $\bar{J}$)

Contains Virasoro : Sugawara construction

$\operatorname{tr} t^a t^b = \ell \delta^{ab}$
$C_r d_r = \ell_r |G|$

$$T(z) = \frac{1}{\beta} \sum_{a=1}^{|G|} :J^a J^a:$$

↑ constant to be fixed

$$\sum_{a,c,d} i f_{abc}\, i f_{acd} J^d \to \underline{\underline{C_A}}\, \delta_{bd}$$

(quadratic Casimir or dual Coxeter ×2)

$i f_{abc} J^c(w)$

$$T(z) J^b(w) = :J^a(z) J^a(z): J^b(w)$$

$$= \frac{1}{\beta} \sum_a 2 \frac{k\delta^{ab}}{(z-w)^2} J^a(z) + \sum_c \frac{C_A}{z-w} \frac{J^d(w)}{z-w}$$

$$= \frac{J^b(w)}{(z-w)^2} + \frac{\partial_w J^b}{z-w} + \cdots \quad \Rightarrow \beta = 2\left(k + \frac{C_A}{2}\right)$$

$\frac{C_A}{2} \to n$ for $su(n)$

$$T(z)T(w) = \frac{1}{\beta} T(z)\, J^b J^b\, \beta = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T}{(z-w)}$$

$k\sum_b 1 \times 2$

with $c = \frac{k\,|G|}{k + C_A/2} = \frac{k(n^2-1)}{k+n}$ for $SU(n)$

$$L_n = \frac{1}{2(k+C_A)} \sum_a \sum_m :J^a_m J^a_{n-m}:$$

neg. ⟵ ⟶ positive

$$[L_n, J^a_m] = -m J^a_{n+m} \quad \left([L_0, J^a_{-m}] = m J^a_{-m} \right)$$

Primary fields : highest weight of Kac-Moody representation

$\phi_{\lambda,\mu}$

$$J^a(z)\,\phi_{\lambda\mu}(w,\bar{w}) \sim -\frac{t^a_\lambda \cdot \phi_{\lambda\mu}}{z-w}$$

$$\Rightarrow \quad J^a_0 \phi_\lambda = -t^a_\lambda \phi_\lambda, \quad J^a_n \phi_\lambda = 0 \quad (n>0)$$

$$h_\lambda = \frac{\sum_a t^a_\lambda t^a_\lambda}{2(k + C_A/2)} = \frac{(\lambda, \lambda+2\rho)}{2(k+C_A/2)} = \frac{C_\phi}{2(k+C_A/2)}$$

(C_ϕ: quadratic Casimir of Rep λ)

Knizhnik - Zamolodchikov Eq.

$$\oint dz \sum_a \omega^a \langle J^a(z)\,\phi_1(z_1)\cdots\phi_n(z_n)\rangle$$
$$= -\sum_{i=1}^{n} \oint \frac{dz}{2\pi i} \frac{1}{z-z_i} \sum_a \omega^a t^a_i \langle \phi_1 \cdots \phi_n \rangle$$

using $L_{-1} = \frac{1}{k+g} \sum_a J^a_{-1} J^a_0$

null state $\left(L_{-1} + \frac{1}{k+g} \sum J^a_{-1} t^a_i \right) |\phi_i\rangle = 0$

$$\langle \phi_1 \cdots J^a_{-1}\phi_i \cdots \phi_n \rangle = \oint \frac{dz}{2\pi i}\,\frac{1}{z-z_i}\,\underbrace{\langle J^a(z)\,\phi_1 \cdots \phi_n \rangle}_{\sum_{j(\neq i)} \frac{t^a_j}{z-z_j}}$$

$$= \sum_{j\neq i} \frac{t_j^a}{z_i-z_j}\,\langle \phi_1 \cdots \phi_n \rangle$$

$$\therefore \left[\partial_{z_i} + \frac{1}{k+C_g}\sum_{j\neq i} \frac{\sum_a t_i^a \otimes t_j^a}{z_i - z_j} \right] \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle = 0$$

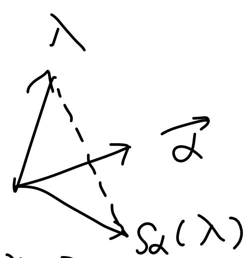
extended character

$$\chi^k_{(\lambda)}(\theta^i, \tau) = q^{-C_g/24}\,\mathrm{Tr}_{(\lambda),k}\left(e^{2\pi i \tau L_0}\, e^{-2\pi i \sum_j z_j H^j_0}\right)$$

H_0^i : Cartan subalgebra generators

Weyl character formula

$$\chi_{(\lambda)} = \frac{\sum_{w\in W} \epsilon(w)\,\Theta^{(k+\tilde{h})}_{w(\lambda+\rho)}}{\sum_{w\in W} \epsilon(w)\,\Theta^{(\tilde{h})}_{w(\rho)}}$$



$$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$$

$$\Theta^{(k)}_\lambda \equiv \sum_{\alpha^\vee} e^{-\pi i\left[-\tau k\left|\alpha^\vee + \frac{\lambda}{k}\right|^2 + 2k(\alpha^\vee, z) + 2(\lambda, z)\right]}$$

generalized theta function

$SU(2)_k$ WZW

Representation : spin-j $\rightarrow$ $h_j = \frac{j(j+1)}{k+2}$

$c = \frac{3k}{k+2}$ $\quad \left\{ J^{a_1}_{-n_1} \cdots J^{a_n}_{-n_n} L_{-m_1} \cdots L_{-m_l} |j\rangle \right\}$

$$\Theta^{(k+2)}_{\lambda} = \sum_{\alpha^\vee} e^{-\pi i \left[-\tau(k+2) \underbrace{\left| \alpha^\vee + \frac{\lambda}{k+2} \right|^2}_{4\left(n+\frac{\lambda}{N}\right)^2} + 2(k+2)(\alpha^\vee, z) + 2(\lambda, z) \right]} \quad \alpha^\vee = 2n,\ N = 2(k+2)$$

$$= \sum_{n=-\infty}^{\infty} e^{2\pi i \left[\tau N \left(n+\frac{\lambda}{N}\right)^2 - \frac{N}{2}\left(n+\frac{\lambda}{N}\right) z \right]} \equiv \Theta_{\lambda, \frac{N}{2}}(\tau, z)$$

$$\left(\Theta_{n,m}(\tau, z) \equiv \sum_{j \in \mathbb{Z} + \frac{n}{2m}} e^{2\pi i \tau \left(j^2 - 2j\frac{z}{\tau}\right) m} \right)$$

Weyl-formula

$$\chi_\lambda(\tau, z) = \frac{\Theta_{\lambda, \frac{N}{2}}(\tau, z) - \Theta_{-\lambda, \frac{N}{2}}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)}$$

$\lambda = 2j+1$, $0 \leq 2j \leq k$, $\underline{1 \leq \lambda \leq k+1 = \frac{N}{2} - 1}$

Sym: $\underbrace{\chi_\lambda(\tau, z) = \chi_{\lambda+N}(\tau, z)}_{\lambda = \{1, \cdots, N\}}$, $\chi_\lambda(\tau, z) = -\chi_{-\lambda}(\tau, -z)$

$\lambda = \frac{N}{2}$, $z = 0 \rightarrow \chi_\lambda = 0$

$\lambda \cong N - \lambda \longrightarrow \left\{ 1, \cdots, \frac{N}{2} - 1 \right\}$ fundamental domain

$\tau \rightarrow \tau + 1$

$$h_\lambda = \frac{N-4}{8N} - \frac{1}{8} + \frac{\lambda^2}{2N} = \frac{\lambda^2 - 1}{2N} = \frac{j(j+1)}{k+2} \ //$$

Modular transform of $SU(2)_k$ character

$T: \tau \to \tau+1$; $\quad \chi_\lambda(\tau+1) = q^{h_\lambda} \chi_\lambda(\tau)$

$S: \tau \to -\frac{1}{\tau}$

using Poisson resum. $(z=0)$

$$\chi_\lambda\left(-\frac{1}{\tau}\right) = \sum_{\mu=1}^{\frac{N}{2}-1} S_{\lambda\mu} \chi_\mu(\tau)$$

$$S_{\lambda\mu} = \frac{2}{\sqrt{N}} \sin\left(\frac{2\pi}{N} \lambda\mu\right)$$

A-D-E Classification

partition function

$$Z = \sum_{\lambda,\bar{\lambda}=1}^{k+1} N_{\lambda\bar{\lambda}} \chi_\lambda(q) \bar{\chi}_{\bar{\lambda}}(\bar{q})$$

$N_{\lambda\bar{\lambda}}$ → # of primary fields $\phi_{h_\lambda, \bar{h}_{\bar{\lambda}}}$

should be invariant under modular invariance

Considering only T & S transform, SU(2) WZW is completely classified.

① A_{k+1} : $N_{\lambda\bar{\lambda}} = \delta_{\lambda\bar{\lambda}}$ "diagonal"

k=even

② $D_{\frac{k}{2}+2}$: $\displaystyle\sum_{\substack{\lambda=1 \\ (\lambda=\mathrm{odd})}}^{\frac{k}{2}-1} |\chi_\lambda + \chi_{k+2-\lambda}|^2 + 2|\chi_{\frac{k}{2}+1}|^2$ $k=4\rho$

$D_{\frac{k}{2}+1}$: $\displaystyle\sum_{\substack{\lambda=1 \\ \lambda=\mathrm{odd}}}^{k+1} |\chi_\lambda|^2 + |\chi_{\frac{k}{2}+1}|^2 + \sum_{\substack{\lambda=2 \\ \mathrm{even}}}^{\frac{k}{2}-1} (\chi_\lambda \chi^*_{k+2-\lambda} + c.c.)$ $k=4\rho-2$

③ $k+2=12$

$$|\chi_1+\chi_7|^2 + |\chi_4+\chi_8|^2 + |\chi_5+\chi_{11}|^2 \quad E_6$$

$k+2=18$

$$|\chi_1+\chi_{17}|^2 + |\chi_5+\chi_{13}|^2 + |\chi_7+\chi_{11}|^2 + |\chi_9|^2 + [(\chi_3+\chi_{15})\chi_9^* + c.c.] \quad E_7$$

$k+2=30$

$$|\chi_1+\chi_{11}+\chi_{19}+\chi_{29}|^2 + |\chi_7+\chi_{13}+\chi_{17}+\chi_{23}|^2$$

- "D" series are related to orbifolding

- Minimal CFT's Partition function can be classified by coseting

- $SU(3)_k$ WZW has been classified

Multichannel Kondo effect.

$\psi_{\alpha i}(\vec{k})$

α ↑ spin; i ↑ channel : different d-shell orbitals $i = 1, \cdots, k$

- Construct currents.

Total central charge $\quad C = 2 \times k \times 1$ (2 ← $\alpha = \uparrow, \downarrow$; k ← i; 1 ← Dirac fermion)

① U(1) charge

$$J = \sum_{\alpha, i} : \psi^{\alpha i \dagger} \psi_{\alpha i} : \quad \rightarrow \quad C = 1$$

② spin current

$$\vec{J} = \sum_{\alpha, \beta, i} \psi^{\alpha i \dagger} \frac{\vec{\sigma}_\alpha^{\ \beta}}{2} \psi_{\beta i} \quad \rightarrow \quad SU(2)_k \rightarrow C = \frac{3k}{k+2}$$

③ flavour current

$$J^A = \sum_{\alpha, i, j} \psi^{\alpha i \dagger} (T^A)_i^{\ j} \psi_{\alpha j} \quad \rightarrow \quad SU(k)_2 \rightarrow C = \frac{2(k^2-1)}{2+k}$$

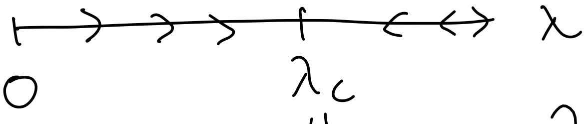
should be SU(k)

$$1 + \frac{3k}{k+2} + \frac{2(k^2-1)}{2+k} = 2k \quad !$$

Only spin current will couple with the magnetic impurity

① overscreened : $\frac{1}{2}k \left\{ \downarrow\downarrow\downarrow\downarrow \right. \Uparrow S = \downarrow$ if $\frac{k}{2} > S$

RG flow



$\lambda_c = ?$

$$H_S = \frac{1}{2\pi(k+2)} \vec{\mathcal{J}}^2 + \lambda \vec{J} \cdot \vec{S}\, \delta(x)$$

$\vec{\mathcal{J}} = \vec{J} + \delta(x) \vec{S}$

by same trick ; $\lambda_c = \frac{2}{k+2}$

② Impurity entropy

$$\tilde{q} = e^{-\frac{4\pi l}{\beta}}$$

$$\tilde{Z}_{\alpha\beta}(\tilde{q}) = \sum_j \langle \alpha | j \rangle \langle j | \beta \rangle \underbrace{\chi_j(\tilde{q})}_{\tilde{q}^{-\frac{c}{24} + h_j}(1 + q + \cdots)}$$

as $\tilde{q} \to 0$ $\quad \cong \langle \alpha | 0 \rangle \langle 0 | \beta \rangle e^{\frac{\pi l c}{6\beta}} + \cdots$

$$\to F = -\pi c \frac{T^2 l}{6} - T \underbrace{\ln \langle \alpha | 0 \rangle \langle 0 | \beta \rangle}_{\text{impurity entropy}}$$

$F = \frac{1}{\beta} \log \tilde{Z}$

$$\downarrow$$

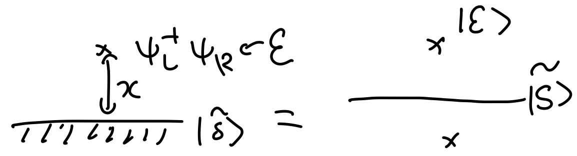
$$C = \frac{\pi c l}{3} T$$

$$\langle \alpha | 0 \rangle = \langle \tilde{S} | 0 \rangle = \frac{S_S^0}{\sqrt{S_0^0}} \leftarrow \text{modular S-matrix}$$

$\tilde{S}$: spin-S boundary state

$$S_j^{j'} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi(2j+1)(2j'+1)}{k+2} \right)$$

③ resistivity from boundary 1-pt



= 2-pt on $\mathbb{C}$

$$\langle \psi_L^+ \psi_R \rangle_{UHP} = \frac{1}{(2x)} \cdot \underbrace{\left(\frac{S_S^\varepsilon / \sqrt{S_0^\varepsilon}}{S_S^0 / \sqrt{S_0^0}} \right)}$$

$$S^{(1)} = \frac{\cos \frac{\pi(2S+1)}{k+2}}{\cos \frac{\pi}{k+2}}$$

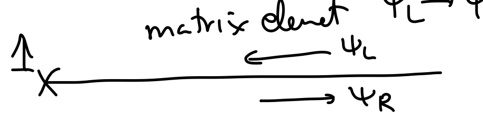
T=0 resistivity, (Kubo formula)

$$\rho(0) \propto \left[\frac{1 - |S^{(1)}|}{2} \right]$$

(1) $k=1, S=\frac{1}{2}$ $\quad \rho(0) = 0$ (Fermi liquid)

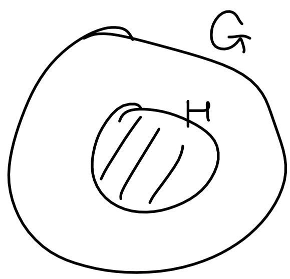
(2) others ; interesting non-Fermi. (ex) $k=2, S=\frac{1}{2}$ $\quad S^{(1)} = 0$!

$|S^{(1)}| < 1$: matrix element $\psi_L \to \psi_R$



Coset CFT

consider G & its subgroup H



$$\hat{G}_{k_G} \to T_G = \frac{\sum_{a=1}^{|G|} :J_G^a J_G^a:}{2(k_G+\tilde{h}_G)}$$

$$\hat{H}_{k_H} \to T_H = \frac{\sum_{a=1}^{|H|} :J_H^a J_H^a:}{2(k_H+\tilde{h}_H)}$$

$$T_G(z) J_H^i(w) = \frac{J_H^i(w)}{(z-w)^2} + \frac{\partial_w J_H^i}{z-w} + \cdots$$

&

$$T_H(z) J_H^i(w) = \frac{J_H^i(w)}{(z-w)^2} + \frac{\partial_w J_H^i}{z-w} + \cdots$$

$$\therefore (T_G - T_H) J_H^i(w) = \text{nonsingular}$$

$$\overset{\shortparallel}{T_{G/H}} \quad \to \quad T_{G/H}(z)["H"](w) = \text{nonsglr}$$

(ex) $[T_{G/H}, T_H] = 0$

Writing $T_G = T_{G/H} + T_H$

$$T_G T_G = (T_{G/H} + T_H)(T_{G/H} + T_H)$$

$$= T_{G/H} T_{G/H} + T_H T_H + \overset{\text{nonsglr}}{2 \cdots}$$

$$\therefore T_{G/H}(z) T_{G/H}(w) = \frac{(c_G - c_H)/2}{(z-w)^4} + \frac{\overbrace{T_G - T_H}^{T_{G/H}}}{(z-w)^2} + \frac{\partial_w (\overbrace{T_G - T_H}^{T_{G/H}})}{z-w} + \cdots$$

$$\overset{\shortparallel}{T_G T_G} - T_H T_H$$

$$\therefore c_{G/H} = c_G - c_H = \frac{k_G |G|}{k_G + \tilde{h}_G} - \frac{k_H |H|}{k_H + \tilde{h}_H}$$

Primaries

$\hat{G}_{k_G}$: $\phi^{(G)}_{\lambda_G}$ $\quad T_G \phi^{(G)}_{\lambda_G} = \frac{h_{\lambda_G}}{(z-w)^2}\phi^{(G)}_{\lambda_G} + \frac{\partial_w \phi^{(G)}_{\lambda_G}}{z-w} + \cdots$

$\hat{H}_{k_H}$: $\phi^{(H)}_{\lambda_H}$ $\quad T_H \phi^{(H)}_{\lambda_H} = \frac{h_{\lambda_H}}{(z-w)^2}\phi^{(H)}_{\lambda_H} + \frac{\partial_w \phi^{(H)}_{\lambda_H}}{z-w} + \cdots$

$\{\phi^{(G)}_{\lambda_G}\} \supset \{\phi^{(H)}_{\lambda_H}\} \to T_{G/H}\{\phi^{(H)}_{\lambda_H}\} =$ nonsgla

$\Rightarrow \{\phi^{(G)}_{\lambda_G}\} \Big/ \{\phi^{(H)}_{\lambda_H}\}$ "primaries" of G/H coset.

with $h^{(G/H)}_{(\lambda_G,\lambda_H)} = h^{(G)}_{\lambda_G} - h^{(H)}_{\lambda_H}$

Character

$$\chi^{(G)}_{\lambda_G}(q,z) = \sum_{\lambda_H} \underbrace{\chi^{G/H}_{\lambda_G,\lambda_H}(\tau)}_{\text{Branching function}} \chi^{(H)}_{\lambda_H}(q,z)$$

Branching function
$\shortparallel$
character of G/H

very powerful way of constructing CFT's & their spectra

① $SU(2)_k \supset U(1)$ → $J^3 = i\sqrt{k}\partial\phi$, $J^+ = \psi\, e^{\sqrt{2}i\frac{\phi}{\sqrt{k}}}$

↑ PF $\quad 1-\frac{1}{k}+\frac{1}{k}=1$

$$\frac{SU(2)_k}{U(1)} = \mathbb{Z}_k \text{ parafermion (PF) CFT.}$$

PF currents $\{\psi_\ell, \psi_\ell^+\} \to h = \frac{\ell(k-\ell)}{k}$

Central change

$$c_k = \frac{2(k-1)}{k+2}$$

OPE: $\psi_1(z)\psi_1^+(w) = (z-w)^{-2h_1}\left[\mathbb{1} + (z-w)^2 T(w) + \cdots\right]$

$k=2$ Maj. fermion, ...

Virasoro primary of $SU(2)_k$: $(J_0^-)^N \psi^\ell = \psi^\ell_m$ $\quad \ell-m=2N, -\ell\le m\le\ell$ $\to h = \frac{\ell(\ell+2)}{4(k+2)}$

↑ SU(2) lowering op. (no dim)

Primary of PF:

$$\psi^\ell_m = \Phi^\ell_m\, e^{\sqrt{2}i\frac{m\varphi}{2\sqrt{k}}} \Rightarrow h(\Phi^\ell_m) = \frac{\ell(\ell+2)}{4(k+2)} - \frac{m^2}{4k}$$

↑ $SU(2)_k$ $\quad$ ↑ PF $\quad$ ↑ U(1)

Character

$$\chi^{(k)}_\ell(\tau,z) = \sum_{\substack{m=\ell+2N \\ N=-\infty}}^{\infty} \eta\, C^\ell_m(\tau)\, \frac{q^{\frac{m^2}{4k}} - z\, m}{\eta} = \sum_{m=-\ell}^{2k-\ell-1} C^\ell_m(\tau)\, \Theta_{m,k}(\tau,z)$$

PF "string function"

Correlators $\langle \Phi \cdots \Phi \rangle = \dfrac{\langle \psi \cdots \psi \rangle}{\langle e \cdots e \rangle}$

② $\dfrac{SU(2)_1 \times SU(2)_k}{SU(2)_{k+1}}$

$$C = 1 + \frac{3k}{k+2} - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)}$$

$k+2 \equiv m$

$\Rightarrow$ Unitary Minimal $\mathcal{M}_{m,m+1}$

$D = \Theta_{1,2} - \Theta_{-1,2}$

$$\chi^{(k)}_\lambda(\tau, z) = \frac{1}{D}\left(\Theta^{(\tau,z)}_{\lambda,\frac{N}{2}} - \Theta_{-\lambda,\frac{N}{2}}(\tau,z)\right)$$

$N = 2(k+2)$, $1 \leq \lambda \leq k+1$

$$\left(\Theta_{\lambda,\frac{N}{2}}(\tau,z) \equiv \sum_{n=-\infty}^{\infty} e^{2\pi i\left[\tau\left(n+\frac{\lambda}{N}\right)^2 - 2\left(n+\frac{\lambda}{N}\right)z\right]\frac{N}{2}}\right)$$

$- (nN+\lambda)z \Rightarrow x^{(nN+\lambda)}$

$$= \frac{1}{D}\sum_n q^{\frac{(\lambda+nN)^2}{2N}}\left(x^{+\frac{(nN+\lambda)}{2}} - x^{-\frac{(nN+\lambda)}{2}}\right)$$

$\widehat{SU(2)}_1$

$$\hookrightarrow \chi^{(1)}_{\lambda_1} = \frac{1}{D}\left(\Theta_{\lambda_1,3} - \Theta_{-\lambda_1,3}\right) = \frac{1}{\eta}\sum_{k=-\infty}^{\infty} q^{\left(k+\frac{\lambda_1}{2}\right)^2} x^{\left(k+\frac{\lambda_1}{2}\right)}$$

(k here is not the level k)

$\lambda_1 = 1, 2 (\equiv 0)$

$$\chi^{(k+1)}_\mu = \frac{1}{D}\left(\Theta_{\mu,\frac{N}{2}+1} - \Theta_{-\mu,\frac{N}{2}+1}\right)$$

$$= \frac{1}{D}\sum_l q^{\frac{(\mu+l(N+2))^2}{2(N+2)}}\left(x^{+\frac{(l(N+2)+\mu)}{2}} - x^{-\frac{(l(N+2)+\mu)}{2}}\right)$$

$m \equiv k+2$

$$\frac{2nm+\lambda}{2} + \left(k+\frac{\lambda_1}{2}\right) = \frac{2l(m+1)+\mu}{2}$$

$$\mu = \lambda + \lambda_1 + 2(nm + k - l(m+1))$$

$(n = l) \Rightarrow (k - l)$

$$k + \frac{\lambda_1}{2} = \frac{\mu - \lambda}{2} + l$$

$$\chi^{(3)}_{\lambda_1}\,\chi^{\overbrace{(k+2)}^{m}}_{\mu} = \sum_{\lambda} \chi^{\overset{m+1}{(k+3)}}_{\lambda} \underbrace{\sum_{t=-\infty}^{\infty} \frac{1}{\eta}\left(q^{\Lambda^{(t)}_{\mu\lambda}} - q^{\Lambda^{(t)}_{\mu,-\lambda-2}}\right)}_{\chi^{[m,m+1]}_{\mu,\lambda}(q)}$$

$$\Lambda^{(t)}_{\mu\lambda} = \frac{\left[\mu(m+1) - \lambda m + 2tm(m+1)\right]^2}{4m(m+1)}$$

$1 \leqq \lambda \leqq m$

$1 \leqq \mu \leqq m-1$

---

modular trasf

$$S^{[m]}_{\lambda\lambda'} = \sqrt{\frac{2}{m}}\,\sin\left(\frac{\pi}{m}\lambda\lambda'\right)$$

$$S^{[3]}_{\lambda_1\lambda_1'} = \sqrt{\frac{2}{3}}\,\sin\left(\frac{\pi}{3}\underbrace{\lambda_1\lambda_1'}_{\pm 1 \bmod 3}\right) \qquad \lambda_1, \lambda_1' = 1, 2$$

$$= \frac{1}{\sqrt{2}}(-)^{\lambda_1\lambda_1'} \hookrightarrow (\lambda+\mu)(\lambda'+\mu')$$

$$S^{[m+1]}_{\mu\mu'} = \sqrt{\frac{2}{m+1}}\,\sin\left(\frac{\pi}{m+1}\mu\mu'\right)$$

& using $S^{[m+1]^2} = \mathbb{1}$

$$S^{[m,m+1]}_{(\mu,\lambda)(\mu',\lambda')} = \sqrt{\frac{2}{m(m+1)}}\,(-1)^{(\lambda+\mu)(\lambda'+\mu')}\sin\frac{\pi\lambda\lambda'}{m}\sin\frac{\pi\mu\mu'}{m+1}$$

(cf) minimal $\sqrt{\frac{2}{pp'}}\cdot(-1)^{1+r\sigma+s\rho}\sin\left(\pi\frac{p}{p'}r\rho\right)\sin\left(\pi\frac{p'}{p}s\sigma\right)$ $\quad p = m+1$, $p' = m$

✓

"Conformal Zoo"

minimal CFT M_{k+2}

↑ $l=1$

$N=1$ super minimal ← $l=2$

$$\frac{SU(2)_k \otimes SU(2)_l}{SU(2)_{k+l}} \rightarrow c = \frac{3k}{k+2}\left(1 - \frac{2(k+2)}{(l+2)(k+l+2)}\right)$$

↓ k

"fractional" minimal CFT

$G^+ \sim \psi_1^{(k)} : e^{i\sqrt{\frac{k+2}{k}}\varphi} :$, $J \sim i\partial\varphi$

↑ boson with different radius

$G^- = \psi_1^+ : e^{-i\sqrt{\ }\varphi} :$ $h = \frac{3}{2}$

satisfy $T G^{\pm} \sim \frac{3/2}{z^2} G^{\pm} + \frac{1}{z}\partial G^{\pm} + \cdots$

$l \to \infty$

[$SU(2)_k$ WZW] ≡ [$N=2$ SUSY minimal CFT]

---

nonunitary series from k = fractional #

$SU(2)_k$ ← "admissible Rep"

$$\frac{SU(N)_1 \times SU(N)_k}{SU(N)_{k+1}}$$: spin N generalizat'n "W_N" model

level-rank duality

- Same CFT is represented by different cosets.

(Ex) IM : $\frac{SU(2)_1 \times SU(2)_1}{SU(2)_2}$ TIM $\frac{SU(2)_1 \times SU(2)_2}{SU(2)_3}$

M_4 M_5

but $c = \frac{1}{2}$ by $\frac{(E_8)_1 \times (E_8)_1}{(E_8)_2}$, $\frac{(E_7)_1 \times (E_7)_1}{(E_7)_2}$

(more later)

$$2\frac{1\cdot 248}{1+30} - \frac{2\cdot 248}{2+30} = \frac{1}{2}$$

$$2\times\frac{1\cdot 133}{1+18} - \frac{2\cdot 133}{2+18} = \frac{7}{10}$$

$$\frac{SU(3)_1 \times SU(3)_1}{SU(3)_2} = \frac{2\cdot 8}{4} - \frac{2\cdot 8}{5} = \frac{4}{5} \in M_6 \text{ (3-state Potts)}$$

* Degeneracies are lifted at off-criticality

Z_kPF : $\frac{SU(2)_k}{U(1)}$

& $\frac{SU(k)_1 \times SU(k)_1}{SU(k)_2} \rightarrow c = \frac{2(k-1)}{k+2}$

k : either level or rank → "level-rank" duality

[Zamolodchikov I] Away from criticality

conformal symmetry is broken by adding an relevent operator ("primary")

formally $S_g = S_{CFT} + \sum g_a \int \Phi_a(x)\, d^2x$

Consider two-point functions

$$\langle T(z,\bar z)\, T(0,0)\rangle = \frac{F(t)}{z^4} \qquad t = \log z\bar z$$

$$\langle T(z,\bar z)\, \Theta(0,0)\rangle = \frac{H(t)}{z^3 \bar z} \leftarrow \qquad [\Theta] = (1,1)$$

$$\langle \Theta(z,\bar z)\, \Theta(0,0)\rangle = \frac{G(t)}{z^2 \bar z^2} \qquad F, H, G \geq 0$$

$$\partial_{\bar z} T = \partial_z \Theta \;, \quad \partial_z T = \partial_{\bar z} \Theta$$

$$\langle \partial_{\bar z} T\, T(0)\rangle = \partial_z \langle \Theta T \rangle = \frac{\dot F}{\bar z z^4}$$

$$\partial_z \langle \Theta T\rangle = \frac{\dot H}{z^4 \bar z} - \frac{3H}{z^4 \bar z} \Rightarrow \dot F = \dot H - 3H$$

$$\partial_{\bar z} \langle T \Theta\rangle = \frac{\dot H}{z^3 \bar z^2} - \frac{H}{z^3 \bar z^2} = \partial_z \langle \Theta\Theta\rangle$$

$$= \frac{\dot G}{z^3 \bar z^2} - \frac{2G}{z^3 \bar z^2}$$

$$\Rightarrow \dot H - H = \dot G - 2G$$

$$\left.\begin{aligned} \dot F - \dot H &= 3H \\ \dot H - \dot G &= H - 2G \end{aligned}\right\} \Rightarrow \frac{C}{2} \equiv F - H + 3H - 3G$$

$$\to C = 2F + 4H - 6G$$

$$\dot{c} = -12G \leq 0$$

"c-theorem"

"=" when $G = 0 \Rightarrow \Theta \equiv \beta^a \Phi_a = 0 \Rightarrow \beta^a = 0$
↓
fixed pt.

$$\dot{c} = \frac{dc}{dt} = \beta^i \frac{\partial c(g)}{\partial g^i} = -12G$$

$$= -12\, z^2 \bar{z}^2 \langle \Theta \Theta \rangle$$

$$= -12\, z^2 \bar{z}^2 \beta^a \beta^b \langle \Phi_a(z,\bar{z})\, \Phi_b(0,0) \rangle$$

$$\langle \Phi_a(z,\bar{z})\, \Phi_b(0,0) \rangle = \frac{1}{z^2 \bar{z}^2} G_{ab}(t,g)$$

$$= -12\, G_{ab}(g)\, \beta^a \beta^b$$

$$G_{ab}(g) = G_{ab}(0,g)$$

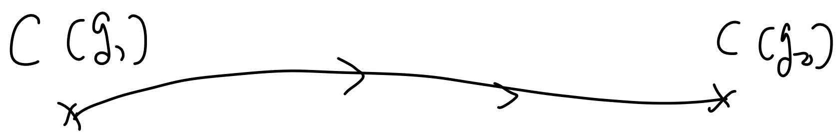
$$\frac{\partial c}{\partial g^a} = -12 G_{ab} \beta^b$$

G_{ab}: metric in parameter space ("Zamolodchikov" metric)

$$\therefore \quad \beta^a(g) = -\frac{1}{12} G^{ab}(g) \frac{\partial c}{\partial g^b}$$

if $\frac{\partial c}{\partial g^i} = 0$ at $g = g^*$ for all i

$\Rightarrow \quad \beta^a(g^*) = 0$ for all a RG FP



$$c_1 > c_2$$

Explicit example: $\mathcal{M}_p + g\,\Phi_{1,3} \equiv \Phi$ "least relevant"

$$\Theta = \beta(g)\,\Phi$$

$$h = 1 - \frac{2}{p+1} \equiv 1 - \epsilon$$

(set $\epsilon = 0$)

$$\beta(g) = \epsilon g - \frac{1}{2}\,\mathbb{C}\,g^2 - \frac{4}{3}\,g^3 + \cdots, \quad p \gg 1 \to \epsilon \ll 1$$

↑ pert. in g

$$\mathbb{C} = C_{(1,3)(1,3)}^{(1,3)}$$

$$[\text{Dotsenko-Fateev}] = \frac{4}{\sqrt{3}}\left(1 - \frac{3\epsilon}{2} + O(\epsilon^2)\right)$$

$$\beta(g) = \epsilon g - \frac{2}{\sqrt{3}}\left(1 - \frac{3\epsilon}{2}\right) g^2 - \frac{4}{3}\,g^3 + \cdots \quad (g \sim \epsilon)$$

$= 0$ for fixed points.

$g = 0$ (trivial, UV FP)

$$g_* = \frac{\sqrt{3}}{2}\,\epsilon\left(1 + \frac{\epsilon}{2}\right) + O(\epsilon^3) \quad \text{(nontrivial, Wilson-Fisher FP)}$$

$$\boxed{g_* > 0}$$

$$\beta^a(g) = -\frac{1}{12}\,G^{ab}(g)\,\frac{\partial C}{\partial g^b} \to \beta(g) = -\frac{1}{12}\frac{dC}{dg}$$

$$\therefore\ \Delta C = C_{IR} - C_{UV} = -12\int_0^{g_*} \beta(g)\,dg$$

$$= -12\left(\frac{\epsilon}{2}\,g_*^2 - \frac{2}{3\sqrt{3}}\left(1 - \frac{3\epsilon}{2}\right) g_*^3 - \frac{1}{3}\,g_*^4 + \cdots\right)$$

$$= -\frac{3}{2}\epsilon^3 - \frac{9}{4}\epsilon^4$$

$$\therefore\ C_{IR} = \underbrace{C_{UV}}_{1 - \frac{6}{p(p+1)}} - \frac{3}{2}\left(\frac{2}{p+1}\right)^3 - \frac{9}{4}\left(\frac{2}{p+1}\right)^4 + \cdots$$

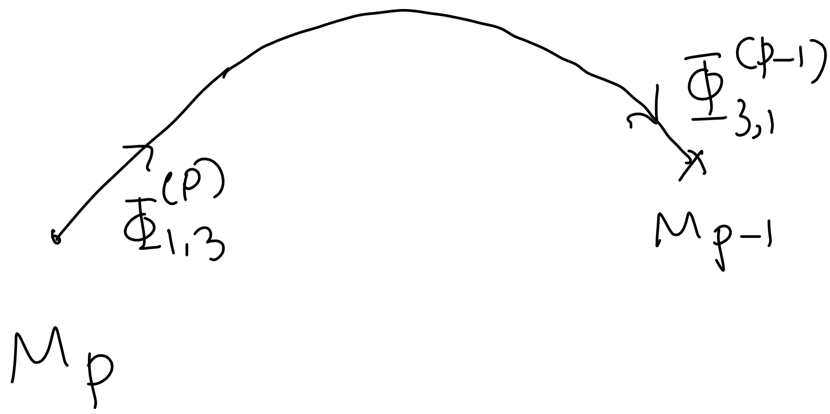
$$= 1 - \frac{6}{p(p-1)}$$

$\mathcal{M}_p \to \mathcal{M}_{p-1}$

$$\left.\frac{d\beta}{dg}\right|_{g_*} = -\epsilon - \epsilon^2 \longrightarrow h(\Phi)_* = 1+\epsilon+\epsilon^2 = 1+\frac{2}{p-1}$$

least irrelevant!

$\equiv \Phi^{(p-1)}_{(3,1)}$



Is this only valid for large p?

No. This is exact, non-pert. result! valid for any p.

(ex) 3-state Potts $\longrightarrow$ TIM

TIM $\longrightarrow$ IM etc.

Need "Integrability"

- more advanced topic than CFT

- will be two steps
  - show integrability i.e. ∞ # of conserved charges
  - use integrability to solve the QFT [only results]

Integrability from CFT [Zamolodchikov II]

$$S = S_{CFT} + \lambda \Phi \quad ; \quad \Phi = \text{some of relevant } (h<1)$$

If a pair $(T_{s+1}, \Theta_{s-1}) \ni$: $\partial_{\bar z} T_{s+1} = \partial_z \Theta_{s-1}$ for $s \in \mathbb{Z}$

$$\Rightarrow \quad P_s = \oint \left(T_{s+1}\, dz + \Theta_{s-1}\, d\bar z \right) \quad : \text{integrals of motion}$$

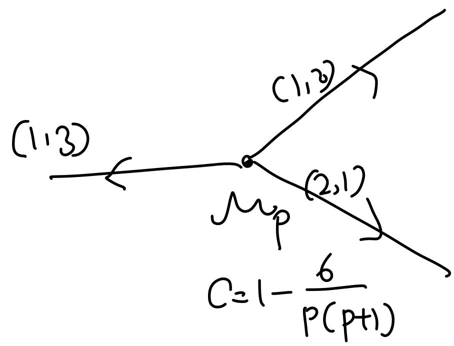
(ex) $s=1$ $\quad \partial_{\bar z} T = \partial_z \Theta$ $\qquad$ s : spin $= h - \bar h$

Concrete cases.

$$S_p^{(1,3)} = S_{M_p} + \lambda \int \Phi_{(1,3)}\, d^2x$$

$$S_p^{(1,2)} = S_{M_p} + \lambda \int \Phi_{(1,2)}\, d^2x$$

$$S_p^{(2,1)} = S_{M_p} + \lambda \int \Phi_{(2,1)}\, d^2x$$



above pair exists for

$S_p^{(1,3)}$: $\quad s = \text{odd}$

$S_p^{(1,2)}$ & $S_p^{(2,1)}$; $\quad s = 1,\ 6n \pm 1 = 1, 5, 7, 11, \cdots$

How to use Integrability?

- Exact S-matrix & spectrum
- Thermodynamic Bethe ansatz
- Correlation functions

will skip details but show examples

[Example 1]

$$\mathcal{M}_{p} + \lambda \Phi_{1,3} \longrightarrow \mathcal{M}_{3}$$

$p = 4$ (TIM) $\qquad \mathcal{M}_3$ = IM

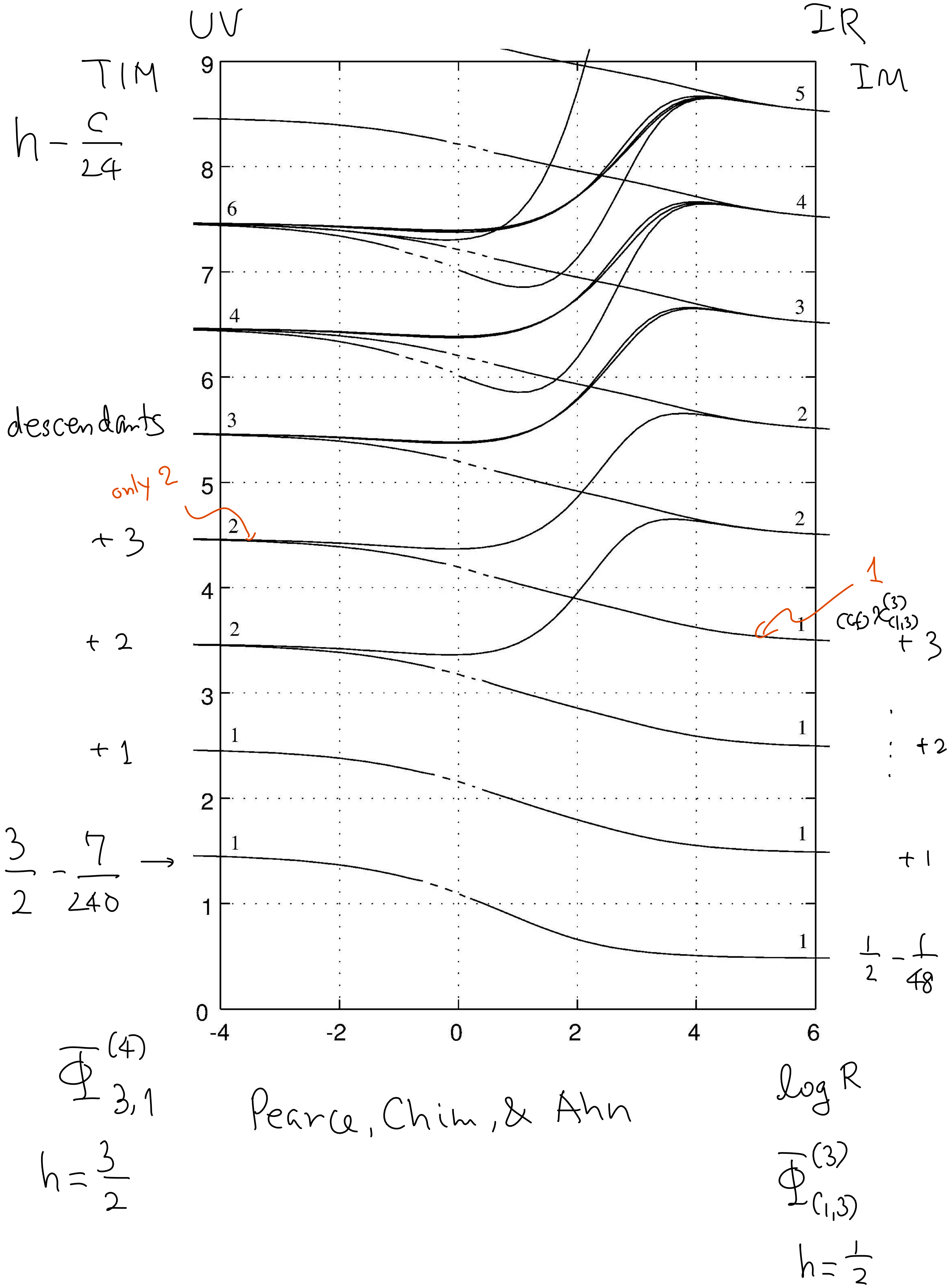
One can obtain EXACT RG flows validinary scale from UV to IR for all operators.

[Example 2]

$$\underset{\text{IM}}{\mathcal{M}_3} \cong \frac{E_{8,1} \times E_8'}{E_8^{\,2}} + \lambda \Phi \Rightarrow \text{massive scattering theory}$$

(Φ: least relevant; $\lambda < 0$)

S-matrix bootstrap fixes all mass spectrum.


UV
IR
TIM
IM
h - c/24
descendants
only 2
+ 3
+ 2
+ 1
3/2 - 7/240
(4) ℓ χ^(3)_(1,3)
+ 3
+ 2
+ 1
1/2 - 1/48
9
8
7
6
5
4
3
2
1
0
-4
-2
0
2
4
6
log R
Φ^(4)_{3,1}
h = 3/2
Pearce, Chim, & Ahn
Φ^(3)_(1,3)
h = 1/2

[Example 2] E_8 symmetry in Nature.

for $IM + \lambda \Phi_{1,2}$ $(\lambda < 0)$

its E_8 spectrum (can be seen from $\frac{E_8 \times E_8}{E_8}$)

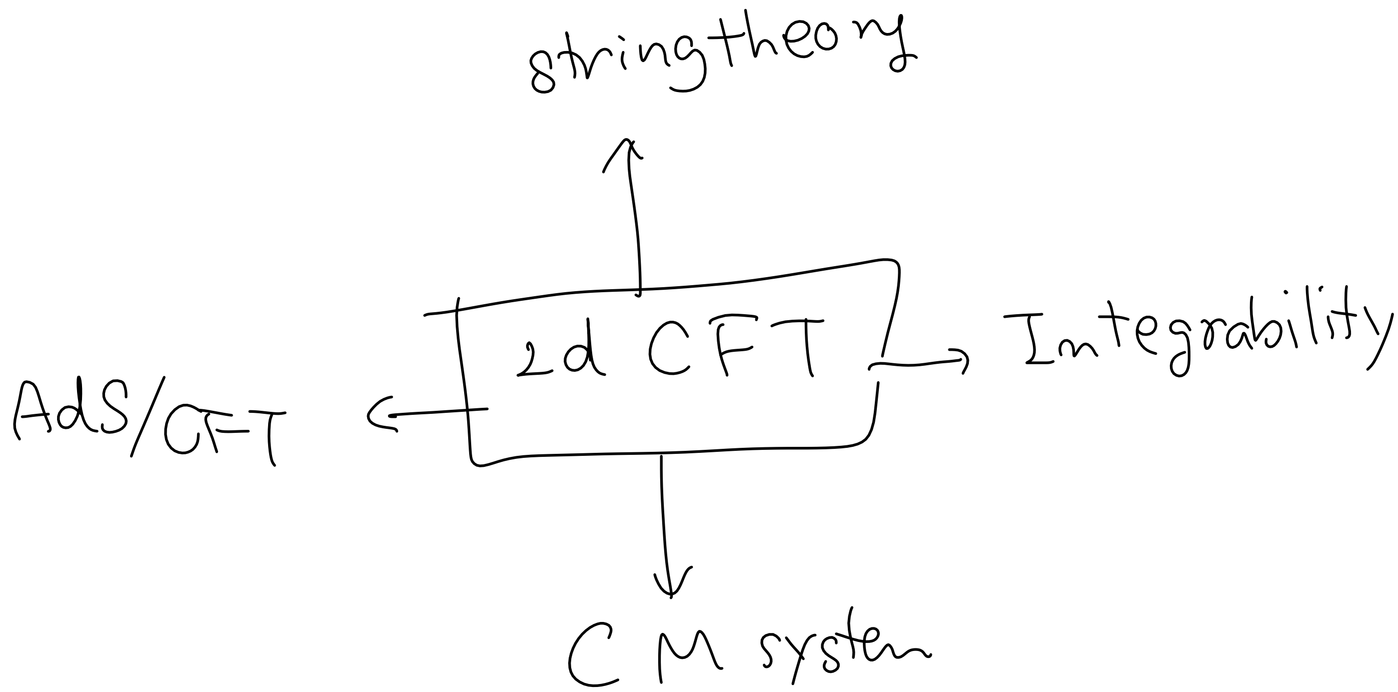
IM + external magnetic field

$$H = -J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j} + h \sum_i \sigma_i$$ (let $J = J_c$)

$J_c = \frac{1}{2} \sinh^{-1}(1)$

[pdf file]

[Conclusion]



Modern "Hydrogen atom"
for theoretical physicists.